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Synthesis of a class of two-variable positive-real functions†

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A synthesis procedure for realizing a class of two-variable positive-real functions bilinear in one variable will be given here. The synthesis technique can be considered as a generalization of Ansell's synthesis procedure of two-variable reactance functions. Two examples will be given to illustrate the applicability of the above result in the synthesis of lumped-distributed structures and variable parameter networks.

1. Introduction

The realization procedures of multivariable positive-real functions have been the subject of interest in network theory during the last decade due to their wide applications in the realization of variable parameter networks, distributed and lumped-distributed structures.

Lunelli (1955) outlined a synthesis procedure for realizing a positive real impedance function $Z(p_1)$, by decomposing the numerator polynomial of $dZ(p_1)/dp_1$, into sums and differences of squares of polynomials. Ansell (1964) gave a two-variable extension of the above result for the case of lossless reciprocal two-variable networks. The applicability of the above result in the realization of a high-degree two-variable reactance functions is limited in scope, as it involves the development of methods for finding the relevant polynomial decomposition, which is a difficult problem. The above problem, however, does not exist in the case of minimal realization of a class of two-variable reactance functions, bilinear in one variable. Here the restriction imposed by Ansell, namely, that the two-variable positive-real function has to be a reactance function, will be overcome, and a more general result will be arrived at, which will be used in realizing a class of two-variable positive-real functions (not necessarily reactance) bilinear in one variable.

2. Definitions

Definition 1. The function $Z(p_1 \, p_2)$ is said to be a two-variable positive-real function if (*a*) $Z(p_1, p_2)$ is a rational-real function of p_1, p_2, and (*b*) $\text{Re}\, Z(p_1, p_2) \geqslant 0$ for $\text{Re}\, p_1 > 0$ and $\text{Re}\, p_2 > 0$.

Definition 2. The function $Z(p_1, x)$ is called a variable parameter positive-real function, if it is a positive-real function of p_1 for $x \geqslant 0$.

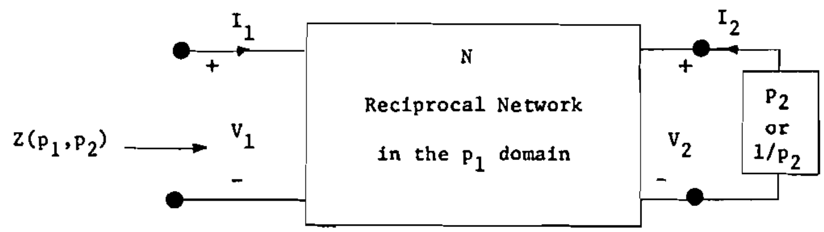
3. Analysis

Consider in general a lossy two-port network N, characterized by its open circuit impedance matrix,

$$[Z(p_1)] = \begin{bmatrix} z_{11}(p_1) & z_{12}(p_1) \\ z_{21}(p_1) & z_{22}(p_1) \end{bmatrix} \tag{1}$$

† Communicated by the Author.

Fig. 1



as shown in fig. 1, and consider first the case where the output port is terminated by p_2. Then, the driving-point impedance at port 1, is given by

$$Z(p_1, p_2) = z_{11}(p_1) - \frac{z_{12}(p_1)\, z_{21}(p_1)}{z_{22}(p_1) + p_2}. \tag{2}$$

By differentiating the above function, partially with respect to p_2, one gets

$$\frac{\partial Z(p_1, p_2)}{\partial p_2} = \frac{z_{12}(p_1)\, z_{21}(p_1)}{(z_{22}(p_1) + p_2)^2}. \tag{3}$$

A simple analysis will also lead to

$$\left(\frac{I_2}{I_1}\right)^2 = \frac{(z_{21}(p_1))^2}{(z_{22}(p_1) + p_2)^2}. \tag{4}$$

If the network N, is reciprocal, that is $z_{12}(p_1) = z_{21}(p_1)$, then by comparing eqns. (3) and (4), one gets

$$\frac{\partial Z(p_1, p_2)}{\partial p_2} = \left(\frac{I_2}{I_1}\right)^2. \tag{5}$$

Similarly if the reciprocal network N is terminated at its output port by $1/p_2$, one can show that

$$\frac{\partial Z(p_1, p_2)}{\partial p_2} = -\left(\frac{V_2}{I_1}\right)^2. \tag{6}$$

The above result can be summarized in the form of a theorem.

4. Theorem

If $Z(p_1, p_2)$ is a two-variable positive-real function (not necessarily reactance) bilinear in the variable p_2, then $Z(p_1, p_2)$ is realizable as the driving-point impedance of a reciprocal network in the p_1 domain, terminated at its output port by p_2 or $1/p_2$, if and only if, the numerator of $\partial Z(p_1, p_2)/\partial p_2 = \pm$ (a perfect square). The positive sign corresponds to a load of p_2, and the negative sign corresponds to a load of $1/p_2$. It is also noted that, in general, the two-port will be lossy, unless $Z(p_1, p_2)$ is a two-variable reactance function, in which case the two-port is lossless.

5. Synthesis

The synthesis procedure is summarized below :

5.1

Any two-variable positive-real function

$$Z(p_1, p_2) = \frac{N_1(p_1, p_2)}{M_1(p_1, p_2)} \tag{7}$$

bilinear in the variable p_2, is to be realized as shown in fig. 1 (if possible).

Since $Z(p_1, p_2) = V_1/I_1$, comparing with eqn. (7), one identifies

$$V_1 = \frac{N_1(p_1, p_2)}{Q(p_1, p_2)}, \tag{7 a}$$

$$I_1 = \frac{M_1(p_1, p_2)}{Q(p_1, p_2)}, \tag{7 b}$$

where $Q(p_1, p_2)$ is a polynomial in the variables p_1, p_2.

5.2

If

$$\frac{\partial Z(p_1, p_2)}{\partial p_2} = +\left(\frac{M_2(p_1, p_2)}{M_1(p_1, p_2)}\right)^2 = \left(\frac{M_2/Q}{M_1/Q}\right)^2, \tag{8 a}$$

or if

$$\frac{\partial Z(p_1, p_2)}{\partial p_2} = -\left(\frac{N_2(p_1, p_2)}{M_1(p_1, p_2)}\right)^2 = -\left(\frac{N_2/Q}{M_1/Q}\right)^2, \tag{8 b}$$

then the synthesis procedure is definitely possible, and in the first case the output port load termination is p_2. Comparing eqns. (5) and (8 a) one gets

$$I_2 = \pm\frac{M_2(p_1, p_2)}{Q(p_1, p_2)}. \tag{9 a}$$

In the second case, however, the output port is terminated by $1/p_2$, and comparing with (6),

$$V_2 = \pm\frac{N_2(p_1, p_2)}{Q(p_1, p_2)}. \tag{9 b}$$

5.3

(a) Consider the case, when the output port is terminated by p_2. In this case the port currents and voltages can be written as

$$[I] = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{Q(p_1, p_2)} \begin{bmatrix} M_1(p_1, p_2) \\ \pm M_2(p_1, p_2) \end{bmatrix}, \tag{10 a}$$

$$[V] = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{Q(p_1, p_2)} \begin{bmatrix} N_1(p_1, p_2) \\ \mp p_2 M_2(p_1, p_2) \end{bmatrix}. \tag{10 b}$$

Since the elements of $[I]$ and $[V]$ are polynomials in p_1 and p_2, one can write matrices $[A(p_1)]$ and $[B(p_1)]$ from the coefficients of the powers of p_2 in $[M_1, \pm M_2]$ and $[N_1, \mp p_2 M_2]$, such that

$$[I]^t = \frac{1}{Q(p_1, p_2)} [1 \quad p_2][A(p_1)], \tag{11 a}$$

$$[V]^t = \frac{1}{Q(p_1, p_2)} [1 \quad p_2][B(p_1)]. \tag{11 b}$$

The scattering matrix of the reciprocal two-port network is found from

$$[[V]-[I]]=[S(p_1)][[V]+[I]]. \tag{12 a}$$

From the fact that $[S(p_1)]$ is symmetric for reciprocal two-port networks, one gets

$$[S(p_1)]=[B(p_1)+A(p_1)]^{-1}[B(p_1)-A(p_1)]. \tag{12 b}$$

When $[A(p_1)]$ is non-singular, the impedance matrix $[Z(p_1)]$ is found from

$$[V]=[Z(p_1)][I], \tag{13 a}$$

$$[Z(p_1)]=[A(p_1)]^{-1}[B(p_1)]. \tag{13 b}$$

Similarly when $[B(p_1)]$ is non-singular, the admittance matrix results from

$$[I]=[Y(p_1)][V], \tag{14 a}$$

$$[Y(p_1)]=[B(p_1)]^{-1}[A(p_1)]. \tag{14 b}$$

(b) Next, consider the case when the output port is terminated by p_2. In this case, one gets

$$[V]=\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}=\frac{1}{Q(p_1,p_2)}\begin{bmatrix} N_1(p_1,p_2) \\ \pm N_2(p_1,p_2) \end{bmatrix}=\frac{1}{Q(p_1,p_2)}[B(p_1)]^t\begin{bmatrix} 1 \\ p_2 \end{bmatrix}, \tag{15 a}$$

$$[I]=\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}=\frac{1}{Q(p_1,p_2)}\begin{bmatrix} M_1(p_1,p_2) \\ \mp p_2 N_2(p_1,p_2) \end{bmatrix}=\frac{1}{Q(p_1,p_2)}[A(p_1)]^t\begin{bmatrix} 1 \\ p_2 \end{bmatrix}. \tag{15 b}$$

Hence one can determine $[S(p_1)]$, $[Z(p_1)]$ or $[Y(p_1)]$.

5.4

Realize the reciprocal, two-port network in the p_1 domain following any classical synthesis procedure, of a single-variable two-port network.

The above synthesis procedure is best illustrated by the following examples.

6. Examples

6.1. *Example* 1

Consider the realization of the variable parameter positive-real function

$$Z(p_1,x)=\frac{2p_1{}^2x+p_1x+p_1{}^2+2p_1+1}{p_1x+p_1{}^2+2p_1+1}. \tag{16}$$

Since $Z(p_1,x)=V_1/I_1$, one can identify

$$V_1=\frac{2p_1{}^2x+p_1x+p_1{}^2+2p_1+1}{Q(p_1,x)}, \tag{17}$$

$$I_1=\frac{p_1x+p_1{}^2+2p_1+1}{Q(p_1,x)}, \tag{18}$$

$$\frac{\partial Z(p_1,x)}{\partial x}=\frac{(\sqrt{2p_1}(p_1+1))^2}{(p_1x+p_1{}^2+2p_1+1)^2}. \tag{19}$$

Using the result of the above theorem, it follows that $Z(p_1, x)$ can be realized as the driving-point impedance of a lossy reciprocal network in the p_1 domain, terminated by x, and one identifies

$$I_2 = -\frac{\sqrt{2}p_1(p_1+1)}{Q(p_1, x)}, \tag{20}$$

whence

$$V_2 = -xI_2 = \frac{\sqrt{2}p_1 x(p_1+1)}{Q(p_1, x)}. \tag{21}$$

The port voltages and currents can be written as

$$[V] = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{1}{Q}\begin{bmatrix} 2p_1^2x + p_1x + p_1^2 + 2p_1 + 1 \\ \sqrt{2}p_1x(p_1+1) \end{bmatrix}, \tag{22}$$

$$[I] = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{1}{Q}\begin{bmatrix} p_1x + p_1^2 + 2p_1 + 1 \\ -\sqrt{2}p_1(p_1+1) \end{bmatrix}. \tag{23}$$

Following the synthesis procedure, it follows that

$$[V]^t = \frac{1}{Q}[1 \quad x][B(p_1)], \tag{24 a}$$

where

$$[B(p_1)] = \begin{bmatrix} p_1^2 + 2p_1 + 1 & 0 \\ 2p_1^2 + p_1 & \sqrt{2}p_1(p_1+1) \end{bmatrix} \tag{24 b}$$

and

$$[I]^t = \frac{1}{Q}[1 \quad x][A(p_1)], \tag{25 a}$$

where

$$[A(p_1)] = \begin{bmatrix} p_1^2 + 2p_1 + 1 & -2\sqrt{p_1}(p_1+1) \\ p_1 & 0 \end{bmatrix}. \tag{25 b}$$

As $[V] = [Z][I]$, and since $[A(p_1)]$ is non-singular, therefore the open circuit impedance matrix of the lossy reciprocal network is given by

$$\begin{aligned} [Z(p_1)] &= [A(p_1)]^{-1}[B(p_1)] \\ &= \begin{bmatrix} 2p_1 + 1 & \sqrt{2}p_1 + \sqrt{2} \\ \sqrt{2}p_1 + \sqrt{2} & p_1 + \dfrac{1}{p_1} + 2 \end{bmatrix}, \end{aligned} \tag{26}$$

and the final network realization is shown in fig. 2.

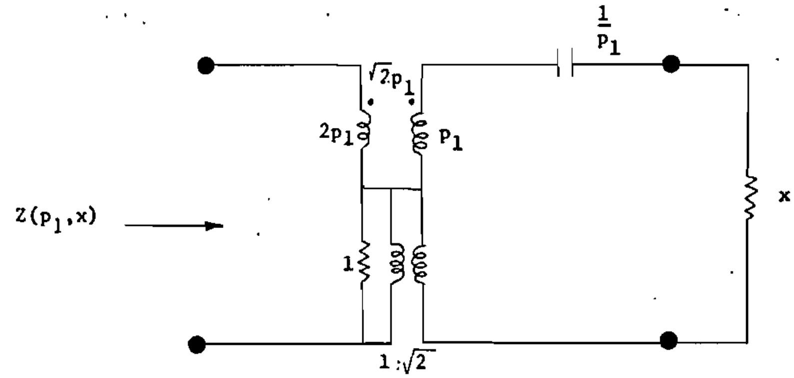
6.2. *Example* 2

Consider the bounded real two-variable function

$$W(p_1, p_2') = \frac{(2p_1^3 - p_1^2 + 3p_1 - 1)p_2' + (p_1^2 - p_1 + 1)}{(2p_1^3 + 3p_1^2 + 5p_1 + 3)p_2' + (p_1^2 + p_1 + 1)}, \tag{27}$$

which is to be realized as the input reflection coefficient of a one-port network,

Fig. 2



assuming unit normalization for W, where p_1 is the complex frequency variable, and $p_2' = \exp(2\tau p_1)$ where τ is a positive-real constant.

$$Z(p_1, p_2') = \frac{1 + W(p_1, p_2')}{1 - W(p_1, p_2')}. \tag{28}$$

Introducing the transformation,

$$p_2 = \frac{p_2' - 1}{p_2' + 1} = \tanh(\tau p_1). \tag{29}$$

Hence, the corresponding driving-point impedance is,

$$Z(p_1, p_2) = \frac{(4p_1^3 + 4p_1^2 + 8p_1 + 4) + p_2(4p_1^3 + 8p_1)}{(4p_1^2 + 4p_1 + 4) + p_2(4p_1^2 + 4)}, \tag{30}$$

$$\frac{\partial Z(p_1, p_2)}{\partial p_2} = -(4)^2 = -(\text{a perfect square}). \tag{31}$$

Indicating that the given function is realizable by the given technique, and the load impedance is $1/p_2$.

$$[V] = \frac{1}{Q(p_1, p_2)} \begin{bmatrix} (4p_1^3 + 4p_1^2 + 8p_1 + 4) + p_2(4p_1^3 + 8p_1) \\ 4 \end{bmatrix} \tag{32 a}$$

$$= \frac{1}{Q(p_1, p_2)} \begin{bmatrix} (4p_1^3 + 4p_1^2 + 8p_1 + 4) & 4p_1^3 + 8p_1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ p_2 \end{bmatrix}, \tag{32 b}$$

$$[I] = \frac{1}{Q(p_1, p_2)} \begin{bmatrix} (4p_1^2 + 4p_1 + 4) + p_2(4p_1^2 + 4) \\ -4p_2 \end{bmatrix} \tag{33 a}$$

$$= \frac{1}{Q(p_1, p_2)} \begin{bmatrix} (4p_1^2 + 4p_1 + 4) & (4p_1^2 + 4) \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ p_2 \end{bmatrix}. \tag{33 b}$$

Identifying

$$[B(p_1)] = \begin{bmatrix} 4p_1^3 + 4p_1^2 + 8p_1 + 4 & 4 \\ 4p_1^3 + 8p_1 & 0 \end{bmatrix}, \tag{34}$$

$$[A(p_1)] = \begin{bmatrix} 4p_1^2 + 4p_1 + 4 & 0 \\ 4p_1^2 + 4 & -4 \end{bmatrix}. \tag{35}$$

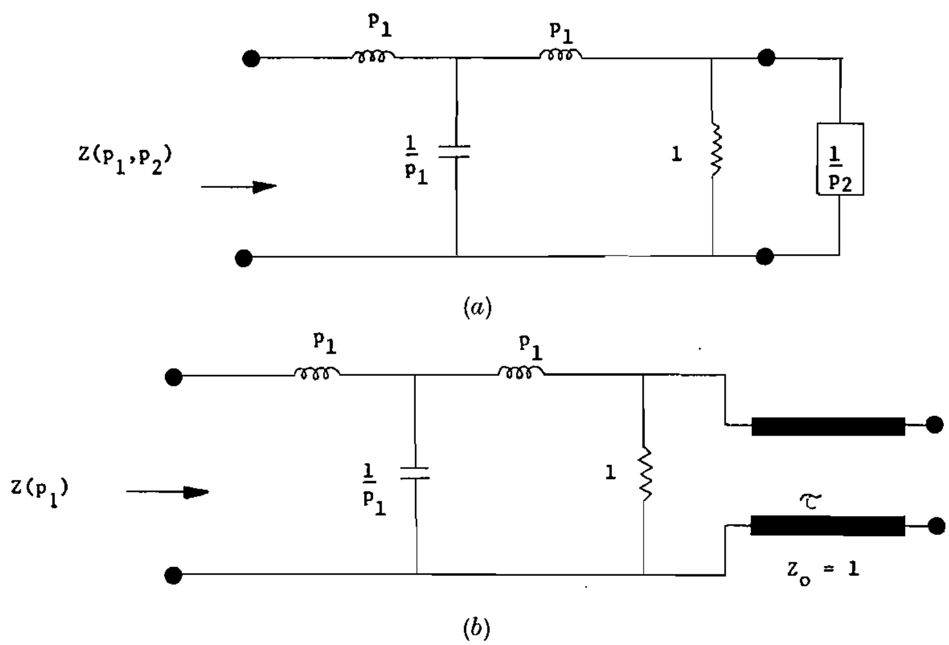
Therefore, the open circuit impedance matrix of the lossy reciprocal network is given by

$$[Z(p_1)] = [A(p_1)]^{-1}[B(p_1)]$$

$$= \begin{bmatrix} \dfrac{p_1^3+p_1^2+2p_1+1}{p_1^2+p_1+1} & \dfrac{1}{p_1^2+p_1+1} \\ \dfrac{1}{p_1^2+p_1+1} & \dfrac{p_1^2+1}{p_1^2+p_1+1} \end{bmatrix}, \tag{36}$$

and the network realization is shown in fig. 3 (*a*). The final network realization in the p_1 domain is shown in fig. 3 (*b*), where the load impedance cosh (τp_1) is realized as the driving-point impedance of an open circuit unit element with characteristic impedance = 1, and one way time delay τ, which is the same as Youla's (1966) realization.

Fig. 3



7. Conclusions

A synthesis procedure for realizing a class of two-variable positive-real functions bilinear in one variable was given. The procedure is illustrated by two examples which demonstrate the application of the above method in the realization of variable parameter networks and lumped-distributed networks.

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