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A new generation of positive real functions using the Bessel polynomials†

AHMED M. SOLIMAN

Electrical Engineering Department, College of Steubenville,
Steubenville, Ohio 43952, U.S.A.

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A new generation of single variable positive real functions using the Bessel polynomials is given. It will be shown that the real rational function $sB_{n-1}(s)/B_n(s)$ is realizable as the driving point impedance (or admittance) of a finite ladder of lumped inductances and capacitances terminated at the output port by a 1 ohm resistor.

1. Introduction

Generation of positive real functions is a very important subject in n etwork synthesis, since it may lead to new synthesis techniques (Bott and Duffin 1949) or in some cases may lead to new coefficient relations (Reza and Bose 1968). The generation of a special class of positive real functions from the real orthogonal polynomials was discussed by Reza and Bose (1967). The main idea in the generation was based on the interlacing of the simple zeros of two successive orthogonal polynomials from the same family. The above idea can be used only in generating a special class of positive real functions, namely, RL, RC or LC functions, where the locations of the poles and zeros are restricted to the real or the imaginary axes respectively. One link between the Bessel polynomials and network theory is well known. It was shown by Thomson (1959) that the linear phase filter of order n with all of its transmission zeros at infinity has a transfer characteristic that can be represented by

$$T(s)=\frac{K}{B_n(s)} \tag{1}$$

where $B_n(s)$ is the Bessel polynomial (1949) of order n. The present paper presents another link between the Bessel polynomials and realizability theory. The main result of this paper is written below in the form of a theorem.

2. Theorem

The real rational function $F_n(s)=sB_{n-1}(s)/B_n(s)$ is a positive real function, realizable as the driving point impedance (or admittance) of a finite ladder of lumped inductances and capacitances, terminated at the output port by a 1 ohm resistor, for all $(n=1, 2, 3, \ldots)$ where $B_n(s)$ is the Bessel polynomial of order n, and s is the complex frequency variable.

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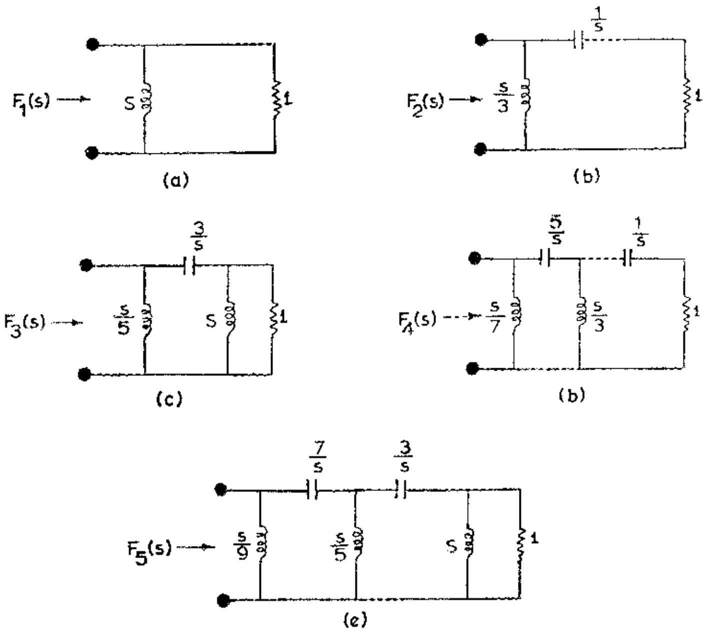
Fig. 1



Realization of $F_n(s) = sB_{n-1}(s)/B_n(s)$ as a driving point impedance, for $n = 1, 2, 3, 4$ and 5. All elements are in ohms.

3. Proof

The concept of mathematical induction is used here to prove the above theorem. It is clear that for $n = 1$, the real rational function

$$F_1(s) = \frac{sB_0(s)}{B_1(s)} = \frac{s}{s+1} \tag{2}$$

is a positive real function, and its realization as a driving point impedance is shown in fig. 1 (*a*).

Next, assuming that

$$F_n(s) = \frac{sB_{n-1}(s)}{B_n(s)} \tag{3}$$

is a positive real function, it will be shown that

$$F_{n+1}(s) = \frac{sB_n(s)}{B_{n+1}(s)} \tag{4}$$

is also a positive real function.

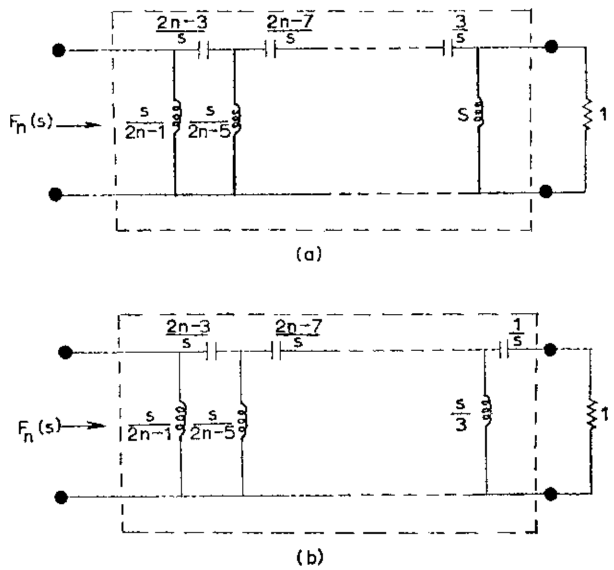
From the recursion formula of the Bessel polynomials, namely :

$$B_{n+1}(s) = (2n+1)B_n(s) + s^2 B_{n-1}(s), \tag{5}$$

dividing both sides by $sB_n(s)$, one gets

$$F_{n+1}(s) = \frac{1}{\dfrac{2n+1}{s} + F_n(s)}. \tag{6}$$

Fig. 2



Realization of $F_n(s)=sB_{n-1}(s)/B_n(s)$ as a driving point impedance. All elements are in ohms. (*a*) n is odd. (b) n is even.

Since the sum of two positive real functions and the reciprocal of a positive real function is positive real (Seshu and Balabanian 1957), it follows that $F_{n+1}(s)$ is positive real.

The realization of $F_n(s)$ as the driving point impedance is shown in fig. 2 (*a*) and (*b*) for n odd and n even respectively. It should be noted that the realization is always in the form of a lossless two port network, consisting of a finite number of lumped inductances and capacitances in the form of a ladder structure, terminated at the output port by a 1 ohm resistor. Furthermore, it is noted that the first element in the realization is always a parallel inductor of inductance equal to $1/(2n-1)$ $(n=1, 2, 3, \ldots)$, and the last element in the lossless two port is either a parallel unit inductor if n is odd or a series unit capacitor if n is even.

The realization of $F_n(s)$ as a driving point admittance is also given in fig. 3.

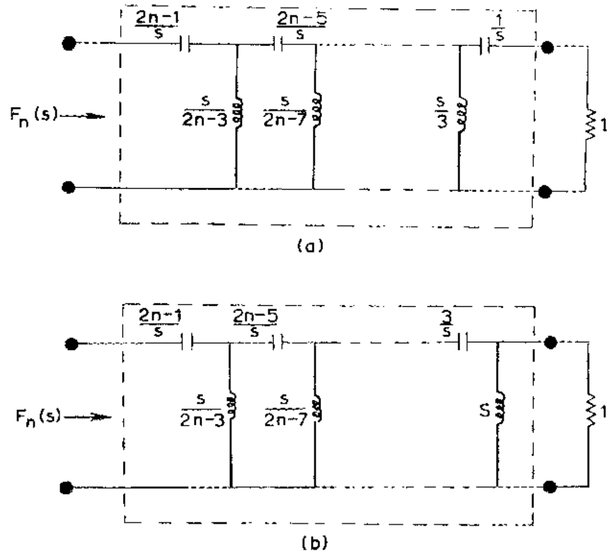
It is emphasized here that the given realization is minimal and the total number of elements is equal to $(n+1)$.

It is of interest to compare the result arrived at here using the Bessel polynomials, with the well known result of Reza and Bose (1967) namely : the real rational function $jJ_{n-1}(js)/J_n(js)$ is LC, and is realizable as an infinite ladder of lumped inductances and capacitances, where $J_n(s)$ is the Bessel function of order n.

4. Conclusions

A new generation of positive real functions using the Bessel polynomials was given. It was shown that the real rational function $F_n(s)=sB_{n-1}(s)/B_n(s)$ is realizable as the driving point impedance (or admittance) of a lossless finite

Fig. 3



Realization of $F_n(s) = sB_{n-1}(s)/B_n(s)$ as a driving point admittance. All elements are in ohms. (*a*) n is odd. (*b*) n is even.

two port network, terminated at its output port by a 1 ohm resistor. The overall structure in this case is in the form of the Darlington's well known configuration (1939), and the lossless two port is in the Cauer's form (1926).

References

Bott, R., and Duffin, R. J., 1949, *J. appl. Phys.*, **20**, 816.
Cauer, W., 1926, *Arch. Elektrotech.*, **15**, 355.
Darlington, S., 1939, *J. Math. Phys.*, **18**, 257.
Krall, H. L., and Fink, O., 1949, *Am. math. Soc.*, **65**, 100.
Reza, F. M., and Bose, N. K., 1967, *Proc. Tenth Midwest Symposium on Circuit Theory*, Purdue University, Lafayette, Indiana, U.S.A., pp. V.2.1–V.2.11; 1968, *I.E.E.E. Trans. Circuit Theory*, **15**, 76.
Seshu, S., and Balabanian, N., 1957, *I.R.E. Trans. Circuit Theory*, **4**, 306.
Thomson, W. E., 1959, *I.R.E. Trans. Circuit Theory*, **6**, 235.