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Gyratorless realization of a class of three-variable positive real functions†

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The necessary and sufficient conditions for realizing a class of three-variable positive real functions of degree one in two variables as the driving-point impedance at port 1 of a reciprocal three-port network in the domain of the higher degree variable, and with a unit inductor or a unit capacitor termination at port 2 in the domain of the second variable, and a unit inductor or a unit capacitor termination at port 3 in the domain of the last variable, are given in this paper.

An example is given to illustrate the synthesis procedure. Extension of the synthesis technique to a class of multivariable positive real functions is possible.

1. Introduction

The synthesis technique of multivariable positive real functions is a very important subject in realizability theory, due to its applications in the realization of variable parameter networks and lumped-distributed structures (Koga 1968, Soliman and Bose 1971).

The realization of a single variable positive real function $Z(p_1)$ of the complex frequency variable p_1 by decomposing the numerator polynomial of $dZ(p_1)/dp_1$ into sums and differences of squares of polynomials is well known (Lunelli 1955). The extension of the above result to cover a class of two-variable reactance functions was given by Ansell (1964).

The present paper deals with the realization of a class of three-variable positive real rational functions $Z(p_1, p_2, p_3)$, not necessarily reactance, with $\deg_{p_i} Z(p_1, p_2, p_3) = 1$ $(i = 2, 3)$.

2. Definition (Ozaki and Kasami 1960)

The function $Z(p_1, p_2, p_3)$ is said to be a three-variable positive real rational function, if

(a) $Z(p_1, p_2, p_3)$ is a rational real function of p_1, p_2, p_3.
(b) $\text{Re } Z \geqslant 0$ in the domain $\text{Re } p_i > 0$ $(i = 1, 2, 3)$.

3. Analysis

Consider in general a lossy three-port network N in the p_1 domain, characterized by its open-circuit impedance matrix

$$[Z(p_1)] = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}, \quad (1)$$

and consider first the case where the output ports are terminated by p_2 and p_3,

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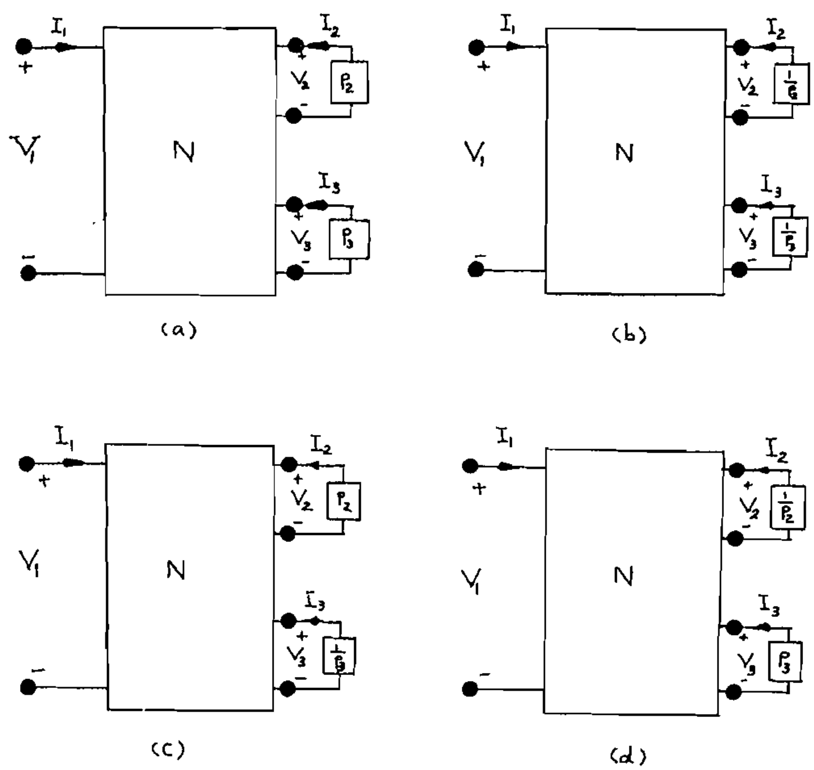
respectively, as shown in fig. 1 (*a*). Substituting the terminal relations at the output ports, one gets

$$\begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22+P_2} & z_{23} \\ z_{31} & z_{32} & z_{33+P_3} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}. \tag{2}$$

Fig. 1



Solving for the port currents, hence

$$\frac{I_2}{I_1} = -\frac{z_{21}p_3 + (z_{21}z_{33} - z_{23}z_{31})}{p_2p_3 + z_{33}p_2 + z_{22}p_3 + (z_{22}z_{33} - z_{23}z_{32})}. \tag{3}$$

Similarly,

$$\frac{I_3}{I_1} = -\frac{z_{31}p_2 + (z_{31}z_{22} - z_{32}z_{21})}{p_2p_3 + z_{33}p_2 + z_{22}p_3 + (z_{22}z_{33} - z_{23}z_{32})}. \tag{4}$$

The driving-point impedance at port 1 is given by

$$Z(p_1, p_2, p_3) = \frac{\begin{matrix} z_{11}p_2p_3 + (z_{11}z_{33} - z_{13}z_{31})p_2 + (z_{11}z_{22} - z_{12}z_{21})p_3 \\ + (z_{11}z_{22}z_{33} + z_{13}z_{21}z_{32} + z_{12}z_{31}z_{23} - z_{11}z_{23}z_{32} - z_{12}z_{21}z_{33} - z_{13}z_{31}z_{22}) \end{matrix}}{p_2p_3 + z_{33}p_2 + z_{22}p_3 + (z_{22}z_{33} - z_{23}z_{32})}. \tag{5}$$

Differentiating the above equation partially with respect to p_2, one gets :

Numerator

$$\left(\frac{\partial Z}{\partial p_2}\right) = (z_{12}z_{21})p_3^2 + (2z_{12}z_{21}z_{33} - z_{12}z_{31}z_{23} - z_{13}z_{21}z_{32}) + (z_{33}^2 z_{12}z_{21} + z_{13}z_{31}z_{23}z_{32} - z_{12}z_{31}z_{23}z_{33} - z_{13}z_{21}z_{32}z_{33}). \quad (6)$$

Similarly :

Numerator

$$\left(\frac{\partial Z}{\partial p_3}\right) = (z_{13}z_{31})p_2^2 + (2z_{13}z_{31}z_{22} - z_{13}z_{21}z_{32} - z_{12}z_{31}z_{23}) + (z_{22}^2 z_{13}z_{31} + z_{12}z_{21}z_{32}z_{23} - z_{13}z_{21}z_{32}z_{22} - z_{12}z_{31}z_{23}z_{22}). \quad (7)$$

Comparing eqns. (3), (6) and (4), (7), it is seen that if the network N is reciprocal, that is $[Z(p_1)]$ is symmetric (Belevitch 1968), then

$$\frac{\partial Z}{\partial p_2} = \left(\frac{I_2}{I_1}\right)^2, \quad (8)$$

$$\frac{\partial Z}{\partial p_3} = \left(\frac{I_3}{I_1}\right)^2. \quad (9)$$

On the other hand, it can be easily seen that for

$$\frac{\partial Z}{\partial p_2} = +(\text{a perfect square}) \quad (10)$$

one must have

$$z_{12} = z_{21}, \quad (11)$$

$$z_{13}z_{32} = z_{31}z_{23}, \quad (12)$$

and for

$$\frac{\partial Z}{\partial p_3} = +(\text{a perfect square}) \quad (13)$$

one must have

$$z_{13} = z_{31}, \quad (14)$$

$$z_{12}z_{23} = z_{21}z_{32}. \quad (15)$$

Therefore, for eqns. (10) and (13) to be satisfied simultaneously, the open-circuit impedance matrix $[Z]$ must be symmetric, and in this case

$$\frac{\partial Z}{\partial p_2} = +\left(\frac{I_2}{I_1}\right)^2, \quad (8)$$

$$\frac{\partial Z}{\partial p_3} = +\left(\frac{I_3}{I_1}\right)^2. \quad (9)$$

Similar analysis for the circuit in fig. 1 (*b*) will lead to the following result; the network N is reciprocal, if, and only if

$$\frac{\partial Z}{\partial p_2} = -\left(\frac{V_2}{I_1}\right)^2, \tag{16 a}$$

$$\frac{\partial Z}{\partial p_3} + -\left(\frac{V_3}{I_1}\right)^2. \tag{16 b}$$

Similar conditions can be obtained for the networks in figs. 1 (*c*) and (*d*). The above results are summarized below in the form of a theorem.

4. Theorem

The necessary and sufficient conditions for a positive real rational function $Z(p_1, p_2, p_3)$ to be realizable as the driving-point impedance at port 1 of a reciprocal three-port network N in the p_1 domain, with load termination at the second port equal to p_2 or $1/p_2$ and load termination at the third port equal to p_3 or $1/p_3$, are

$$(1) \quad \deg_{p_i} Z(p_1, p_2, p_3) = 1 \quad (i = 2, 3), \tag{17}$$

$$(2) \quad \frac{\partial Z}{\partial p_2} = \pm\,(\text{a perfect square}), \tag{18}$$

$$(3) \quad \frac{\partial Z}{\partial p_3} = \pm\,(\text{a perfect square}). \tag{19}$$

The positive sign in the above two equations corresponds to a load equal to p_i ($i = 2, 3$), and the negative sign corresponds to a load equal to $1/p_i$ ($i = 2, 3$) respectively.

Corollary

In the above theorem if $Z(p_1, p_2, p_3)$ is a reactance function, then N will be lossless.

5. Synthesis

The synthesis procedure is summarized below.

5.1.

Any three-variable positive real rational function

$$Z(p_1, p_2, p_3) = \frac{N_1(p_1, p_2, p_3)}{D_1(p_1, p_2, p_3)} \tag{20}$$

with $\deg_{p_i} Z = 1$ ($i = 2, 3$), is to be realized as the driving-point impedance at port 1 of the network N shown in fig. 1 (if possible). Since $Z(p_1, p_2, p_3) = V_1/I_1$, comparing with eqn. (20), one identifies

$$V_1 = \frac{N_1(p_1, p_2, p_3)}{Q(p_1, p_2, p_3)}, \tag{21 a}$$

$$I_1 = \frac{D_1(p_1, p_2, p_3)}{Q(p_1, p_2, p_3)} \tag{21 b}$$

where Q is a polynomial in the variables p_1, p_2 and p_3.

5.2

If

$$\frac{\partial Z}{\partial p_i} = +\left(\frac{D_i}{D_1}\right)^2 = \left(\frac{D_i/Q}{D_1/Q}\right)^2, \tag{22 a}$$

or if

$$\frac{\partial Z}{\partial p_i} = -\left(\frac{N_i}{D_1}\right)^2 = -\left(\frac{N_i/Q}{D_1/Q}\right)^2 \quad (i=2,\ 3), \tag{22 b}$$

then the synthesis procedure is definitely possible, and here one can classify between four different cases.

Case (*a*) :

$$\frac{\partial Z}{\partial p_2} = +\left(\frac{D_2/Q}{D_1/Q}\right)^2, \quad \frac{\partial Z}{\partial p_3} = +\left(\frac{D_3/Q}{D_1/Q}\right)^2.$$

In this case the load termination at ports 2 and 3 are p_2 and p_3, respectively, and

$$I_2 = \pm\frac{D_2}{Q}, \quad I_3 = \pm\frac{D_3}{Q}. \tag{23}$$

Case (*b*) :

$$\frac{\partial Z}{\partial p_2} = -\left(\frac{N_2/Q}{D_1/Q}\right)^2, \quad \frac{\partial Z}{\partial p_3} = -\left(\frac{N_3/Q}{D_1/Q}\right)^2.$$

Here the terminations at ports 2 and 3 are $1/p_2$ and $1/p_3$, respectively, and

$$V_2 = \pm\frac{N_2}{Q}, \quad V_3 = \pm\frac{N_3}{Q}. \tag{24}$$

Case (*c*) :

$$\frac{\partial Z}{\partial p_2} = +\left(\frac{D_2/Q}{D_1/Q}\right)^2, \quad \frac{\partial Z}{\partial p_3} = -\left(\frac{N_3/Q}{D_1/Q}\right)^2.$$

Here the terminations at ports 2 and 3 are p_2 and $1/p_3$, respectively, and

$$I_2 = \pm\frac{D_2}{Q}, \quad V_3 = \pm\frac{N_3}{Q}. \tag{25}$$

Case (*d*) :

$$\frac{\partial Z}{\partial p_2} = -\left(\frac{N_2/Q}{D_1/Q}\right)^2, \quad \frac{\partial Z}{\partial p_3} = +\left(\frac{D_3/Q}{D_1/Q}\right)^2.$$

Here the terminations at ports 2 and 3 are $1/p_2$ and p_3, respectively, and

$$V_2 = \pm\frac{N_2}{Q}, \quad I_3 = \pm\frac{D_3}{Q}. \tag{26}$$

5.3

For brevity, only case (a) will be considered here. The port currents and voltages can be written as

$$[I]=\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}=\frac{1}{Q(p_1, p_2, p_3)}\begin{bmatrix} D_1(p_1, p_2, p_3) \\ \pm D_2(p_1, p_2, p_3), \\ \pm D_3(p_1, p_2, p_3) \end{bmatrix}, \tag{27}$$

$$[V]=\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}=\frac{1}{Q(p_1, p_2, p_3)}\begin{bmatrix} N_1(p_1, p_2, p_3) \\ \mp p_2 D_2(p_1, p_2, p_3) \\ \mp p_3 D_3(p_1, p_2, p_3) \end{bmatrix}. \tag{28}$$

Since the elements of $[I]$ and $[V]$ are polynomials in p_1, p_2 and p_3, one can write matrices $[A]$ and $[B]$ as follows :

$$[I]^t=\frac{1}{Q}\,[1\ p_2\ p_3][A], \tag{29}$$

$$[V]^t=\frac{1}{Q}\,[1\ p_2\ p_3][B]. \tag{30}$$

The scattering matrix of the reciprocal three-port is found from

$$[[V]-[I]]=[S(p_1)][[V]+[I]]. \tag{31}$$

Since $[S(p_1)]$ is symmetric for reciprocal networks, one gets

$$[S(p_1)]=[B+A]^{-1}[B-A]. \tag{32}$$

When $[A]$ is non-singular, the open-circuit impedance matrix is found from

$$[Z(p_1)]=[A]^{-1}[B]. \tag{33}$$

Similarly, when $[B]$ is non-singular, the admittance matrix results from

$$[Y(p_1)]=[B]^{-1}[A]. \tag{34}$$

5.4

Realize the reciprocal three-port network in the p_1 domain following any classical synthesis procedure. The above synthesis procedure is best illustrated by the following example.

6. Example

Consider the following three-variable reactance function :

$$Z(p_1, p_2, p_3)=\frac{2p_1^2p_2+p_1p_2p_3+p_1+2p_2}{4p_1^2+2p_1p_2+2p_1p_3+p_2p_3+1}. \tag{35}$$

Numerator

$$\left(\frac{\partial Z}{\partial p_2}\right)=(\sqrt{2}(2p_1^2+p_1p_3+1))^2. \tag{36}$$

Numerator

$$\left(\frac{\partial Z}{\partial p_3}\right) = -(\sqrt{2}(p_1+p_2))^2. \tag{37}$$

Indicating that the above synthesis procedure is possible, and the terminations at ports 2 and 3 are p_2 and $1/p_3$, respectively,

$$[V] = \frac{1}{Q}\begin{bmatrix} 2p_1^2p_2 + p_1p_2p_3 + p_1 + 2p_2 \\ -\sqrt{2}p_2(2p_1^2 + p_1p_3 + 1) \\ +\sqrt{2}(p_1+p_2) \end{bmatrix}, \tag{38}$$

$$[I] = \frac{1}{Q}\begin{bmatrix} 4p_1^2 + 2p_1p_2 + 2p_1p_3 + p_2p_3 + 1 \\ \sqrt{2}(2p_1^2 + p_1p_3 + 1) \\ -\sqrt{2}p_3(p_1+p_2) \end{bmatrix}, \tag{39}$$

$$[V]^t = \frac{1}{Q}[1\ p_2\ p_3]\begin{bmatrix} p_1 & 0 & \sqrt{2}p_1 \\ 2(p_1^2+1) & -\sqrt{2}(2p_1^2+1) & \sqrt{2} \\ p_1p_2 & -\sqrt{2}p_1p_2 & 0 \end{bmatrix}, \tag{40}$$

$$[I]^t = \frac{1}{Q}[1\ p_2\ p_3]\begin{bmatrix} 4p_1^2+1 & \sqrt{2}(2p_1^2+1) & 0 \\ 2p_1 & 0 & 0 \\ 2p_1+p_2 & \sqrt{2}p_1 & -\sqrt{2}(p_1+p_2) \end{bmatrix}, \tag{41}$$

$$[Z(p_1)] = \begin{bmatrix} 0 & \dfrac{1}{2p_1} & 0 \\ \dfrac{1}{\sqrt{2}(2p_1^2+1)} & -\dfrac{4p_1^2+1}{2\sqrt{2}p_1(2p_1^2+1)} & 0 \\ \dfrac{p_1}{\sqrt{2}(p_1+p_2)(2p_1^2+1)} & \dfrac{p_1+2p_1^2p_2+p_2}{2\sqrt{2}p_1(2p_1^2+1)(p_1+p_2)} & -\dfrac{1}{\sqrt{2}(p_1+p_2)} \end{bmatrix} \times \begin{bmatrix} p_1 & 0 & \sqrt{2}p_1 \\ 2(p_1^2+1) & -\sqrt{2}(2p_1^2+1) & \sqrt{2} \\ p_1p_2 & -\sqrt{2}p_1p_2 & 0 \end{bmatrix}, \tag{42}$$

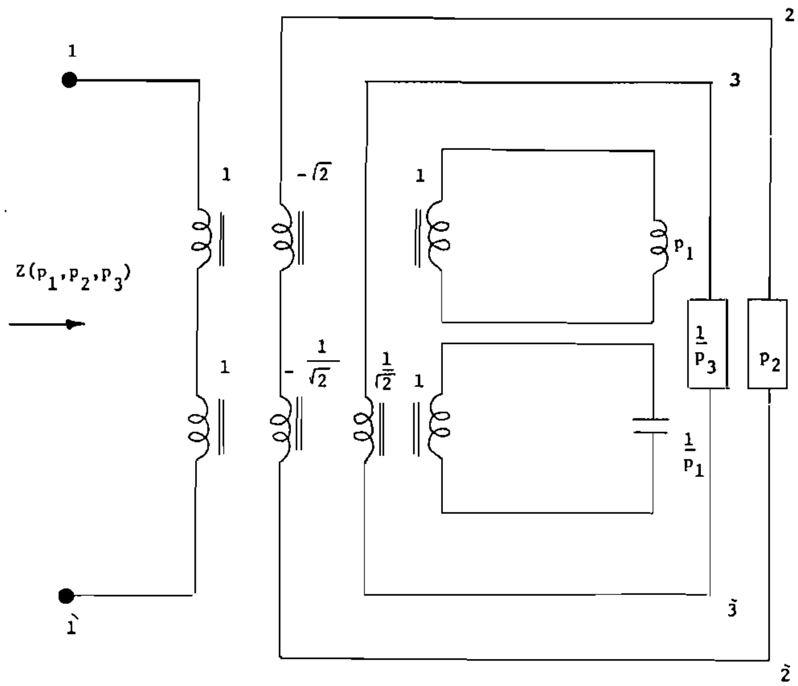
$$[Z(p_1)] = \begin{bmatrix} p_1 + \dfrac{1}{p_1} & -\sqrt{2}p_1 - \dfrac{1}{\sqrt{2}p_1} & \dfrac{1}{\sqrt{2}p_1} \\ -\sqrt{2}p_1 - \dfrac{1}{\sqrt{2}p_1} & 2p_1 + \dfrac{1}{2p_1} & -\dfrac{1}{2p_1} \\ \dfrac{1}{\sqrt{2}p_1} & -\dfrac{1}{2p_1} & \dfrac{1}{2p_1} \end{bmatrix}, \tag{43}$$

$$= \begin{bmatrix} 1 \\ -\sqrt{2} \\ 0 \end{bmatrix} [p_1] \quad [1 \quad -\sqrt{2} \quad 0]$$

$$+ \begin{bmatrix} 1 \\ -\dfrac{1}{\sqrt{2}} \\ +\dfrac{1}{\sqrt{2}} \end{bmatrix} \left[\frac{1}{p_1}\right] \left[1 \quad -\frac{1}{\sqrt{2}} \quad +\frac{1}{\sqrt{2}}\right] \tag{44}$$

and the final network realization is shown in fig. 2.

Fig. 2



7. Conclusions

A synthesis procedure for realizing a class of three-variable positive real functions was given. Extension of the synthesis technique to a class of multi-variable positive real functions is possible.

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