Ch 3: Fuzzy Relations

Classical Relation

Definition 3.1. *(Classical relation). A subset $R \subseteq X \times Y$ where X and Y are classical sets, is a classical relation.*

A classical relation can be characterized by a function $R : X \times Y \to \{0, 1\}$,

$$R(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in R \\ 0 & \quad \text{otherwise} \end{cases} .$$

Fuzzy Relation

Definition 3.2. *(Fuzzy Relation, Sanchez [129], Di Nola-Sessa-Pedrycz-Sanchez [48], De Baets [42]) Let* X, Y *be two classical sets. A mapping* $R : X \times Y \to [0,1]$ *is called a* ***fuzzy relation****. The number* $R(x,y) \in [0,1]$ *can be interpreted as the degree of relationship between* x *and* y.

Remark 3.3. *A fuzzy relation can be seen as a fuzzy subset of the set* $X \times Y$. *We denote by* $\mathcal{F}(X \times Y)$ *the family of all fuzzy relations between elements of* X *and* Y.

Example

Example 3.4. $R =$ "*much greater than*"

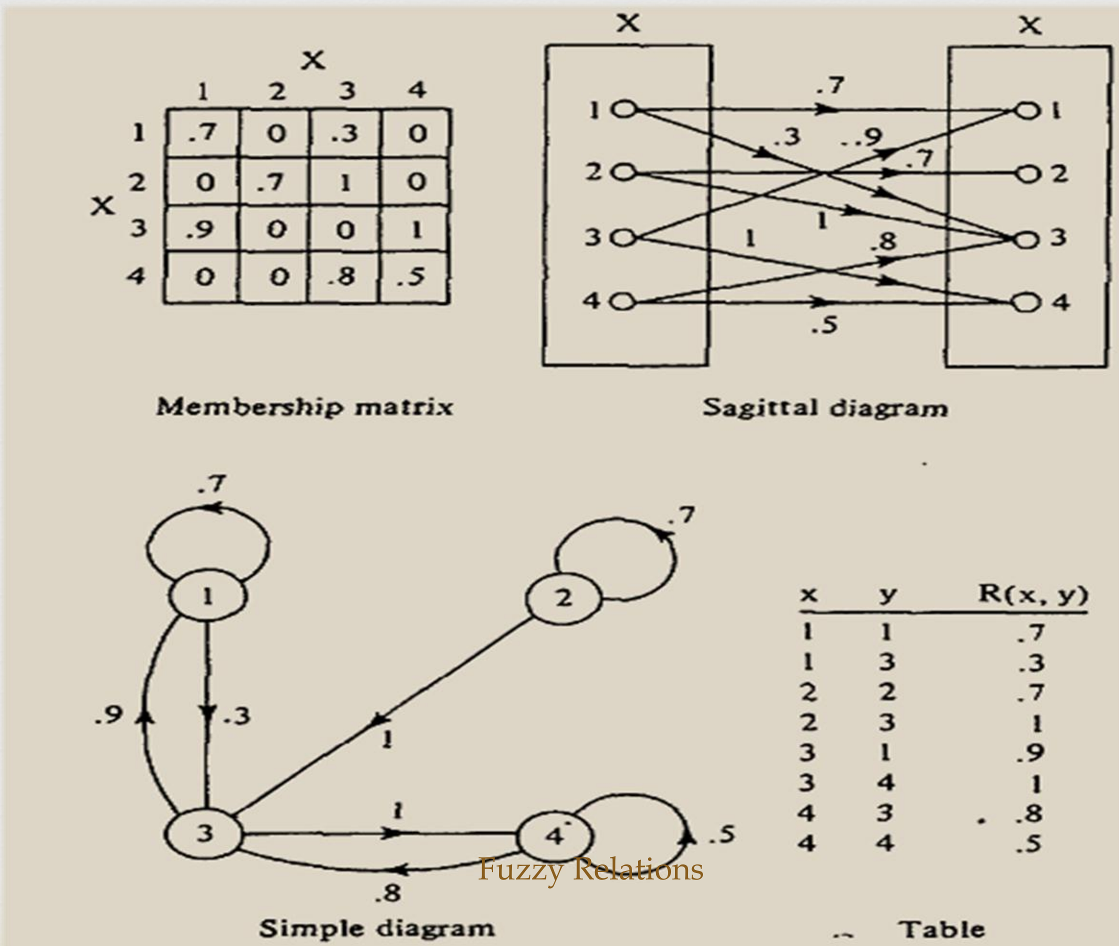
$$R(x,y) = \begin{cases} \frac{1}{1+\frac{100}{(x-y)^2}} & if \quad x > y \\ 0 & otherwise \end{cases}.$$

Representation

A fuzzy relation between elements in two finite sets $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ can be represented as a matrix

$$R = \begin{pmatrix} R(x_1, y_1) & R(x_1, y_2) & \dots & R(x_1, y_n) \\ R(x_2, y_1) & R(x_2, y_2) & \dots & R(x_2, y_n) \\ \dots & \dots & \dots & \dots \\ R(x_m, y_1) & R(x_m, y_2) & \dots & R(x_m, y_n) \end{pmatrix}.$$

Different Representations



Example

Let R be a fuzzy relation between the two sets X = {New York City, Paris} and Y = {Beijing, New York City, London}, which represents the relational concept "very far." This relation can be written in list notation as

$$R(X, Y) = 1/\text{NYC, Beijing} + 0/\text{NYC, NYC} + .6/\text{NYC, London} + .9/\text{Paris, Beijing} + .7/\text{Paris, NYC} + .3/\text{Paris, London}.$$

This relation can also be represented by the following two-dimensional membership array (matrix):

	NYC	*Paris*
Beijing	1	.9
NYC	0	.7
London	.6	.3

Fuzzy Set Operations

$$N(R(x,y)) = \bar{R}(x,y) = 1 - R(x,y),$$

$$(R \vee S)(x,y) = R(x,y) \vee S(x,y),$$

$$(R \wedge S)(x,y) = R(x,y) \wedge S(x,y).$$

T-Norm, T-Conorm, and The Inverse (Transpose)

$$R^{-1}(x, y) = R(y, x),$$

$$T(R, P)(x, y) = T(R(x, y), P(x, y)),$$

$$S(R, P)(x, y) = S(R(x, y), P(x, y)),$$

where T, S are a t-norm and a t-conorm respectively.

Max-Min Composition

Definition 3.5. *Let $R \in \mathcal{F}(X \times Y)$ and $S \in \mathcal{F}(Y \times Z)$ be fuzzy relations. Then $R \circ S \in \mathcal{F}(X \times Z)$, defined as*

$$R \circ S(x, z) = \bigvee_{y \in Y} R(x, y) \wedge S(y, z),$$

is called the ***max-min composition*** *of the fuzzy relations R and S.*

If $R \in \mathcal{F}(X \times X)$ then we can define $R^2 = R \circ R$, and generally $R^n = R \circ R^{n-1}$, $n \geq 2$.

Max-Min Composition: Discrete Case

Let $X = \{x_1, ..., x_n\}$, $Y = \{y_1, ..., y_m\}$, and $Z = \{z_1, ..., z_p\}$ be finite sets. If $R = (r_{ij})_{i=1,\dots,n, j=1,\dots,m} \in \mathcal{F}(X \times Y)$, and $S = (s_{jk})_{j=1,\dots,m,k=1,\dots,p} \in \mathcal{F}(Y \times Z)$ are discrete fuzzy relations then the composition $T = (t_{ik})_{i=1,\dots,n,k=1,\dots,p} = R \circ S \in \mathcal{F}(X \times Z)$ is given by

$$t_{ik} = \bigvee_{j=1}^{m} r_{ij} \wedge s_{jk},$$

$i = 1, ..., n,\ k = 1, ..., p.$

Example

Example 3.7. *If* $R = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1 & 0 & 0.9 \end{pmatrix}$ *and* $S = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0 \\ 0.5 & 0.6 \end{pmatrix}$ *then* $R \circ S = \begin{pmatrix} 0.3 & 0.3 \\ 0.8 & 0.6 \end{pmatrix}$.

Example

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} .8 & .3 & .5 & .5 \\ 1 & .2 & .5 & .7 \\ .5 & .4 & .5 & .6 \end{bmatrix}.$$

For example,

$$\begin{aligned} .8(= r_{11}) &= \max[\min(.3, .9), \min(.5, .3), \min(.8, 1)] \\ &= \max[\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})], \\ .4(= r_{32}) &= \max[\min(.4, .5), \min(.6, .2), \min(.5, 0)] \\ &= \max[\min(p_{31}, q_{12}), \min(p_{32}, q_{22}), \min(p_{33}, q_{32})]. \end{aligned}$$

Properties

Proposition 3.8. *(i) The max-min composition is associative, i.e.,*

$$(R \circ S) \circ Q = R \circ (S \circ Q),$$

where $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ *and* $Q \in \mathcal{F}(Z \times U)$.

(ii) Let $R_1, R_2 \in \mathcal{F}(X \times Y)$ *and* $Q \in \mathcal{F}(Y \times Z)$. *If* $R_1 \leq R_2$ *then*

$$R_1 \circ Q \leq R_2 \circ Q.$$

Properties

Proposition 3.9. *For any* $R, S \in \mathcal{F}(X \times Y)$ *and* $Q \in \mathcal{F}(Y \times Z)$ *we have*

(i) $(R \vee S) \circ Q = (R \circ Q) \vee (S \circ Q)$

(ii) $(R \wedge S) \circ Q \leq (R \circ Q) \wedge (S \circ Q)$.

Remark 3.10. *Equality in (ii) does not hold. Indeed, if we consider* $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *then* $(R \wedge S) \circ Q = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ *while* $(R \circ Q) \wedge (S \circ Q) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Min-Max Composition

Definition 3.11. *(e.g. Nobuhara-Bede-Hirota [118]) Let* $R \in \mathcal{F}(X \times Y)$ *and* $S \in \mathcal{F}(Y \times Z)$*. Then* $R \bullet S \in \mathcal{F}(X \times Z)$*, defined as*

$$R \bullet S(x, z) = \bigwedge_{y \in Y} R(x, y) \vee S(y, z)$$

is called the ***min-max composition*** *of the fuzzy relations R and S.*

Example

Example 3.12. *If* $R = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1 & 0 & 0.9 \end{pmatrix}$ *and* $S = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0 \\ 0.5 & 0.6 \end{pmatrix}$ *then*

$R \bullet S = \begin{pmatrix} 0.5 & 0.3 \\ 0.1 & 0 \end{pmatrix}$.

Properties

Proposition 3.13. *(i) The min-max composition is associative, i.e., for any* $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ *and* $T \in \mathcal{F}(Z \times U)$ *we have*

$$(R \bullet S) \bullet T = R \bullet (S \bullet T).$$

(ii) Consider $R_1, R_2 \in \mathcal{F}(X \times Y)$, $Q \in \mathcal{F}(Y \times Z)$. *If* $R_1 \leq R_2$ *then*

$$R_1 \bullet Q \leq R_2 \bullet Q.$$

Properties

Proposition 3.14. *For any* $R, S \in \mathcal{F}(X \times Y)$ *and* $T \in \mathcal{F}(Y \times Z)$ *we have*

(i) $(R \wedge S) \bullet T = (R \bullet T) \wedge (S \bullet T)$.

(ii) $(R \vee S) \bullet T \geq (R \bullet T) \vee (S \bullet T)$.

Remark 3.15. *Equality in (ii) does not hold. Indeed,* $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *then* $(R \vee S) \bullet T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *while* $(R \bullet T) \vee (S \bullet T) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.

Properties

Proposition 3.16. *If we consider the standard negation we have* $\overline{R \circ S} = \bar{R} \bullet \bar{S}$ *and* $\overline{R \bullet S} = \bar{R} \circ \bar{S}$.

Generalization

Remark 3.17. *The min-max composition can be naturally generalized to min-t-conorm compositions*

$$R \bullet_S P(x,z) = \bigwedge_{y \in Y} R(x,y) S P(y,z),$$

where S is an arbitrary t-conorm.