Butterworth Passive Filter in the Fractional - Order

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Abstract—**In this paper, the generalized analysis of the first Butterworth filter based on two passive elements is introduced in the fractional-order sense. The fractional-order condition of the Butterworth circuit is presented for the first time where it will lead us back to the known condition of the integer-order circuit when the two fractional-orders equal one. Therefore, the conventional behavior of the integer-order circuit is a narrow subset of the fractional-order ones. The circuit is studied under same and different order cases, and verified through their numerical simulations. Stability analysis is also introduced showing the poles location in the fractional-order versus integer order cases.**

I. INTRODUCTION

The concept of the differential operator $\mathrm{D} = \mathrm{d/dx}$ has been known for a long time and reflects on the analysis and design of the classical linear circuit theory based on the three conventional elements, which are the resistor, capacitor and inductor. Recently, huge attentions have been concentrated on fractional calculus and its applications in various fields such as the electrical, mechanical, electromagnetic, and bioengineering fields [1]–[8]. This revolution is increased due to the huge effort by many scientists in many fields to realize the fractional-element or the so called constant phase element (CPE) [9] – [13]. The generalized analysis of the first and second order filters are introduced in the fractional-order domain as in [3] – [5].

The Riemann–Liouville definition of fractional [3]-[6] derivative is given by:

$$\frac{\mathrm{d}^\alpha}{\mathrm{dt}^\alpha}\mathrm{f(t)} \equiv \mathrm{D}^\alpha \mathrm{f(t)} = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{dt}}\int_0^{\mathrm{t}} (\mathrm{t}-\tau)^{-\alpha}\mathrm{f}(\tau)\mathrm{d}\tau \quad (1)$$

Where $0 < \alpha \leq 1$. The Laplace transform of (1) is

$$L\{d_t^\alpha f(t)\} = s^\alpha F(s) \quad (2)$$

Therefore it is possible to define a fractance element as one whose impedance $Z(j\omega)$ is proportional to s^α [9]–[13] as follows

$$Z(j\omega) \propto \omega^\alpha e^{\frac{j\alpha\pi}{2}} \quad (3)$$

Conventionally, α has been restricted to $\{-1, 0, +1\}$ for the well known circuit components capacitor, resistor and inductor respectively [10]-[11]. However, the fractional-element has many interesting properties such as: the phase angle is constant independent of the frequency, its magnitude versus frequency is nonlinear ($\alpha \neq 1$) which can magnify or decrease the effect of frequency for $\alpha > 1$ and $\alpha < 1$ respectively, and the extra parameter α added to the circuit design can be used for further optimization or design control.

The importance of filter design in engineering applications such as signal processing and integrated RF transceivers [14], [15] is indubitable. Previously, some fractional-order filter characteristics have a unique response and can't be achieved by the integer order elements [3] – [5].

Butterworth filter approximation is one of the most widely used filter type because it has a number of interesting properties where the frequency response of the Butterworth low-pass filter is maximally flat in the pass-band [15] – [16]. In addition, the response has unity gain at DC, and the gain at the normalized pass-band edge frequency of $\omega_n = 1$ depends on the value of ε [17]–[21]. The magnitude squared function of the characteristic equation was introduced in [18] as

$$|D(j\omega)|^2 = 1 + \varepsilon\omega_n^{2N} \quad (4)$$

Where N is a positive integer and represents the filter order and ω_n is the normalized frequency and ε is the attenuation in the pass-band and it is assumed to be unity in the paper.

In this paper, we will discuss the generalized condition for the Butterworth filter design in the fractional order domain instead of the integer order filter design [3]–[6], where the characteristic equation of Butterworth filter given in (4) will become as follow

$$|D(j\omega)|^2 = 1 + \varepsilon\omega_n^{2m} \quad (5)$$

where m is any real positive value. This paper is organized as follow; the second section presents the proposed design procedure, while sections III and IV discuss the fractional order Butterworth filters for same and different fractional orders respectively. Mathematical formulas, stability analysis and numerical simulations are presented for the first time with different examples. Finally the conclusion exists in section V.

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II. DESIGN PROCEDURE

To study Butterworth filter, we start the analysis from a simple implementation of Butterworth of two passive elements as shown in Fig 1 [18]. Then it will be easy to generalize the design using the cascaded stages. For the circuit to work as a low-pass filter, assume $Z_1 = s^{\beta} L$ and $Z_2 = 1/s^{\alpha}C$. Then, the transfer function of Fig.1 is given by

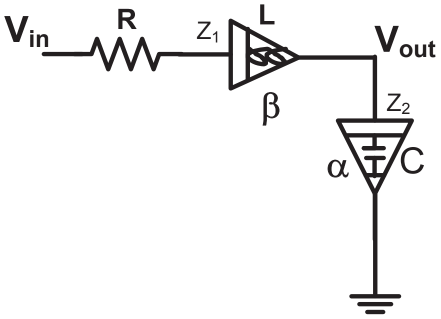

Fig 1. Butterworth filter of two passive elements

$$\frac{V_{out}}{V_{in}} = \frac{1}{LCs^{\alpha+\beta} + RCs^{\alpha} + 1} \tag{6}$$

Where α and β are the fractional-order parameters of the capacitor and the inductor respectively. From (6), the characteristic equation can be obtained as follow

$$D(j\omega) = \\ \left(1 + LC\cos(0.5(\alpha+\beta)\pi)\,\omega^{\alpha+\beta} + RC\cos(0.5\alpha\pi)\omega^{\alpha}\right) + \\ j\left(LC\sin(0.5(\alpha+\beta)\pi)\,\omega^{\alpha+\beta} + RC\sin(0.5\alpha\pi)\,\omega^{\alpha}\right) \tag{7}$$

So, the magnitude squared of the characteristic equation $|D(j\omega)|^2$can be obtained as

$$|D(j\omega)|^2 = 1 + (LC)^2\omega^{2(\alpha+\beta)} + (RC)^2\omega^{2\alpha} \\ + 2LC^2R\cos(0.5\beta\pi)\,\omega^{2\alpha+\beta} \\ + 2LC\cos(0.5(\alpha+\beta)\pi)\,\omega^{\alpha+\beta} \\ + 2RC\cos(0.5\alpha\pi)\,\omega^{\alpha} \tag{8}$$

So, to make the transfer function (6) work as Butterworth filters, the characteristic equation given in (8) should be similar to that given in (5) as follow:

$$|D(j\omega)|^2 = 1 + (LC)^2\omega^{2(\alpha+\beta)} \tag{9}$$

Then the cut-off frequency (ω_o) is given by

$$\omega_o = \left(\frac{1}{LC}\right)^{\frac{1}{\alpha+\beta}} \tag{10}$$

Therefore, the remaining terms of (8) should be equal to zero which represents the condition to obtain Butterworth response for a fractional order filter.

$$2LC^2R\cos(0.5\beta\pi)\,\omega^{2\alpha+\beta} + 2LC\cos(0.5(\alpha+\beta)\pi)\,\omega^{\alpha+\beta} \\ + (RC)^2\omega^{2\alpha} + 2RC\cos(0.5\alpha\pi)\,\omega^{\alpha} = 0 \tag{11}$$

It is clear that, equation (11) is nonlinear and has six variables which are $\{\omega, L, C, R, \alpha, \text{and } \beta\}$. To simplify the analysis, let us define $\lambda^{\alpha} = RC\omega^{\alpha}$ and $\rho\lambda^{\beta} = \omega^{\beta}L/R$, so (11) can be written as follow:

$$\lambda^{\alpha} + 2\rho\cos(0.5\beta\pi)\lambda^{\alpha+\beta} + 2\rho\cos(0.5(\alpha+\beta)\pi)\,\lambda^{\beta} \\ + 2\cos(0.5\alpha\pi) = 0 \tag{12a}$$

$$\rho = \frac{L}{R}(RC)^{-\beta/\alpha}, \qquad \omega = \lambda\left(\frac{1}{RC}\right)^{\frac{1}{\alpha}} = \lambda\left(\frac{R\rho}{L}\right)^{\frac{1}{\beta}} \tag{12b}$$

Since λ is the only parameter that depends on the frequency ω in (12). Then, the condition for Butterworth filter will be frequency dependent unless all coefficients become zero. However, the free term will be zero only if $\alpha = 1$, the second term will be zero if $\rho = 0$ which is impractical or when $\alpha = \beta = 1$ (conventional case). Therefore (12a) can be written as

$$(1 + 2\rho\cos\pi)\lambda = 0 \tag{13}$$

Then the condition will be $\rho = 0.5$ which means $R = \sqrt{2L/C}$ and $\omega_o = 1/\sqrt{LC}$; same as known in the integer order Butterworth filters [15]-[18]. In the next two subsections, we will study (12) in its generic form in the special case when $\beta = \alpha$, and then in general when $\alpha \neq \beta$.

III. EQUAL ORDER FRACTIONAL BUTTERWORTH FILTER

In the case $\beta = \alpha$ for equal orders, equation (9) can be written as

$$|D(j\omega)|^2 = 1 + (LC)^2\omega^{2N\alpha} \tag{14}$$

where N is the number of reactive elements in the circuit and for the case of interest $N = 2$ and $0 \leq \alpha \leq 2$ .Using the definitions of λ and ρ, then (8) and (12) become

$$|D(\rho,\lambda)|^2 = 1 + \rho^2\lambda^{4\alpha} + \lambda^{2\alpha} + \rho\cos(0.5\alpha\pi)\,\lambda^{3\alpha} \\ + 2\rho\cos(\alpha\pi)\,\lambda^{2\alpha} + 2\cos(0.5\alpha\pi)\,\lambda^{\alpha} \tag{15}$$

$$\rho\lambda^{2\alpha} + \left(\frac{1 + 2\rho\cos(\alpha\pi)}{2\cos(0.5\alpha\pi)}\right)\lambda^{\alpha} + 1 = 0 \tag{16}$$

From the above equations, we can deduce that ω_o will be given as follow:

$$\omega_o = \frac{1}{\sqrt[2\alpha]{LC}} \tag{17}$$

Consequently, the cut-off frequency depends on the value of L, C and α, which adds an extra degree of design freedom. From (12b) and (17), the relation between λ and ρ at the cut–off frequency is given by (18) which simplifies (16) into (19) which can be solved as shown in (20). Then according to α the number of roots can be determined as indicated in (20). Three different cases are observed where we obtain no roots, single root, and double roots in the case of $0 < \alpha < 0.5$, $0.5 \leq \alpha < 1.5$, and $1.5 \leq \alpha < 2.0$ respectively as shown in Fig 2.

$$\rho = \frac{1}{\lambda^{2\alpha}} \quad (18)$$

$$\lambda^{2\alpha} + 4\cos(0.5\alpha\pi)\,\lambda^{\alpha} + 2\cos\alpha\pi = 0 \quad (19)$$

$$\lambda^{\alpha} = -2\cos(0.5\alpha\pi) \pm \sqrt{2} \quad (20a)$$

$$\lambda = \begin{cases} 0 & 0 < \alpha < 0.5 \\ \left(-2\cos(0.5\alpha\pi) + \sqrt{2}\right)^{\frac{1}{\alpha}} & 0.5 \le \alpha < 1.5 \\ \left(-2\cos(0.5\alpha\pi) \pm \sqrt{2}\right)^{\frac{1}{\alpha}} & 1.5 \le \alpha < 2.0 \end{cases} \quad (20b)$$

Therefore, if the fractional order α is known, λ can be calculated, then the value of R, L and C can be calculated at a cut-off frequency ω_o using (12b). In spite of (19) gives two values of λ at certain cases, equation (14) gives the same poles for both values of λ. The Butterworth poles of the fractional order can be calculated at the condition of (19) as listed in Table 1, where the poles of the fractional order Butterworth filter moves on a circle like the integer order case [15]-[17]. Also, from table 1 the values of the poles for $\alpha = 1$ *and* 2 are the same for the conventional integer order Butterworth filter of order 2 and 4 [18]. Also, when $\alpha > 0.5$ two conjugate poles (equivalent to single λ) while, when $\alpha > 1.5$ two different conjugate pairs (equivalent to two different values of λ). Therefore, the values in Table 1 validate the results of Fig.2.

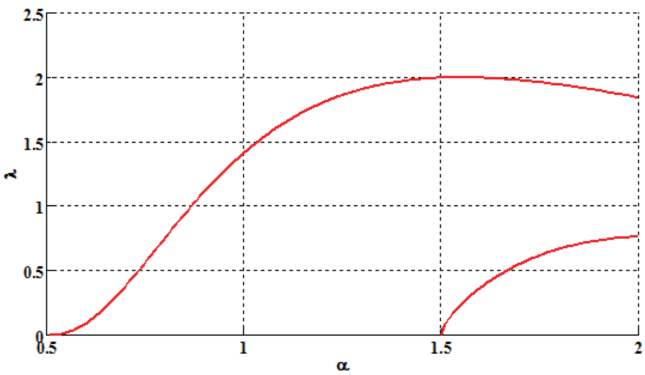

Fig 2 The change in λ with respect to α

Table 1:
The poles location for different values of α.

α	Poles			
0.1	------	------	------	------
0.2	------	------	------	------
0.3	$0.866 \pm 0.5i$	------	------	------
0.4	$0.3827 \pm 0.9239i$	------	------	------
0.5	$\pm 1\,i$	------	------	------
0.6	$-0.2588 \pm 0.9659i$	------	------	------
0.7	$-0.4339 \pm 0.901i$	------	------	------
0.8	$-0.5556 \pm 0.8315i$	$0.9808 \pm 0.1951i$	------	------
0.9	$-0.6428 \pm 0.766i$	$0.866 \pm 0.5i$	------	------
1.0	$-0.7071 \pm 0.7071i$	$0.7071 \pm 0.7071i$	------	------
1.1	$-0.7557 \pm 0.6549i$	$0.5406 \pm 0.8413i$	------	------
1.2	$-0.7934 \pm 0.6088i$	$0.3827 \pm 0.9239i$	------	------
1.3	$-0.823 \pm 0.5681i$	$0.2393 \pm 0.9709i$	$0.993 \pm 0.12i$	------
1.4	$-0.8467 \pm 0.532i$	$0.112 \pm 0.9937i$	$0.944 - 0.33i$	------
1.5	$-0.866 \pm 0.5i$	$\pm 1\,i$	$0.866 \pm 0.5i$	------
1.6	$-0.8819 \pm 0.4714i$	$-0.098 \pm 0.9952i$	$0.773 \pm 0.6344i$	------
1.7	$-0.8952 \pm 0.4457i$	$-0.1837 \pm 0.983i$	$0.6737 \pm 0.739i$	------
1.8	$-0.9063 \pm 0.4226i$	$-0.2588 \pm 0.9659i$	$0.996 \pm 0.087i$	$0.5736 \pm 0.819i$
1.9	$-0.9158 \pm 0.4017i$	$-0.3247 \pm 0.9458i$	$0.476 \pm 0.8795i$	$0.9694 \pm 0.246i$
2.0	$-0.9239 \pm 0.3827i$	$-0.3827 \pm 0.9239i$	$0.383 \pm 0.9239i$	$0.9239 \pm 0.383i$

To complete the filter design, the values of L, C and R need to be calculated. So, from the original definition of λ and the cut-off frequency given in (17), the components value will given by

$$C = \frac{1}{R}\left(\frac{\lambda}{\omega_o}\right)^{\alpha}, L = \frac{1}{C\omega^{2\alpha}} \quad (21)$$

Table 2 gives the value of L and C at $R = 50\Omega$ and at the normalized frequency $\omega_n = 1\,rad/sec$, and by using the frequency transformation the value of L and C can be obtained at any cut-off frequency. From Table 2, it can be noticed that for the fractional orders that have two values of λ they also show two combinations of the values for L and C that satisfy the Butterworth response.

To verify the design procedure of the Butterworth filter, the simulation of (14) is shown in Fig 3 when $(\alpha, \lambda) = (0.7, 0.3781)$, $(1.0, 1.4142)$, $(1.7, 1.9808)$ and $(1.7, 0.5552)$ at $\omega_o = 2\pi * 10000\,rad/sec$, $R = 50\Omega$, and the L and C values are equal ($43mH$, 4.4μF), ($0.56mH$, $0.45\mu F$), (0.1μH, 0.44nF) and (0.95μH, 0.21pF) respectively.

Table 2:
The values of L and C for different α at $R = 50\Omega$ and $\omega_n = 1$ rad/sec

α	L(H)		C(F)	
0.7	98.77		0.01012	
0.8	62.8		0.0159	
0.9	45.4		0.0220	
1	35.36		0.02828	
1.1	28.95		0.03454	
1.2	24.6		0.0406	
1.3	21.53		0.0464	
1.4	19.31		0.0518	
1.5	17.68		0.0567	
1.6	16.49	245.28	0.0606	0.004077
1.7	15.64	135.96	0.0639	0.00736
1.8	15.077	102.48	0.0663	0.009758
1.9	14.75	89.1	0.06779	0.01122
2	14.64	85.35	0.06829	0.011717

It's interesting to note here that the performance of (15) is approximately equal to the performance of the canonical equation of Butterworth given in (14) for $m = 2\alpha$ for different values of α as shown in Fig 3. The magnitude responses of five different cases are shown in Fig. 4. Therefore, the condition of (19) is almost satisfied for the whole bandwidth of interest. The stability study of the system is introduced in Fig.5, which introduces the poles location for different fractional order cases.

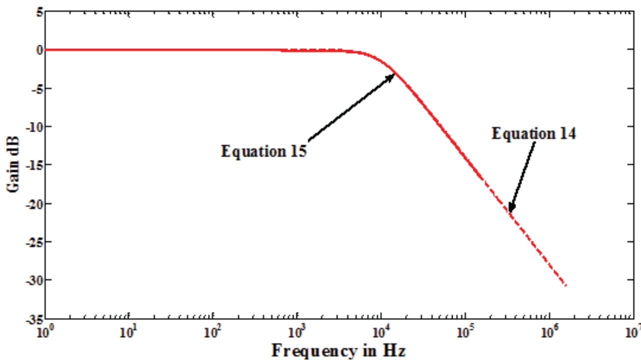

Fig. 3. The magnitude response of equations 14 and 15 for $\alpha = 0.7$

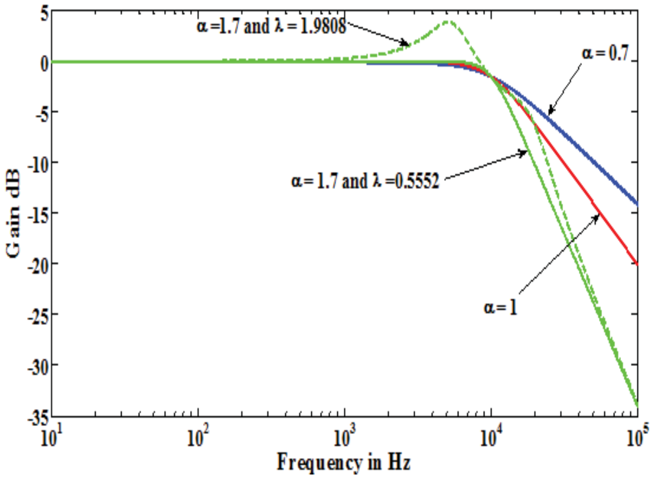

Fig 4. The magnitude response using (14) for different cases.

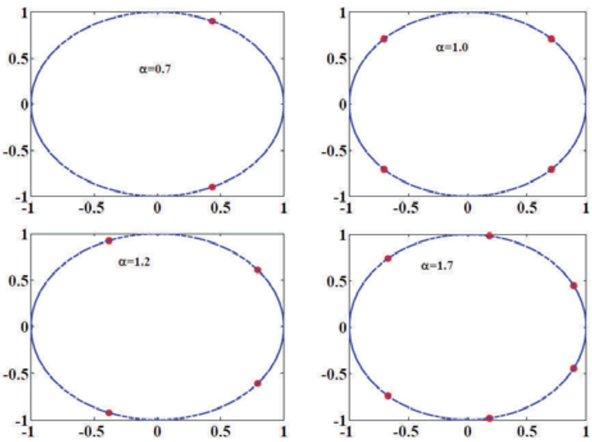

Fig 5. The pole locations for the filters of interest

IV. DIFFERENT ORDER FRACTIONAL BUTTERWORTH FILTER

When $\boldsymbol{\beta \neq \alpha}$, which represents the most general case for Butterworth filter design, as it depends on using two different orders α and β where (8) becomes

$$|D(\rho,\lambda)|^2 = 1 + \rho^2\lambda^{2(\alpha+\beta)} + \lambda^{2\alpha} + 2\rho\cos(0.5\beta\pi)\,\lambda^{2\alpha+\beta} + 2\rho\cos(0.5(\alpha+\beta)\pi)\,\lambda^{\alpha+\beta} + 2\cos(0.5\alpha\pi)\,\lambda^{\alpha} \quad (22)$$

where α and β are any positive real number ($0 \leq \alpha, \beta \leq 2$). The condition and the cut-off frequency are given by (12) and (10) respectively. Then, the relation between λ and ρ is given by (23) which simplifies (12a) into (24) that represents the fractional-order Butterworth filter condition when $\alpha \neq \beta$. It is worthy to note that (24) is similar to (19), and the value of λ^{α} can be obtained as given in (25).

$$\rho = \frac{1}{\lambda^{\alpha+\beta}} \quad (23)$$

$$\lambda^{2\alpha} + 2(\cos(0.5\beta\pi) + \cos(0.5\alpha\pi))\lambda^{\alpha} + 2\cos(0.5(\alpha+\beta)\pi) = 0 \quad (24)$$

$$\lambda^{\alpha} = -(\cos(0.5\beta\pi) + \cos(0.5\alpha\pi)) \pm \sqrt{2 - (\sin(0.5\alpha\pi) - \sin(0.5\beta\pi))^2} \quad (25)$$

Then λ equals zero if $\tan(0.5\alpha\pi)\tan(0.5\beta\pi) = 1$ which means $\alpha + \beta = 1$ *or* 3. Therefore, the previous equation has three cases as follows:

- No solution when $\alpha + \beta < 1$
- Single solution when $1 \leq \alpha + \beta < 3$
- Two solutions if $3 \leq \alpha + \beta$

The dependence of λ on α at specific values for β is shown in Fig. 6, and the values of λ versus β for fixed values of α are shown in Fig. 7. After these analyses, it's important to determine the values of R, L and C at a specific cut-off frequency as follows:

$$C = \frac{1}{R}\left(\frac{\lambda}{\omega_o}\right)^{\alpha}, L = \frac{1}{C\omega^{\beta+\alpha}} \quad (26)$$

To verify the previous analysis of the general Butterworth filter, the frequency responses using (8) and (9) are shown in Figs. 8 and 9 respectively, where the values of L, C and λ for $R = 50\Omega$ and at $\omega_o = 2\pi \times 10000\, rad/sec$ are given in Table 3. It is clear from Fig. 8 that a small difference appears between the two curves due to the additional terms.

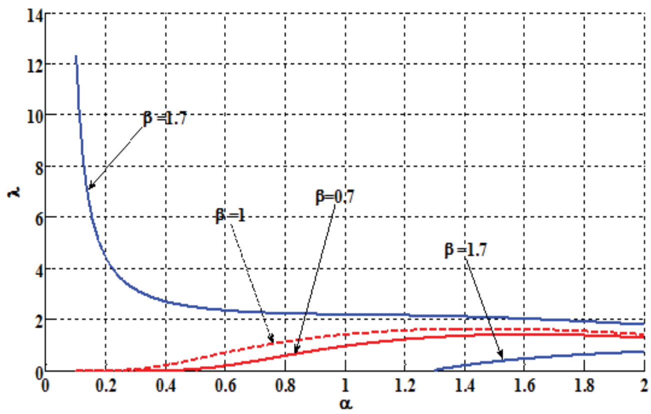

Fig 6. The change in λ with respect to α for specific values of β

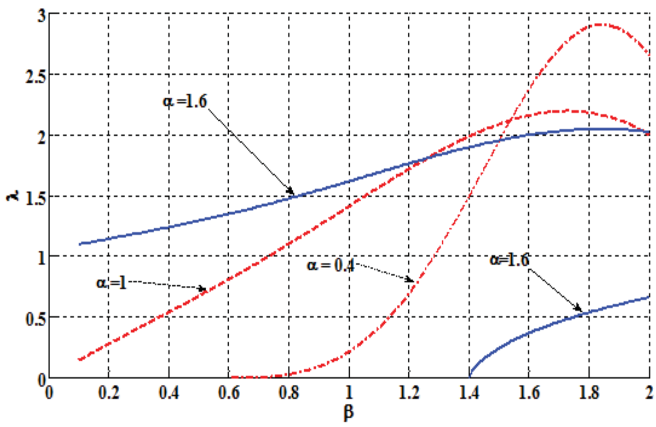

Fig 7. The change in λ with respect to β for specific values of α

Table 3.
The L, C and pole locations used in simulations for $R = 50\Omega$

(α,β)	λ	C	L(H)
(0.1, 1.5)	1.2433	0.0068	3μ
(0.7, 0.9)	0.781	0.868n	0.423μ
(1.2,1.6)	2.135	7n	5.3μ
(1.1,0.7)	1.102534	117n	0.0197

The simulation results for the fractional order Butterworth filter using the parameters given in Table 3 is shown in Fig.9, where the magnitude response suffers from a small attenuation in the pass-band for small values of α but it still flat in the pass-band. On the contrary for $\alpha > 0.5$, the filter gives minimum loss in the pass-band.

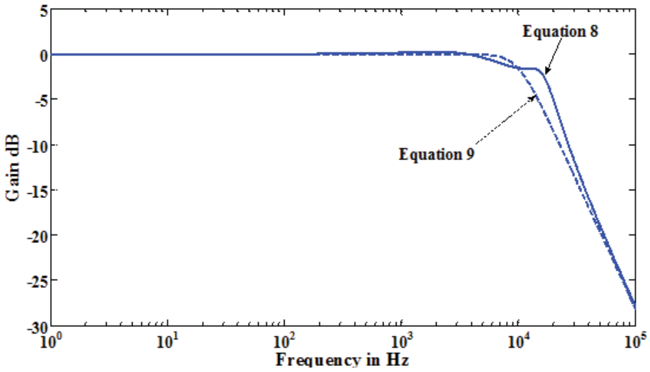

Fig 8. The magnitude responce of equations 8 and 9

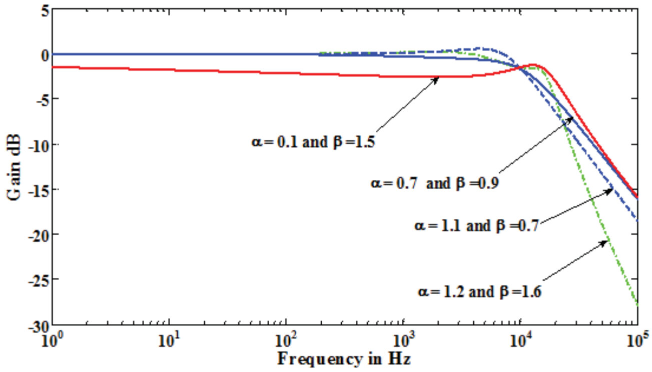

Fig 9. The magnitude response for different values of α and β

V. Conclusion

In this work two procedures to design a fractional order Butterworth filter are presented. The first procedure derived the Butterworth condition and the cut-off frequency for a filter with two fractional elements of the same order. On the other hand, the second method proposed the cut-off frequency and Butterworth condition for a fractional order filter with two fractional elements of different orders. Also, the stability analysis is discussed with the mathematical formulas for each case. Several examples are introduced to validate the theoretical analysis showing that the poles always move on a circle in all cases as known in the conventional case.

References

[1] R. Caponetto, G. Dongola, L. Fortuna, I. Petráš, Fractional Order Systems - Modeling and Control Applications, World Scientific Publishing Co., 2010.

[2] R. P. Agarwal, and D. O'Regen, Ordinary and Partial Differential Equations with Special Functions, Fourier Series and Boundary Value Problems, Springer , New York, 2000.

[3] T.J. Freeborn, B. Maundy, A.S Elwakil, "Field programmable analogue array implementation of fractional step filters," IET Circuits, Devices& Systems, vol. 4, Issue. 6, pp. 514 - 524, 2010.

[4] A. G. Radwan, A. M. Soliman, A.S. Elwakil, A. Sedeek, "On the stability of linear systems with fractional-order elements," Chaos, Solitons and Fractals, vol. 40, pp. 2317 - 2328, 2009.

[5] A. G. Radwan, A.S. Elwakil, A. M. Soliman, "On the Generalization of Second Order Filters to the Fractional Order Domain," J. of Circuits, Systems, and Computers, vol. 18, No 2, pp. 361 - 386, 2009.

[6] A. G. Radwan, A.S. Elwakil, A. M. Soliman, "Fractional – order sinusoidal oscillator: Design procedure and practical examples," IEEE Trans. Circuit and Sys. I, vol. 55, No 7, pp. 2051 - 2063, 2008.

[7] A. G. Radwan, K. N. Salama," Passive and Active Elements Using Fractional $L_\beta C_\alpha$ Circuit," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 58, no. 10, pp. 2388 – 2397, October 2011.

[8] A. G. Radwan, A. Shamim, K. N. Salama, "Theory of fractional order elements based impedance matching networks," IEEE Microwave Wireless Compon. Lett., vol. 21, no. 3, pp. 120–122, 2011.

[9] M. Sugi, Y. Hirano, K. Saito, "Simulation of fractal immittance by analog circuits: An approach to the optimized circuits," IEICE Trans. Fund., vol.E82-A, no. 8, pp.1627 - 1635, AUGUST 1999.

[10] M. Nakagawa, K. Sorimachi, "Basic characteristic of fractance device," IEICE Trans. Fund., vol.E75-A, no 12, pp.1814 -1819, DEC 1992

[11] K. Saito, M. Sugi, "Simulation of power low relaxation by analog circuits: Fractal distribution of relaxation times and non – integer exponents," IEICE Trans. Fund., vol.E76-A, no. 2, pp. 204 - 209, 1993.

[12] B. T. Krishna, K. V. V. S. Reddy, "Active and passive realization of fractance device of order 1/2,"Active and Passive Electronic Components, 2008, doi:10.1155/2008/369421.

[13] A. Djouambi, A. Charef, A. V. Besancon, "Optimal approximation, simulation and analog realization of the fundamental fractional order transfer function," Int. J. Appl. Math. Comput. Sci., vol.17, no 4, pp. 455 - 462, 2007.

[14] A. S. Sedra, K. C. Smith, Microelectronic Circuits, Oxford, NewYork, 1998.

[15] A. S. Sedra, P. O. Brackett, Filter Theory and Design: Active and Passive, John Wiley & Sons, Singapore, 1986.

[16] L. Thede, Practical Analog and Digital Filter Design, Artech. House, 2004.

[17] S. Winder, Analog and Digital Filter Design, Newnes, New York, 2002.

[18] S. Butterworth, "Theory of Filter Amplifier," Experimental Wireless & the Wireless Engineer, vol. 7, pp. 536 -541, October, 1930.

[19] H.G. Hoang, H.D. Tuan, and T.Q. Nguyen, "Analog Flat Filter Design," ICASSP, pp. 3225 – 3228, 2009.

[20] J. Lim, and D. C. Park, "A Modified Chebyshev Bandpass Filter with Attenuation Poles in the Stopband," IEEE Trans. On Microwave Theory and Techniques, vol. 45, no. 6, pp. 898 -904, JUNE 1997.

[21] S. W. Choi, D. Y. Kim, and H. K. Kim, "A Modified Low-pass Filter with Progressively Diminishing Ripples," Analog Integrated Cir. and Signal Proc., vol. 6, pp. 95 -103, 1994.