Fractional Order Oscillator with Independent Control of Phase and Frequency

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Abstract—**This paper generalizes the concept of the oscillators based on the operational transresistance amplifier (OTRA) into the fractional order domain. The added parameter in the fractional-order design is an important in practical and can model the non-ideality of the conventional capacitors. The general condition and frequency of oscillation have been derived analytically in terms of the circuit components and the fractional order parameters. Different special cases are introduced based on the extra fractional-order parameters in which some features are appeared in the fractional-order circuit only. Through the proposed circuit, the phase and frequency can be controlled individually via the fractional order parameters. Numerical and Spice simulation results of different fractional-order oscillators are presented to validate the analytical formulas.**

Keywords — fractional calculus; oscillators; OTRA

I. Introduction

The fractional calculus has been known since the traditional calculus since three centuries ago. It is the generalization of the traditional calculus which leads to similar concepts and tools, but with wider generality and applicability. Many phenomena not completely understood before, have complex microscopic behaviors, and that their macroscopic dynamics cannot be modeled anymore via integer-order derivatives that's why fractional order calculus grabbed the attention of researchers due to its tendency to give more accurate descriptions of any real objects [1,2]. It is worth to notice the rapid growth of the fractional calculus applications in the field of science and engineering [3-15]. The integer-order models was used for a very long time not because it is accurate or it is better but because the disability to solve the fractional differential equations. That was changed at the past few years; lots of methods for approximation of the fractional derivative and integral [16-18] so, the fractional calculus can be easily model wide areas of applications. Fractional calculus played major rule in physics [3, 4], control systems [5-7], signal processing [8], biomedical [9] and electrical engineering [10-15].

One of the most frequently used definitions of the general fractional derivative is The Caputo definition which can be written as in the following equation

$$D_{t_0}^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} f^{(m)}(u)(t-u)^{m-\alpha-1}\, du \qquad (1)$$

Where m is an integer such that $(m-1) < \alpha < m$. The Laplace transform of (1) under zero initial conditions is given by [14]:

$$\mathcal{L}\{D_0^{\alpha} f(t)\} = s^{\alpha} \mathcal{L}\{f(t)\} \qquad (2)$$

The concept of fractance device which follows (2) is considered a challenge for many researchers to transform the theoretical basis to a feasible device. The fractional-order element or the so called constant phase element (CPE) with impedance $Z = Qs^{\alpha}$ where Q is the coefficient and α is the fractional order is considered as a general element where the conventional elements such as resistor, capacitor, and inductor are considered special cases corresponding to $\alpha = \{0, -1, 1\}$ respectively. Recently, researchers working in circuit theory have tried to realize a CPE with discrete passive elements [13-15].

In this paper, the implementation of the fractional order capacitor is what's taken into consideration. For example; a finite element approximation of the half-order capacitor ($Z(s) = 1/C\, s^{-0.5}$) was presented in [14], which uses the resistance and the capacitor in a recursive structure as shown in Fig.1a The technique was later developed in [15] to approximate any order with impedance $Z(s) = 1/C\,(j\omega)^{\alpha}$ as shown in Fig.1b.

Sinusoidal oscillators are widely used in wireless communications systems, instrumentation and measurement systems as test oscillators or signal generators [19]. With the use of the fractional elements; the design equations of the well known oscillators could be generalized from the tight integer order domain to the general fractional order domain.

In this paper, a study of a generalized fractional order OTRA-based oscillator circuit is introduced. The general condition and frequency of oscillation for this oscillator are derived with the use of fractance device. It shows how this oscillator can add extra degrees of freedom which are the fractional order parameters that can be used to control the oscillator frequency and phase independently. Phase control is very important for oscillators in wireless communication systems [20], synchronization between phases of signals used in transmission and receiving techniques [21]. Moreover, the controllability on phase and frequency using the fractional order oscillator could help in achieving high accuracy in the testing with more simple designs. Previously, many papers introduced the concept of OTRA-based oscillators in the integer order domain such as in [22]. This paper generalizes the

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same circuits in the fractional-order domain where the Integer-order designs are special cases. Due to the extra degree of freedoms, some unconventional results are obtained such as a peak in the frequency curve which is studied in deep through the paper provided with simulation results.

This paper is organized as follows; section I gives a brief introduction while Section II introduces the OTRA definition and presents a discussion for general fractional order oscillator based on OTRA. Section III presents the analytical analysis of this oscillator with numerical results. Section IV presents the simulation results for some cases to verify the concept, and finally section VI concludes the work.

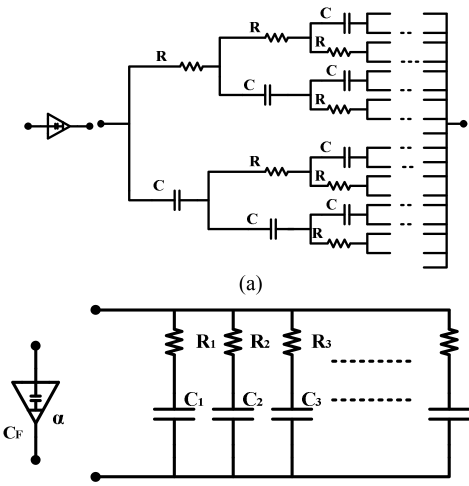


Fig.1. Approximation the capacitor with (a) order 0.5, and (b) general order

II. FRACTIONAL ORDER OTRA BASED OSCILLATOR

A. General theory of fractional order oscillator

A general linear fractional order system with two states is given by:

$$\begin{bmatrix} D^{\alpha}x_1 \\ D^{\beta}x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (3)$$

Based on the theorem presented in [10], this system will oscillate if there is a value for ω which satisfy the following two equations:

$$\omega^{\alpha+\beta}\cos\left(\frac{(\alpha+\beta)\pi}{2}\right) - a_{11}\omega^{\beta}\cos\left(\frac{\beta\pi}{2}\right) - a_{22}\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right) + |A| = 0 \quad (4a)$$

$$\omega^{\beta}\ \sin\left(\frac{(\alpha+\beta)\pi}{2}\right) - a_{11}\omega^{\beta-\alpha}\sin\left(\frac{\beta\pi}{2}\right) - a_{22}\sin\left(\frac{\alpha\pi}{2}\right) = 0 \quad (4b)$$

Where $|A|$ is the determinant of the state matrix and it must be positive to obtain oscillations. The phase difference between the two states x_1 and x_2 is then found from either (5a) or (5b)

$$\varphi = \tan^{-1}\frac{\omega^{\alpha}\sin(0.5\alpha\pi)}{\omega^{\alpha}\cos(0.5\alpha\pi) - a_{11}} - \frac{\pi(1-\mathrm{sign}(a_{12}))}{2} \quad (5a)$$

$$\varphi = \frac{\pi(1-\mathrm{sign}(a_{21}))}{2} - \tan^{-1}\frac{\omega^{\beta}\sin(0.5\beta\pi)}{\omega^{\beta}\ \cos(0.5\beta\pi) - a_{22}} \quad (5b)$$

It is clear that the phase is function of the fractional-order parameters which increases the flexibility of the design where the integer case oscillator is a special case from the fractional-order oscillators at $\alpha = \beta = 1$.

B. OTRA-based fractional oscillator

Current Mode circuits are preferred over voltage mode circuits because of its ability to enhance the amplifier speed, accuracy and bandwidth. One of these circuits is the OTRA; it gains lots of interest in the past few years because its ability to overcome the finite gain–bandwidth product associated with the voltage mode amplifiers like Op-amps. The OTRA (see Fig. 2) is a three port network with terminal characteristics described by the following eqn. [22-25]

$$\begin{bmatrix} V_+ \\ V_- \\ V_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R_m & -R_m & 0 \end{bmatrix} \begin{bmatrix} I_+ \\ I_- \\ I_0 \end{bmatrix} \quad (6)$$

Where R_m is the transresistance gain. The two input terminals of the OTRA are virtually grounded. Ideally the transresistance gain R_m approaches infinity; which forces the two input currents to be equal.

The oscillator in Fig.3; consists of two OTRA's, two fractional capacitors and four resistors. The state matrix equation can be written as:

$$\begin{bmatrix} D^{\alpha}v_1 \\ D^{\beta}v_2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{C_1\,R_3} & \frac{1}{C_1\,R_1} \\ \frac{-1}{C_2\,R_2} & \frac{1}{C_2\,R_4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (7)$$

Where $|A| = (1/C_1C_2R_1R_2) - (1/C_1C_2R_3R_4)$ and it must be positive to obtain oscillations which means that $R_1R_2 < R_3R_4$ to maintain positive determinant.

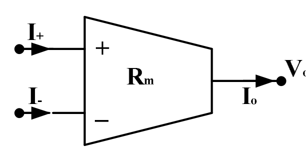


Fig.2. OTRA building block

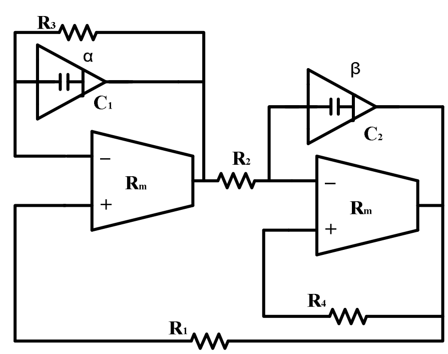


Fig.3. OTRA based oscillator

For example, choosing R_4 to be the design parameter and substituting in the previous conditions; the following equations represent R_4.

TABLE I. ANALYSIS OF FOUR SPECIAL CASES

Case	Oscillation parameters		
	R_4	Freq.	Phase φ
$\beta=\alpha=1$	$\frac{C_1R_3}{C_2}$	$\omega=\sqrt{\frac{1}{C_1C_2R_1R_2}-\frac{1}{(C_1R_3)^2}}$	$tan^{-1}\sqrt{\frac{(C_1R_3)^2}{C_1C_2R_1R_2}-1}$
$\beta=\alpha\neq 1$	$\frac{C_1R_3}{C_2+2C_1C_2R_3\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}$	$\omega^{2\alpha}+2\frac{\omega^{\alpha}}{C_1R_3}\cos\left(\frac{\alpha\pi}{2}\right)+\frac{1}{(C_1R_3)^2}=\frac{1}{C_1C_2R_1R_2}$	$\tan^{-1}\frac{\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)}{\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\frac{1}{C_1R_3}}$
$\beta=1$ $\alpha<1$	$\frac{C_1R_3\sin\left(\frac{\alpha\pi}{2}\right)}{C_2\omega^{1-\alpha}+C_1C_2R_3\omega\cos\left(\frac{\alpha\pi}{2}\right)}$	$\omega^{1+\alpha}+2\frac{\omega}{C_1R_3}\cos\left(\frac{\alpha\pi}{2}\right)+\frac{\omega^{1-\alpha}}{(C_1R_3)^2}=\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{C_1C_2R_1R_2}$	$\tan^{-1}\frac{\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)}{\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\frac{1}{C_1R_3}}$
$\alpha=1$ $\beta<1$	$\frac{C_1R_3}{C_2\omega^{\beta-1}\sin\left(\frac{\beta\pi}{2}\right)+C_1C_2R_3\omega^{\beta}\cos\left(\frac{\beta\pi}{2}\right)}$	$\omega^{\beta+1}+\frac{\omega^{\beta-1}}{(C_1R_3)^2}=\frac{1}{C_1C_2R_1R_2\sin\left(\frac{\beta\pi}{2}\right)}$	$\tan^{-1}(\omega C_1R_3)$

$$R_4=\frac{R_1R_2+C_1R_1R_2R_3\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}{C_1C_2R_1R_2R_3\omega^{\alpha+\beta}\cos\left(\frac{(\beta+\alpha)\pi}{2}\right)+C_2R_1R_2\omega^{\beta}\cos\left(\frac{\beta\pi}{2}\right)+R_3} \quad (8a)$$

$$R_4=\frac{C_1R_3\sin\left(\frac{\alpha\pi}{2}\right)}{C_2\omega^{\beta-\alpha}\sin\left(\frac{\beta\pi}{2}\right)+C_1C_2R_3\omega^{\beta}\sin\left(\frac{(\beta+\alpha)\pi}{2}\right)} \quad (8b)$$

Eliminating R_4 from both equations to get the equation for oscillation frequency

$$\omega^{\beta+\alpha}+2\frac{\omega^{\beta}}{C_1R_3}\cos\left(\frac{\alpha\pi}{2}\right)+\frac{\omega^{\beta-\alpha}}{(C_1R_3)^2}=\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{C_1C_2R_1R_2\sin\left(\frac{\beta\pi}{2}\right)} \quad (9)$$

The phase between the two voltages of this oscillator can be calculated from (5).

III. ANALYTICAL STUDY

Analytical study of some special cases is presented in Table I. In this section, an investigation for each special case in deeper analysis is reported. It is worth to notice that the integer case is now considered as a special case not the only existing case. What makes the fractional elements interesting is not only the order can be manipulated for the same value but also it can be used with integer order so; there are lots of oscillators could be achieved from the same configuration but with different order. Cases 3 and 4 in Table I contains the combinations of using integer and fractional elements together. For using two fractional elements with equal order $\beta=\alpha\neq 1$(case 2); Fig.4 shows the relation between the frequency and the order; it decays with the increase of the fractional order and it is obvious also the reverse relation between the frequency of oscillation and the increase of the value of R_4. The phase is increasing with the increase of the order as well. It shows also the relation between the frequency and the phase which is an advantage the fractional order can provide; design for specific phase. From Fig.4; this configuration can provide a wide frequency range reaches up to GHz range.

For case 3; the oscillator in this case is configured to use fractional order along with integer order. A closer investigation of this case leads to a peak in the frequency of oscillation curve which is unusual result as most of oscillation frequency curves decay with the fractional order as shown in Fig.5. It shows how the peak location changes with the use of different values of R_3. The frequency is increasing till it reaches a maximum value then decreasing again. For fixed α ; the frequency increases with the increment of R_3. This interesting result grants the advantage of the oscillator's designing ability with two different phases at the same frequency simply by changing the fractional order. The phase is increasing with the increase of the fractional order as illustrated in Fig.5. The relation between the condition and frequency of oscillation is proportional relation till the peak is reached then it becomes inverse proportional relation.

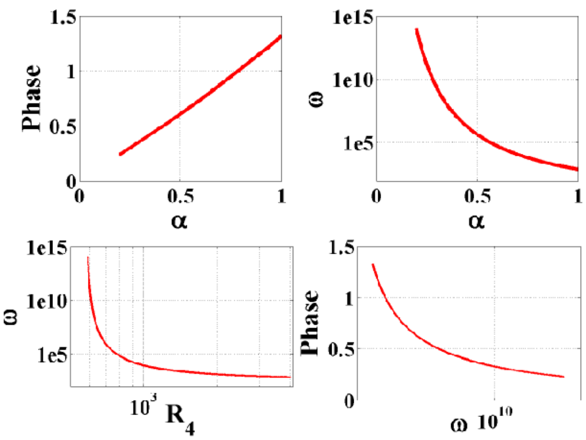


Fig.4.Numerical analysis for special case2

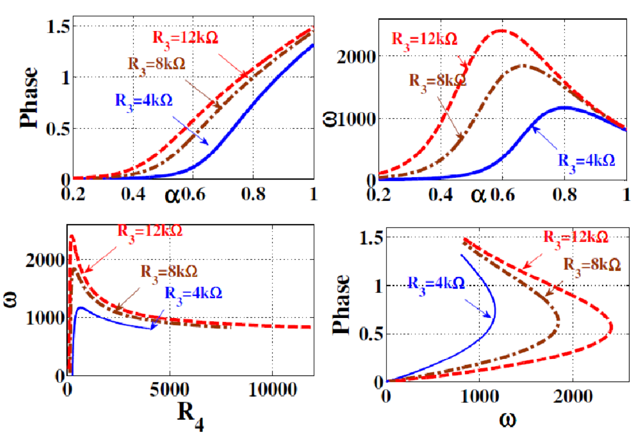


Fig.5. Numerical analysis for special case3

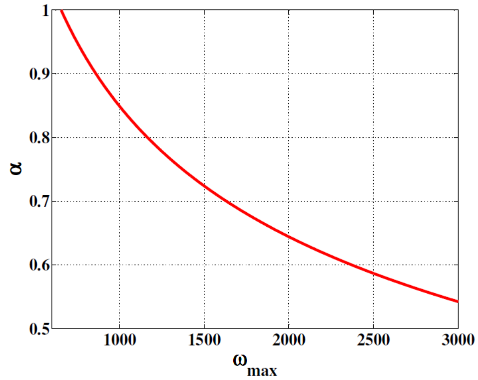


Fig.6. Maximum frequency versus the order (case3)

From Fig.5; this configuration can provide frequencies in KHz ranges. It is clear that the peak is what makes that configuration unique. Investigating the peak at which there is a maximum frequency which can be described by (10). From Table I (case 3); R_3 could be written as a function of the order and the frequency as illustrated in (11).

$$\omega_{max}^{1+\alpha}\ln(\omega_{max}) = \frac{\pi\omega_{max}}{C_1R_3}\sin\left(\frac{\alpha\pi}{2}\right) + \frac{\omega_{max}^{1-\alpha}\ln(\omega_{max})}{(C_1R_3)^2} + \frac{\pi\cos\left(\frac{\alpha\pi}{2}\right)}{2C_1C_2R_1R_2} \quad (10)$$

$$R_3 = \frac{-C_2R_1R_2\omega\cos\left(\frac{\alpha\pi}{2}\right) - \sqrt{C_2R_1R_2\omega\sin\left(\frac{\alpha\pi}{2}\right)\left(\frac{\omega^{-\alpha}}{C_1} - C_2R_1R_2\omega\sin\left(\frac{\alpha\pi}{2}\right)\right)}}{C_1C_2R_1R_2\omega^{1+\alpha} - \sin\left(\frac{\alpha\pi}{2}\right)} \quad (11)$$

To obtain a maximum frequency for each value of the fractional order; it must satisfy the previous two equations. Fig.6 shows the value and location of maximum frequency for different orders. The maximum frequency values decrease with the increase of the fractional order till the integer case is reached which achieve the lowest maximum frequency.

Another interesting advantage the fractional order can provide is phase control. As shown from Table I; the phase can be controlled by changing the values of R_3. To design oscillator for specific frequency and phase; simply choose the suitable value for R_3 which could achieve this phase and accordingly the order of the fractional element used is determined. Fig.7 shows how the phase and R_3 changes with respect to the fractional order.

Exchanging between the fractional and the integer order of case 3; case 4 is created. The frequency of oscillation and the phase are decaying with the order as shown in Fig.8. From which it can be concluded that different frequency could be obtained for same value of R_4 but with different order. The ranges of frequencies achieved could reach MHz ranges.

IV. Simulation Results

Some cases are chosen to be simulated to verify the theoretical concept. For case 2 ($\beta = \alpha \neq 1$); the simulation parameters chosen are $C_1 = C_2 = 1.2 * 10^{\wedge} - 6$, $R_1 = R_2 = 1k\Omega$ and $R_3 = 4k\Omega$. Order 0.7 is used in the simulation for this case. R_4 equals to 969Ω and the frequency of oscillation equals 1.9 kHz. Fig.9 shows the numerical simulation (Matlab simulations) versus circuit simulations (Spice simulations) for the two states which are the voltages of the two fractional order capacitors used in the circuit. For case3 ($\beta = 1, \alpha < 1$); with the same configurations as the previous case; the peak for this case happens for $\alpha = 0.8$, R_4 equals to 652Ω and the frequency of oscillation in this case equals 185.8 Hz. Fig.10 shows the theoretical versus the practical simulations (Matlab simulations versus spice simulations) for the two voltages of the used fractional order capacitor and the integer order capacitor.

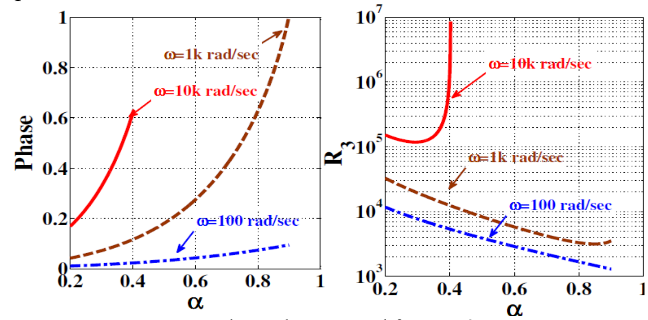


Fig.7. Phase control for case 3

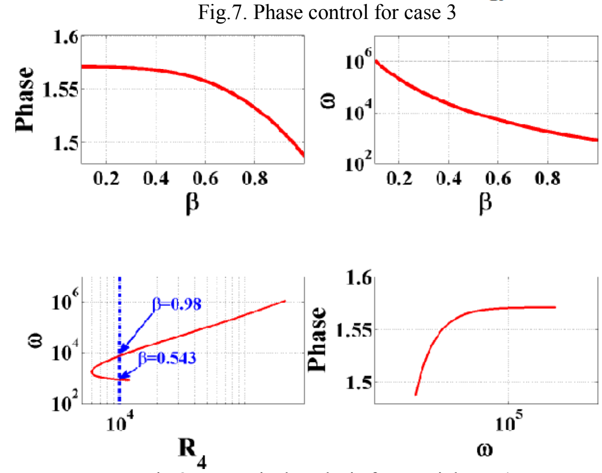
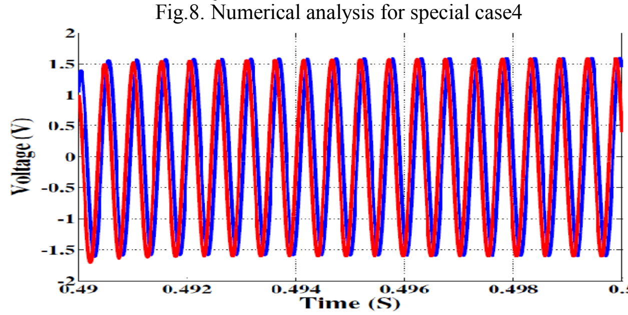
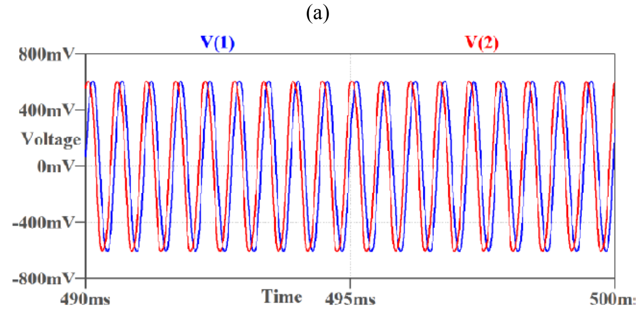


Fig.8. Numerical analysis for special case4

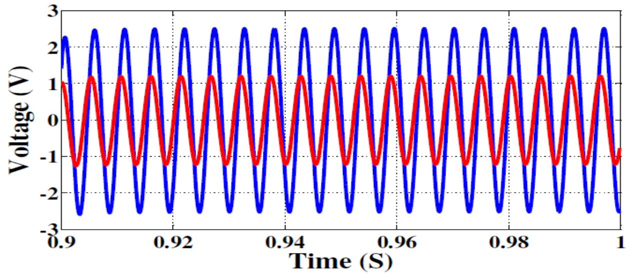


(a)



(b)

Fig.9. (a) Matlab simulations, and (b) Spice Simulations



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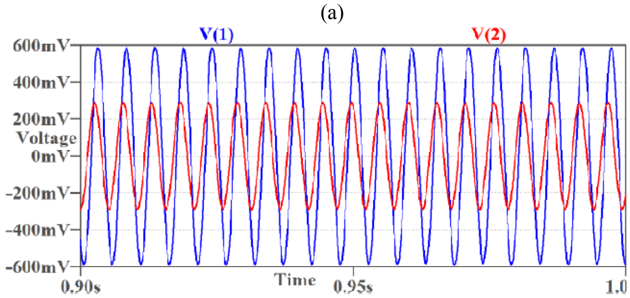
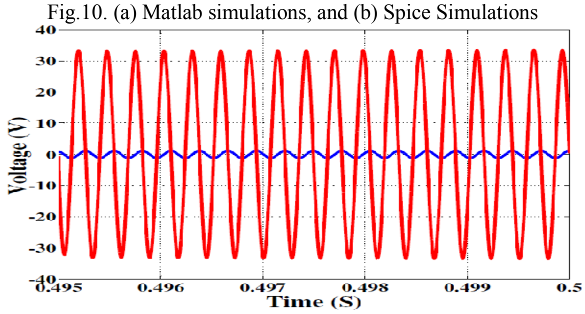
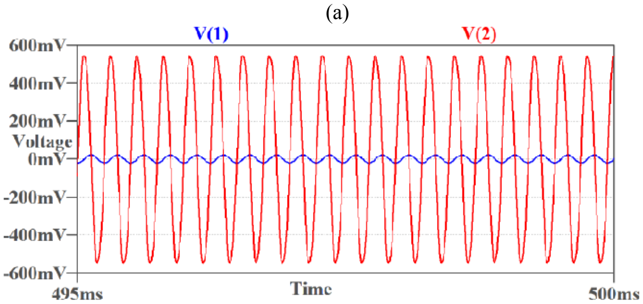


(b)

Fig.10. (a) Matlab simulations, and (b) Spice Simulations



(a)



(b)

Fig.11. (a) Matlab simulations. (b) Spice Simulations

For case 4 ($\alpha = 1, \beta < 1$); with a little change in the previous configuration which is $R_3 = 12k\Omega$. Order 0.4 is used for this case, R_4 equals to $19k\Omega$; the frequency of oscillation in this case equals 3.6 kHz. Fig.11 shows the theoretical versus the practical simulations for the two voltages of the used fractional order capacitor and the integer order capacitor. From Fig. 9 to Fig.11; it is clear that the theoretical waveforms and the simulated waveforms are similar to each other in phase and frequency.

V. CONCLUSION

In this paper, a study of fractional order oscillator based on OTRA was presented. The general condition, frequency of oscillation along as well as the phase between the two output stages was derived for general orders. The fractional order parameter adds extra degree of freedom in the design which increases the design flexibility. Some special cases were introduced such as equal fractional-orders and mixed integer-fractional order with their fundamentals. Numerical and circuit simulation results for these special cases were illustrated which showed some interesting results from which phase and frequency could be controlled independently.

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