The position of a point in the plane can be defined by the use of a **position vector**

- Cartesian coordinates
- Polar coordinates

Each form is directly convertible into the other by:

- Polar form: $|R_A| @ \theta$

- Cartesian form: $R_X, R_Y$

\[
R_A = \sqrt{R_X^2 + R_Y^2}
\]

\[
\theta = \arctan\left(\frac{R_Y}{R_X}\right)
\]
Position

Coordinate Transformation

- If the position of point $A$ is expressed in the local $xy$ system, and it is desired to transform its coordinates to the global $XY$ system, the equations are:

$$
R_X = R_x \cos \delta - R_y \sin \delta
$$

$$
R_Y = R_x \sin \delta + R_y \cos \delta
$$
Displacement

- **Displacement of a point:** is the change in its position and can be defined as the straight-line distance between the initial and final position of a point which has moved in the reference frame.

\[ \mathbf{R}_{BA} = \mathbf{R}_B - \mathbf{R}_A \]
Displacement

• Position difference equation: The position of B with respect to A is equal to the (absolute) position of B minus the (absolute) position of A

\[ \mathbf{R}_{BA} = \mathbf{R}_B - \mathbf{R}_A \]

OR

\[ \mathbf{R}_{BA} = \mathbf{R}_{BO} - \mathbf{R}_{AO} \]
Translation, Rotation, and Complex motion

- Translation

- Rotation

Curvilinear translation path
Translation, Rotation, and Complex motion

• Complex motion
  – Total displacement = translation component + rotation component
Coordinate Systems:

GCS = Global Coordinate System, \((X, Y)\)
LNCS = Local Non-Rotating Coordinate System, \((x, y)\)
LRCS = Local Rotating Coordinate System, \((x', y')\)
Graphical Position Analysis

• The graphical analysis of this problem is trivial and can be done using only high school geometry.
Algebraic Position Analysis

• Complex Numbers as Vectors

  – Remember the Euler identity: 

     \[ e^{\pm j \theta} = \cos \theta \pm j \sin \theta \]
Algebraic Position Analysis

• Complex Numbers as Vectors

We use complex number notation for vectors to develop and derive the equations for position, velocity, and acceleration of linkages.
Algebraic Position Analysis

The Vector loop Equation for a Slider-Crank

1. Write the vector loop equation:

\[ \mathbf{R}_2 - \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \]

2. We substitute the complex number notation for each position vector:

\[ a e^{j\theta_2} - b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0 \]
Algebraic Position Analysis

The Fourbar Slider-Crank Position Solution

\[ R_2 - R_3 - R_4 - R_1 = 0 \]

\[ a e^{j\theta_2} - b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0 \]

\[ a(\cos\theta_2 + j\sin\theta_2) - b(\cos\theta_3 + j\sin\theta_3) \]

\[ -c(\cos\theta_4 + j\sin\theta_4) - d(\cos\theta_1 + j\sin\theta_1) = 0 \]
Algebraic Position Analysis

The Fourbar Slider-Crank Position Solution

Separate the real and imaginary components:

real part (x component):

\[ a \cos \theta_2 - b \cos \theta_3 - c \cos \theta_4 - d \cos \theta_1 = 0 \]

but: \( \theta_1 = 0 \), so:

\[ a \cos \theta_2 - b \cos \theta_3 - c \cos \theta_4 - d = 0 \]

imaginary part (y component):

\[ j a \sin \theta_2 - j b \sin \theta_3 - j c \sin \theta_4 - j d \sin \theta_1 = 0 \]

but: \( \theta_1 = 0 \), and the \( j \)'s divide out, so:

\[ a \sin \theta_2 - b \sin \theta_3 - c \sin \theta_4 = 0 \]
The solution is:

\[ \theta_{31} = \arcsin\left( \frac{a \sin \theta_2 - c}{b} \right) \]

\[ d = a \cos \theta_2 - b \cos \theta_3 \]
Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

1. Write the vector loop equation:
Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

1. Write the vector loop equation: \[ \mathbf{R}_2 + \mathbf{R}_3 - \mathbf{R}_4 - \mathbf{R}_1 = 0 \]

2. We substitute the complex number notation for each position vector: \[ a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0 \]
Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

\[ a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0 \]

We have two unknowns \( \theta_2, \theta_3 \). We need two equations.
Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

Substituting Euler equivalents for the $e^{j\theta}$ into

$$a e^{j\theta_2} + b e^{j\theta_3} - c e^{j\theta_4} - d e^{j\theta_1} = 0$$

we get:

$$a(\cos\theta_2 + j\sin\theta_2) + b(\cos\theta_3 + j\sin\theta_3) - c(\cos\theta_4 + j\sin\theta_4) - d(\cos\theta_1 + j\sin\theta_1) = 0$$

This equation can now be separated into its real and imaginary parts and each set to zero.

real part ($x$ component):

$$a \cos\theta_2 + b \cos\theta_3 - c \cos\theta_4 - d \cos\theta_1 = 0$$

but: $\theta_1 = 0$, so:

$$a \cos\theta_2 + b \cos\theta_3 - c \cos\theta_4 - d = 0$$  \hspace{1cm} (4.6a)
Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

real part (x component):

\[ a \cos \theta_2 + b \cos \theta_3 - c \cos \theta_4 - d = 0 \]

imaginary part (y component):

\[ j a \sin \theta_2 + j b \sin \theta_3 - j c \sin \theta_4 - j d \sin \theta_1 = 0 \]

but: \( \theta_1 = 0 \), and the \( j \)'s divide out, so:

\[ a \sin \theta_2 + b \sin \theta_3 - c \sin \theta_4 = 0 \]  \( (4.6b) \)

We will isolate \( \theta_3 \) and solve for \( \theta_4 \) in this example.

\[ b \cos \theta_3 = -a \cos \theta_2 + c \cos \theta_4 + d \]
\[ b \sin \theta_3 = -a \sin \theta_2 + c \sin \theta_4 \]
Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

\[ b \cos \theta_3 = -a \cos \theta_2 + c \cos \theta_4 + d \]
\[ b \sin \theta_3 = -a \sin \theta_2 + c \sin \theta_4 \]

Now square both sides of equations 4.6c and 4.6d and add them:

\[ b^2 (\sin^2 \theta_3 + \cos^2 \theta_3) = (-a \sin \theta_2 + c \sin \theta_4)^2 + (-a \cos \theta_2 + c \cos \theta_4 + d)^2 \]

The right side of this expression must now be expanded and terms collected.

\[ b^2 = a^2 + c^2 + d^2 - 2ad \cos \theta_2 + 2cd \cos \theta_4 - 2ac (\sin \theta_2 \sin \theta_4 + \cos \theta_2 \cos \theta_4) \]
Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

\[ b^2 = a^2 + c^2 + d^2 - 2ad \cos \theta_2 + 2cd \cos \theta_4 - 2ac(\sin \theta_2 \sin \theta_4 + \cos \theta_2 \cos \theta_4) \]

To further simplify this expression, the constants \( K_1, K_2, \) and \( K_3 \) are defined

\[ K_1 = \frac{d}{a}, \quad K_2 = \frac{d}{c}, \quad K_3 = \frac{a^2 - b^2 + c^2 + d^2}{2ac} \] (4.8a)

and:

\[ K_1 \cos \theta_4 - K_2 \cos \theta_2 + K_3 = \cos \theta_2 \cos \theta_4 + \sin \theta_2 \sin \theta_4 \] (4.8b)

If we substitute the identity \( \cos(\theta_2 - \theta_4) = \cos \theta_2 \cos \theta_4 + \sin \theta_2 \sin \theta_4 \), we get the

\[ K_1 \cos \theta_4 - K_2 \cos \theta_2 + K_3 = \cos(\theta_2 - \theta_4) \] (4.8c)

Freudenstein's equation
Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

\[ K_1 \cos \theta_4 - K_2 \cos \theta_2 + K_3 = \cos(\theta_2 - \theta_4) \]  

(4.8c)

Using the trigonometric rule

\[ \sin \theta_4 = \frac{2 \tan \left( \frac{\theta_4}{2} \right)}{1 + \tan^2 \left( \frac{\theta_4}{2} \right)}; \quad \cos \theta_4 = \frac{1 - \tan^2 \left( \frac{\theta_4}{2} \right)}{1 + \tan^2 \left( \frac{\theta_4}{2} \right)} \]

We get

\[ A \tan^2 \left( \frac{\theta_4}{2} \right) + B \tan \left( \frac{\theta_4}{2} \right) + C = 0 \]

\[ A = \cos \theta_2 - K_1 - K_2 \cos \theta_2 + K_3 \]
\[ B = -2 \sin \theta_2 \]
\[ C = K_1 - (K_2 + 1) \cos \theta_2 + K_3 \]

The solution is:

\[ \tan \left( \frac{\theta_4}{2} \right) = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \]

\[ \theta_{4,1,2} = 2 \arctan \left( \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \right) \]
Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

Back to:

\[ b \cos \theta_3 = -a \cos \theta_2 + c \cos \theta_4 + d \]
\[ b \sin \theta_3 = -a \sin \theta_2 + c \sin \theta_4 \]

\[ c \cos \theta_4 = a \cos \theta_2 + b \cos \theta_3 - d \]
\[ c \sin \theta_4 = a \sin \theta_2 + b \sin \theta_3 \]

Squaring and adding:

\[ K_1 \cos \theta_3 + K_4 \cos \theta_2 + K_5 = \cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \]

\[ K_4 = \frac{d}{b} \]
\[ K_5 = \frac{c^2 - d^2 - a^2 - b^2}{2ab} \]
Algebraic Position Analysis

The Vector loop Equation for a Fourbar linkage

\[ K_1 \cos \theta_3 + K_4 \cos \theta_2 + K_5 = \cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \]

similarly

\[ D \tan^2 \left( \frac{\theta_3}{2} \right) + E \tan \left( \frac{\theta_3}{2} \right) + F = 0 \]

\[ D = \cos \theta_2 - K_1 + K_4 \cos \theta_2 + K_5 \]
\[ E = -2 \sin \theta_2 \]
\[ F = K_1 + (K_4 - 1) \cos \theta_2 + K_5 \]

The solution is:

\[ \theta_{3,2} = 2 \arctan \left( \frac{-E \pm \sqrt{E^2 - 4DF}}{2D} \right) \]
Algebraic Position Analysis

Inverted Slider-Crank Position Solution

- This is inversion #3 of the common fourbar slider-crank linkage in which the sliding joint is between links 3 and 4 at point $B$. This is shown as an offset slider-crank mechanism. The slider block has pure rotation with its center offset from the slide.
Algebraic Position Analysis

Inverted Slider-Crank Position Solution

\[ R_B = R_2 - R_3 \]

\[ R_2 - R_3 - R_4 - R_1 = 0 \]

\[ a \ e^{j\theta_2} - b \ e^{j\theta_3} - c \ e^{j\theta_4} - d \ e^{j\theta_1} = 0 \]

Input

We need 3\textsuperscript{rd} equation

\[ \theta_3 = \theta_4 \pm \gamma \]
Algebraic Position Analysis

Inverted Slider-Crank Position Solution

\[
 a(\cos \theta_2 + j \sin \theta_2) - b(\cos \theta_3 + j \sin \theta_3) \\
- c(\cos \theta_4 + j \sin \theta_4) - d(\cos \theta_1 + j \sin \theta_1) = 0
\]

\[
 a \cos \theta_2 - b \cos \theta_3 - c \cos \theta_4 - d = 0 \\
 a \sin \theta_2 - b \sin \theta_3 - c \sin \theta_4 = 0
\]

\[
b = \frac{a \sin \theta_2 - c \sin \theta_4}{\sin \theta_3}
\]

\[
 a \cos \theta_2 - \frac{a \sin \theta_2 - c \sin \theta_4}{\sin \theta_3} \cos \theta_3 - c \cos \theta_4 - d = 0
\]
Algebraic Position Analysis

Inverted Slider-Crank Position Solution

\[ P \sin \theta_4 + Q \cos \theta_4 + R = 0 \]

\[ P = a \sin \theta_2 \sin \gamma + (a \cos \theta_2 - d) \cos \gamma \]

\[ Q = -a \sin \theta_2 \cos \gamma + (a \cos \theta_2 - d) \sin \gamma \]

\[ R = -c \sin \gamma \]

\[ \theta_{4,1,2} = 2 \arctan \left( \frac{-T \pm \sqrt{T^2 - 4SU}}{2S} \right) \]

\[ S = R - Q; \quad T = 2P; \quad U = Q + R \]
Position of a Point on a Link

Once the angles of all the links are found, it is simple and straightforward to define and calculate the position of any point on any link for any input position of the linkage.

Example: 4-bar linkage

\[
\mathbf{R}_{SO_2} = \mathbf{R}_S = se^{j(\theta_2 + \delta_2)}
\]

\[
\mathbf{R}_{UO_4} = u e^{j(\theta_4 + \delta_4)}
\]

\[
\mathbf{R}_P = \mathbf{R}_A + \mathbf{R}_{PA}
\]

\[
\mathbf{R}_{PA} = pe^{j(\theta_3 + \delta_3)} = p\left[\cos(\theta_3 + \delta_3) + j\sin(\theta_3 + \delta_3)\right]
\]
Transmission Angles

• We will expand that definition here to represent the angle between any two links in a linkage, as a linkage can have many transmission angles.

• It is easy to define the transmission angle algebraically. It is the difference between the angles of the two joined links.
Transmission Angles

Extreme Values of the Transmission Angle

- For a Grashof crank-rocker fourbar linkage the minimum value of the transmission angle occurs when the crank is colinear with the ground link.
Transmission Angles

Extreme Values of the Transmission Angle

\[ \theta_2 = 0, \pi \]

\[ \mu = \theta_4 - \theta_3 \]

Grashof
Transmission Angles

Extreme Values of the Transmission Angle

We label the links \( a = \text{link 2}; \quad b = \text{link 3}; \quad c = \text{link 4}; \quad d = \text{link 1} \)

For the overlapping case (Figure 4-15a) the cosine law gives

\[
\mu_1 = \gamma_1 = \arccos \left( \frac{b^2 + c^2 - (d-a)^2}{2bc} \right)
\]
Transmission Angles

Extreme Values of the Transmission Angle

for the extended case, the cosine law gives

\[ \mu_2 = \pi - \gamma_2 = \pi - \arccos \left( \frac{b^2 + c^2 - (d + a)^2}{2bc} \right) \]

- For a Grashof double-rocker linkage the transmission angle can vary from 0 to 90 degrees because the coupler can make a full revolution
Transmission Angles

Extreme Values of the Transmission Angle

Non-Grashof

\[ \theta_2 = \theta_3 \]

\[ \mu = \theta_4 - \theta_3 \]

(a) Toggle positions for links b and c

(b) Toggle positions for links a and b
Transmission Angles

Extreme Values of the Transmission Angle

- For a non-Grashof triple-rocker linkage the transmission angle will be zero degrees in the toggle positions which occur when the output rocker $c$ and the coupler $b$ are colinear.

\[ \mu = \arccos \left( \frac{(a+b)^2 + c^2 - d^2}{2c(a+b)} \right) \]

Not minimum $\mu$