Design Equations For Fractional-Order Sinusoidal Oscillators: Practical Circuit Examples

A. G. Radwan*, A. M. Soliman** and A. S. Elwakil***

* Dept. of Engineering Mathematics, Faculty of Engineering, Cairo University, Egypt (email: ahmedgom@yahoo.com).

** Dept. of Electronics & Communications, Faculty of Engineering, Cairo University, Egypt (email: asoliman@ieee.org).

*** Dept. of Electrical & Computer Engineering, University of Sharjah, Emirates (email: elwakil@ieee.org).

***Abstract*— Some practical sinusoidal oscillators are studied in the general form where fractional-order energy storage elements are considered. A fractional-order element is one whose complex impedance is given by $Z = a(j\omega)^{\pm\alpha}$; a is a constant and α is not necessarily an integer. As a result, these oscillators are described by sets of fractional-order differential equations. Numerical and PSpice simulation results are shown.**

I. Introduction

A growing interest in the applications of fractional calculus in various disciplines has recently accumulated [1-6]. On the electronic circuits front, the current-voltage relations in the classical inductor and capacitor energy storage elements are governed by first-order differential equations. Hence, any system constructed using n such elements is described by an n^{th}-order system of differential equations. For example, sinusoidal oscillators, which are key electronic building blocks, are classically known to be realizable using at least a second-order circuit. Most famous oscillators are either second-order or third-order oscillators. It is evident that available circuit design techniques are dominantly based on the assumption of a target realizable integer-order circuit.

Four decades ago, several researchers investigated the feasibility of realizing a fractional-order capacitor [7], [8]. A finite element approximation of the special case $Z = 1/C(j\omega)^{1/2}$, was reported in [9]. This finite element approximation relies on the possibility of emulating a fractional-order capacitor via semi infinite self-similar RC trees. The technique was later developed further by the authors of [10-12] in order to approximate any $Z = a(j\omega)^{\pm\alpha}$, where α is arbitrary, and the device with such an impedance was termed "fractance device". Finite element approximations offer a valuable tool by which the effect of a fractance device can be simulated using a standard circuit simulator, or studied experimentally. However, they do not offer a simple practical device. Therefore, investigations of fractional-order circuits remained limited and confined mostly to simulations of special case circuits due to the non existence of a real fractance device and hence the lack of practical motivation [13-15].

Most recently, there has been growing evidence of the possibility of fabricating a real two-terminal fractance device [16], [17]. The fabricated probe described in [16] and [17] is based on a metal-insulator-liquid interface and was used in [17] in a fractional-order differentiator circuit. Although, it is not easy to re-produce the device described by the authors as it involves several chemical procedures, with on going research and more application motivation, the device will surely become commercially available in the near future. Should this happen, circuit designers will be faced with the challenge as to how to utilize the device in constructing their application circuits particularly as the available design techniques have to be generalized from the narrow integer-order subset to the more general fractional-order domain.

In this paper we study some well-known second-order sinusoidal oscillator circuits, assuming they are constructed using fractance devices. We show that these famous oscillators can still be designed to oscillate with fractance devices, given that the classical oscillation conditions are modified as mathematically derived in the paper. We also show that fractional-order oscillators have an advantage which may be exploited. In particular, the oscillation frequency does not only depend on the values of the reactive elements (L and/or C) but also on their fractional-order α, which adds an extra degree of design freedom. Numerical and PSpice simulation results are shown. For the PSpice simulations, the finite element approximations reported in [10-12] were used.

It is worth noting that linear integer or fractional-order differential equations cannot admit sustained oscillations. To obtain an accurate oscillator model requires the modeling equations to be nonlinear. It is also known that the Barkhausen criterion presents a necessary but not sufficient condition for oscillation. In particular, an oscillator might actually latch-up and never oscillate as intended even if the Barkhausen criterion is satisfied [18]. However, circuit designers still apply the Barkhausen criterion to a linearized around the origin equilibrium point model of their oscillator circuit in order to find the oscillation condition and oscillation frequency. The examples in this paper are known to oscillate and not exhibit latch-up. Hence, to compare the derived formulae with existing integer-order ones, the authors have also considered applying the marginal stability condition (Barkhausen criterion) to the oscillator models after being modified to represent non-integer-order oscillators. The stability of fractional order differential equations was studied for example in [19-22] and recently in [23]. It is also worth noting that some fractional-order oscillators were considered by the authors of [5] and [24] and that the fractional-order Fourier transform was considered in [25].

II. BACKGROUND

The Riemann-Liouville definition of a fractional derivative of order $(m-1) < \alpha < m$ is given by [26], [27]:

$$\frac{d^\alpha}{dt^\alpha} f(t) := \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau\right] \quad (1)$$

while the Grünwald-Letnikov approximation of a fractional derivative of order α is given by [28]

$$\frac{d^\alpha}{dt^\alpha} f(t) \equiv \lim_{\Delta T \to 0} \frac{(\Delta T)^{-\alpha}}{\Gamma(-\alpha)} \sum_{j=0}^{\infty} \frac{\Gamma(j-\alpha)}{\Gamma(j+\alpha)} f(t - j\Delta T) \quad (2)$$

where $\Gamma(\cdot)$ is the gamma function and ΔT is a discrete time-step. Throughout this work, numerical simulations are performed using the Grünwald-Letnikov approximation.

According to the theorem developed in [29], a linear fractional order system of the form

$$\begin{pmatrix} \frac{d^\alpha}{dt^\alpha} x_1 \\ \frac{d^\beta}{dt^\beta} x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (A) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3)$$

will oscillate if and only if there exists a value for ω which satisfies simultaneously the two equations

$$\omega^{\alpha+\beta} \cos\frac{(\alpha+\beta)\pi}{2} - a_{11}\omega^\beta \cos\frac{\beta\pi}{2} - a_{22}\omega^\alpha \cos\frac{\alpha\pi}{2} + |A| = 0 \quad (4a)$$

$$\omega^\beta \sin\frac{(\alpha+\beta)\pi}{2} - a_{11}\omega^{\beta-\alpha} \sin\frac{\beta\pi}{2} - a_{22} \sin\frac{\alpha\pi}{2} = 0 \quad (4b)$$

where, $|A|$ is the determinant of the matrix A. Table 1 is a summary of important special cases for such a system.

III. FRACTIONAL-ORDER WIEN OSCILLATORS

In what follows, the finite element approximations of fractional capacitors [10-12], shown in Figs. 1(a) and 1(b) are used for simulations. Consider the fractional-order Wien oscillators shown in Figure 2 which can all be modelled by (3) around the equilibrium point at the origin. The design procedure for all members of this family is similar and may be demonstrated on the circuit of Fig. 2(a) which is modelled by

$$\begin{pmatrix} \frac{d^\alpha}{dt^\alpha} V_{C1} \\ \frac{d^\beta}{dt^\beta} V_{C2} \end{pmatrix} = \begin{pmatrix} -1/R_1C_1 & k/R_1C_1 \\ -1/R_1C_2 & k/R_1C_2 - 1/R_2C_2 \end{pmatrix} \begin{pmatrix} V_{C1} \\ V_{C2} \end{pmatrix} \quad (5)$$

Using (4), we may solve for the design parameter k to obtain the oscillation condition

$$k = \frac{1 + a\omega^{\alpha+\beta}\cos\frac{(\alpha+\beta)\pi}{2} + b\omega^\beta \cos\frac{\beta\pi}{2} + c\omega^\alpha \cos\frac{\alpha\pi}{2}}{d\omega^\alpha \cos\frac{\alpha\pi}{2}} = \frac{a\omega^{\alpha+\beta}\sin\frac{(\alpha+\beta)\pi}{2} + b\omega^\beta \sin\frac{\beta\pi}{2} + c\omega^\alpha \sin\frac{\alpha\pi}{2}}{d\omega^\alpha \sin\frac{\alpha\pi}{2}} \quad (6)$$

$a = R_1R_2C_1C_2$, $b = R_2C_2$, $c = R_1C_1$ and $d = R_2C_1$. Equating both sides of the above equation gives the oscillation frequency as the solution to the following equation

$$a\omega^{\alpha+\beta} \sin\frac{\beta\pi}{2} + b\omega^\beta \sin\frac{(\beta-\alpha)\pi}{2} - \sin\frac{\alpha\pi}{2} = 0 \quad (7)$$

Note that ω does not only depend on the components values but also on α and β, which gives two extra degrees of freedom in the design process. The design steps for this oscillator are as follows: given R_1, R_2, C_1, C_2, α and β equation (7) is solved to obtain the frequency of oscillation and then the necessary value of k can be obtained from (6). Table 2 is a summary of some special cases for this oscillator. The design equations of the last row of the Table are those of the well-known 2^{nd}-order case. Figures 3(a)-(d) show numerical simulations and the corresponding PSpice simulation results for the two cases $(\alpha, \beta) = (0.4, 1)$and $(0.5, 0.5)$ respectively.

IV. NEGATIVE RESISTOR RC OSCILLATOR

An RC oscillator can be obtained from that of Fig. 2(c) by disconnecting R_2 from ground, connecting it to the op amp inverting input terminal and adding an extra resistor R from this same terminal to ground. This oscillator may then be modelled in the op amp linear region by

$$\begin{pmatrix} \frac{d^\alpha}{dt^\alpha} V_{C1} \\ \frac{d^\beta}{dt^\beta} V_{C2} \end{pmatrix} = \begin{pmatrix} -\frac{1+R_1/R_2}{R_1C_1} & \frac{1}{R_1C_1} \\ -\frac{1+[(1+R_1/R_2)R_4/R_3]}{R_2C_2} & \frac{R_4(1+\frac{R}{R_1})}{R_3C_2R} \end{pmatrix} \begin{pmatrix} V_{C1} \\ V_{C2} \end{pmatrix} \quad (8)$$

from which it seen that $|A|$ can be positive or negative, but for stability it must remain positive [23]. Using (4), and assuming that the design parameter is the resistance R, the following condition for oscillation is obtained

$$R = \frac{\frac{R_4\omega^\alpha \sin\alpha\pi/2}{R_3C_2}}{\omega^{\alpha+\beta}\sin\frac{(\beta+\alpha)\pi}{2} + \frac{(R_1+R_2)\omega^\beta \sin\frac{\beta\pi}{2}}{R_1R_2C_1} - \frac{R_4\omega^\alpha \sin\frac{\alpha\pi}{2}}{R_1R_3C_2}} \quad (9)$$

while the oscillation frequency is the solution of

$$\omega^{\alpha+\beta}\sin\frac{\beta\pi}{2} + a\omega^\beta \sin\frac{\beta\pi}{2}\sin\frac{\alpha\pi}{2} + b\omega^{\beta-\alpha}\sin\frac{\beta\pi}{2} - c\sin\frac{\alpha\pi}{2} = 0 \quad (10)$$

where $a = 2(R_1+R_2)/R_1R_2C_1$, $b = [(R_1+R_2)/R_1R_2C_1]^2$ and $c = \frac{1+[(1+R_1/R_2)R_4/R_3]}{R_1R_2C_1C_2}$. Similar design steps as for the Wien oscillator family can be followed. Table 3 summarizes some special cases for this oscillator. The second row of the Table corresponds to the well-known 2^{nd}-order case.

V. CONCLUSION

We have generalized the design equations of the some famous RC oscillators to the fractional-order domain. Should a fractional-order capacitor become commercially available in the near future, using it should not be a challenging matter after developing the appropriate design equations.

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Table 1:- Special cases of the fractional-order system described by (3)

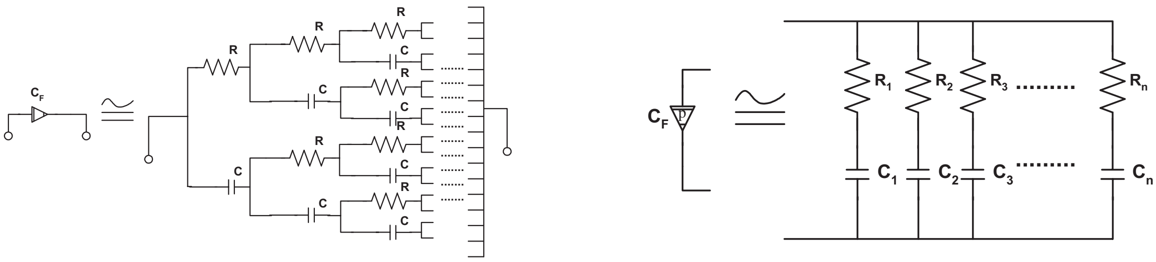
No.	Case	Oscillation condition and oscillation frequency (ω)
1	$\alpha = \beta \neq 1$	$tr(A) = a_{11} + a_{22} = 2\sqrt{\lvert A\rvert}\cos(\frac{\alpha\pi}{2})$ $\omega = \left(\frac{tr(A)}{2\cos(0.5\alpha\pi)}\right)^{\frac{1}{\alpha}} = (\lvert A\rvert)^{\frac{1}{2\alpha}}$
2	$\alpha = \beta = 1$	$tr(A) = a_{11} + a_{22} = 0$ $\omega = \sqrt{\lvert A\rvert}$
3	$\beta = 2\alpha$	$\lvert A\rvert = a_{11}p^{2\alpha}\cos(\alpha\pi) + a_{22}p^{\alpha}\cos(\frac{\alpha\pi}{2}) - p^{3\alpha}\cos(\frac{3\alpha\pi}{2})$ $\omega = \left(\frac{a_{11}\cos(0.5\alpha\pi) + \sqrt{a_{11}^2\cos^2(0.5\alpha\pi) + a_{22}\left(4\cos^2(0.5\alpha\pi) - 1\right)}}{4\cos^2(0.5\alpha\pi) - 1}\right)^{\frac{1}{\alpha}}$
4	$\beta = 1$ $\alpha = 0.5$	$\lvert A\rvert = \sqrt{\frac{\omega}{2}}\left(a_{22} + \omega\right)$ $\omega = \frac{a_{11}^2}{2}\left(1 + \sqrt{1 + \frac{2a_{22}}{a_{11}^2}}\right)^2$
5	$\mathbf{a_{11} = 0}$	$\lvert A\rvert = \omega^{\alpha}\frac{a_{22}\sin\left(0.5\beta\pi\right)}{\sin\left(0.5\left(\alpha+\beta\right)\pi\right)}$ $\omega = \left(\frac{a_{22}\sin\left(0.5\alpha\pi\right)}{\sin\left(0.5\left(\alpha+\beta\right)\pi\right)}\right)^{\frac{1}{\beta}}$
6	$\mathbf{a_{11} = 0}$ $\mathbf{a_{22} = 0}$	$\alpha + \beta = 2$ $\omega = \sqrt{\lvert A\rvert}$
7	$\Phi = \frac{\pi}{2}$	$\omega = \left(\frac{a_{11}}{\cos\left(0.5\alpha\pi\right)}\right)^{\frac{1}{\alpha}} = \left(\frac{a_{22}}{\cos\left(0.5\beta\pi\right)}\right)^{\frac{1}{\beta}} = \left(\frac{\lvert A\rvert}{\cos\left(0.5\left(\alpha-\beta\right)\pi\right)}\right)^{\frac{1}{\alpha+\beta}}$
8	$\mathbf{a_{12} = 0}$ **Or** $\mathbf{a_{21} = 0}$	Impossible to oscillate.

Table 2:- Special cases for the Wien oscillator of Fig. 2(a)

	$a=R_3/R_4$	**ω**	Φ_α
$\alpha = \beta \neq 1$	$\frac{C_2}{C_1} + \frac{R_1}{R_2} + 2\sqrt{\frac{R_1C_2}{R_2C_1}}\cos\left(0.5\alpha\pi\right)$	$\left(\frac{1}{R_1R_2C_1C_2}\right)^{\frac{1}{2\alpha}}$	$\tan^{-1}\frac{\sqrt{R_1C_1}\sin\left(0.5\alpha\pi\right)}{\sqrt{R_1C_1}\cos\left(0.5\alpha\pi\right) + \sqrt{R_2C_2}}$
$\alpha = \beta = 1$	$\frac{C_2}{C_1} + \frac{R_1}{R_2}$	$\frac{1}{R_1R_2C_1C_2}$	$\tan^{-1}\sqrt{\frac{R_1C_1}{R_2C_2}}$
$\alpha = \beta \neq 1$ $R_1=R_2=R$ $C_1=C_2=C$	$2\left(1 + \cos\left(0.5\alpha\pi\right)\right)$	$\left(\frac{1}{RC}\right)^{\frac{1}{\alpha}}$	$0.25\alpha\pi$
$\alpha = \beta = 1$ $R_1=R_2=R$ $C_1=C_2=C$	2	$\frac{1}{RC}$	0.25π

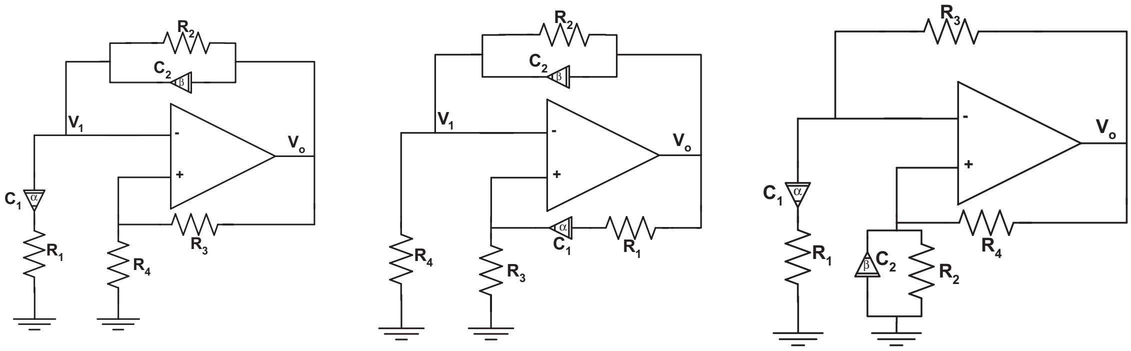
Table 3:- Special cases of the negative resistance oscillator

Cases	Oscillation frequency, oscillation condition and stability condition
$\alpha = \beta < 1$	$\omega^{2\alpha} + 2\omega^{\alpha}\left(\frac{R_1+R_2}{R_1R_2C_1}\right)\cos\left(\frac{\alpha\pi}{2}\right) + \left(\frac{R_1+R_2}{R_1R_2C_1}\right)^2 - \frac{\left(R_1R_3 + R_4\left(R_1+R_2\right)\right)}{R_1^2R_2R_3C_1C_2} = 0$ $R = \frac{R_1R_2C_1R_4}{2R_1R_2R_3C_1C_2\omega^{\alpha}\cos\left(0.5\alpha\pi\right) + R_3C_2\left(R_1+R_2\right) - R_2C_1R_4}$ (for stability) $\left(1+\frac{R_2}{R_1}\right)\frac{C_2}{C_1} \geq \frac{R_4}{R_3} > \left(1+\frac{R_2}{R_1}\right)\frac{C_2}{C_1} - \frac{R_1}{R_1+R_2}$
$\alpha = \beta = 1$	$\omega = \sqrt{\left(\frac{1}{R_1R_2C_1C_2}\left(1+\frac{R_4}{R_3}\left(1+\frac{R_2}{R_1}\right)\right) - \left(\frac{R_1+R_2}{R_1R_2C_1}\right)^2\right)}$ $R = \frac{R_1R_4R_2C_1}{\left(R_1+R_2\right)R_3C_2 - R_4R_2C_1}$ (for stability) $\left(1+\frac{R_2}{R_1}\right)\frac{C_2}{C_1} \geq \frac{R_4}{R_3} > \left(1+\frac{R_2}{R_1}\right)\frac{C_2}{C_1} - \frac{R_1}{R_1+R_2}$
$\alpha = \beta$ & $R_1=R_2=R_3=R_4=R_e$ $C_1=C_2=C$	$R_e^2C^2\omega^{2\alpha} + 4R_eC\omega^{\alpha}\cos\left(0.5\alpha\pi\right) + 1 = 0$ $R = \frac{R_e}{2CR_e\omega^{\alpha}\cos\left(0.5\alpha\pi\right) + 1}$ (for stability) $\alpha > \frac{4}{3}$
$\alpha = \beta$ & $R_1=R_2=R_3=R_4=R_e$	$\omega^{2\alpha} + \left(\frac{4\omega^{\alpha}}{R_eC_1}\right)\cos\left(\frac{\alpha\pi}{2}\right) + \left(\frac{1}{R_eC}\right)^2\left(4 - \frac{3C_1}{C_2}\right) = 0$ $R = \frac{R_eC_1}{2R_eC_1C_2\omega^{\alpha}\cos\left(0.5\alpha\pi\right) + 2C_2 - C_1}$ (for stability) $2 \geq \frac{C_1}{C_2} > \frac{4}{3}$



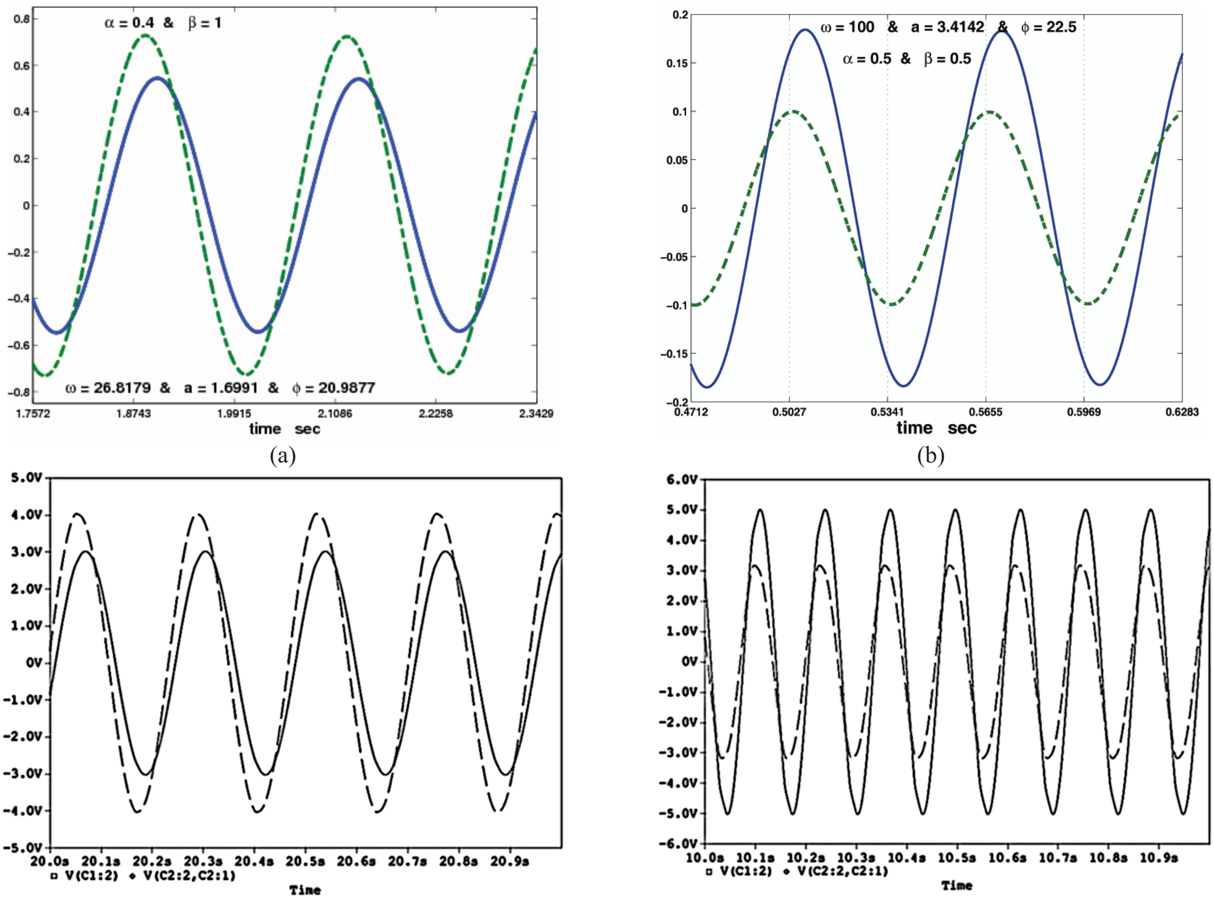
(a) (b)

Figure 1: (a) Finite element approximation of a fractal capacitor of order 0.5, and (b) approximation of a fractal capacitor of any order α<1.



(a) (b) (c)

Figure 2: Fractional-order Wien oscillators with two fractional capacitors.



(c) (d)

Figure 3: Numerical and PSpice simulations for the Wien Oscillator of Fig. 2(a)