Now since $\hat{G}$ and G are related by a nonsingular congruence transformation, they have the same rank. Hence the null space of G has dimension one. It may easily be verified that the null space of G is spanned by a basis vector ε which has only $+1$, -1, and 0 elements.[1] Hence

$$Tu = \alpha\varepsilon.$$

However,

$$T = 1 \dotplus P$$

implies

$$|\alpha| = 1.$$

Thus

$$C_{TF} = u'\hat{C}u = u'T'CTu$$
$$= \varepsilon'C\varepsilon \leq \sum_{k=1}^{n} c_k$$

which establishes the result. Q.E.D.

Summary

An admittance matrix G of an $n+1$ port was created by the reactance extraction method applied to an arbitrary minimal realization of an impedance which does not have a zero at infinity. G was then converted to the Foster canonic form by a linear transformation which preserved the input impedance. The result that the total capacitance was minimum for the Foster form then followed from an investigation of the properties of this transformation. The assumptions of minimality and $z(s)$ not having a zero at infinity were utilized to insure the existence of G. It is clear, however, that the result can be immediately extended to any minimal realization in which the pole at infinity of the admittance is realized by a single capacitor in shunt with an impedance which does not have a zero at infinity. Since the Foster realization contains a shunt capacitor of exactly the same value, the inequality remains the same. Furthermore, it is clear that the result extends to many RC active and passive transfer function cases whose realization is reduced to that of a driving-point impedance.

It is certainly of interest to consider generalizations of the result to realizations which are not minimal [1]. There are two situations which complicate this present procedure. First, G may not exist. Secondly, even if G exists, the rank of $\hat{G}$ is arbitrary. Thus the null space of G contains any vector formed by a linear combination of ε vectors. Hence the conclusion that Tu is an ε vector is no longer valid.

Finally, it should be noted that in many applications it may be more appropriate to minimize a linear combination of the total resistance and total capacitance. The case of minimizing the product of the total resistance and the total capacitance has been investigated for RC distributed networks by Protonotarios and Wing [6].

Appendix

Characterization of the Null Space of the Admittance Matrix of an Arbitrary Transformerless Resistor n-Port

It has been shown that the admittance matrix of a transformerless resistor n-port exists if and only if the ports do not form any closed paths [2], [3]. Furthermore, it may be assumed without loss of generality that the n-port forms a connected network.[2] In this case the admittance matrix G is given by [2], [3].

$$G = B'_{11}(B_2 D_r B'_{12})^{-1} B_{11}.$$

Here $[B_{11} B_{12}]$ is a fundamental circuit matrix of the graph containing the resistor edges as well as edges corresponding to the ports (often called the augmented graph); the submatrix B_{11} corresponds to all the edges which are ports, and D_r is the diagonal edge resistance matrix. The validity of the formula may be established by showing that B_{12} has full rank, cf. [3]. This implies that $(B_{12}D_r B'_{12})^{-1}$ is a positive definite matrix. The proof now proceeds as follows.

The positive definiteness of $(B_{12}D_r B'_{12})^{-1}$ and G implies that the null spaces of G and B_{11} are identical. However, B_{11} is totally unimodular,[3] and the basis vectors of the null space of B_{11} may be taken to have only $+1$, -1, and 0 elements [5].

It should be noted that this surprisingly simple result has probably been overlooked because the popular method of obtaining G is not by means of this simple formula but by augmentation and pivotal condensation, cf. [2] and [3] for a complete discussion.

[1] This result is well known for the case where G is a nodal admittance matrix. To the author's knowledge, the result is not documented for the general n-port case. A proof is, therefore, included in the Appendix.

[2] If this is not the case then the admittance matrix is a direct sum of the admittances matrices of each component and the necessary modifications are obvious.

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[3] A matrix is said to be totally unimodular if all of its submatrices have determinant $+1$, -1, or 0.

Synthesis of a Class of Multivariable Positive Real Functions Using Bott–Duffin Technique

Abstract—The applicability of the classical Bott–Duffin synthesis technique to a class of multivariable positive real functions is discussed, and the results arrived at are illustrated by an example.

I. Introduction

The synthesis of rational positive real functions of the complex frequency variable s by the Bott–Duffin procedure is well established in classical realizability theory [1]. The concept of positive real functions of several variables was introduced in 1960 by Ozaki and Kasami [2]. Since then, a considerable amount of literature has appeared on the subject. Very recently, Koga [3] solved the general problem of synthesizing an arbitrarily prescribed $(n \times n)$ positive real matrix of several variables. The classical importance of the Bott–Duffin procedure and the role of positive real functions of several variables in modern network theory motivates the writing of this correspondence. Here, it is shown that a class of multivariable positive real functions can be synthesized by the Bott–Duffin procedure.

II. Multivariable Bott–Duffin Synthesis

This correspondence is based on a generalization of Saito's extension of the well-known Richards' theorem for a single variable [4]. This generalized version of Saito's result [5] is written in the form of Theorem 1.

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Theorem 1

If $f(p_1, p_2, \cdots, p_n)$ is a positive real function of n variables and if a_1 is any positive real number such that $f(a_1, p_2, \cdots, p_n) = K_1$, a constant independent of $p_2, \cdots, p_n$, then

$$R(p_1, p_2, \cdots, p_n) = K_1 \frac{a_1 f(p_1, p_2, \cdots, p_n) - p_1 K_1}{a_1 K_1 - p_1 f(p_1, p_2, \cdots, p_n)} \tag{1}$$

is also a positive real function, and in general

$$\deg_{p_i} R(p_1, p_2, \cdots, p_n) \leq \deg_{p_i} f(p_1, \cdots, p_n), \qquad i = 1, 2, \cdots, n.$$

The proof follows directly from Saito's special case, when $a_1 = 1$, by using the positive real transformation $p_1 = p_1' a_1$ implying that $p_1 = a_1$, when $p_1' = 1$.

In the preceding theorem, if

$$f(a_1, p_2, \cdots, p_n) + f(-a_1, p_2, \cdots, p_n) = 0$$

then

$$\deg_{p_1} R = (\deg_{p_1} f) - 1.$$

From the foregoing theorem, one directly obtains

$$f(p_1, \cdots, p_n) = \frac{1}{\dfrac{1}{R(p_1, p_2, \cdots, p_n)} + \dfrac{p_1}{a_1 K_1}} + \frac{1}{\dfrac{a_1}{K_1 p_1} + \dfrac{R(p_1, p_2, \cdots, p_n)}{K_1^2}}. \tag{2}$$

It can be seen that sufficient conditions for R to be independent of p_1 are

$$f(a_1, p_2, \cdots, p_n) = -f(-a_1, p_2, \cdots, p_n) = K_1 \quad \text{and} \quad \deg_{p_1} f = 1. \tag{3}$$

In that case R can be written as

$$R = f_1(p_2, p_3, \cdots, p_n). \tag{4}$$

Consequently,

$$f(p_1, p_2, \cdots, p_n) = \frac{1}{\dfrac{1}{f_1(p_2, \cdots, p_n)} + \dfrac{p_1}{a_1 K_1}} + \frac{1}{\dfrac{a_1}{K_1 p_1} + \dfrac{1}{f_2(p_2, \cdots, p_n)}} \tag{5}$$

where

$$f_2(p_2, \cdots, p_n) = \frac{K_1^2}{f_1(p_2, \cdots, p_n)}. \tag{6}$$

If $\deg_{p_2} f = 1$, then $\deg_{p_2} f_1$ and $\deg_{p_2} f_2$ are each either 0 or 1. The former case is trivial, and so the case $\deg_{p_2} f_1 = \deg_{p_2} f_2 = 1$ is considered.

Furthermore, if

$$f(p_1, a_2, \cdots, p_n) = -f(p_1, -a_2, \cdots, p_n) = K_1 \tag{7}$$

where a_2 is a positive real number, then it follows that

$$f_j(a_2, p_3, \cdots, p_n) = -f_j(-a_2, p_3, \cdots, p_n) = K_1, \qquad \text{for } j = 1, 2. \tag{8}$$

Consequently, the Bott–Duffin synthesis procedure can be repeated for f_1 and f_2. The procedure can be continued if analogous conditions as in f_1 and f_2 exist in the remaining functions in the Bott–Duffin cycles for f_1 and f_2, respectively. The foregoing result is summarized below as Theorem 2.

Theorem 2

Sufficient conditions for a n-variable positive real function $f(p_1, p_2, \cdots, p_n)$ to be completely realizable by the multivariable Bott–Duffin procedure with successive applications of the extended Richards' theorem, are

1) $\deg_{p_i} f(p_1, p_2, \cdots, p_n) = 1, \qquad i = 1, 2, \cdots, n-1$ (9)

2) $f(p_1, p_2, \cdots, p_n)|_{p_i = a_i} = -f(p_1, p_2, \cdots, p_n)|_{p_i = -a_i} = K_1,$
$i = 1, 2, \cdots, n-1.$ (10)

The preceding procedure is illustrated by an example at the end of this section.

Another set of sufficient conditions for realizability of $f(p_1, p_2, \cdots, p_n)$ by the multivariable Bott–Duffin procedure, which remove the restriction of a fixed K_1 in condition 2) of Theorem 2, is arrived at next.

If the condition in (3) holds, then using (1) and (4),

$$f_1(p_2, p_3, \cdots, p_n) = f(0, p_2, p_3, \cdots, p_n). \tag{11}$$

Therefore, if

$$f(0, a_2, p_3, \cdots, p_n) = -f(0, -a_2, p_3, \cdots, p_n) = K_2 \tag{12}$$

where a_2 is a positive real number, and K_2, a constant, then it follows that

$$f_1(a_2, p_3, \cdots, p_n) = -f_1(-a_2, p_3, \cdots, p_n) = K_2 \tag{13}$$

and

$$f_2(a_2, p_3, \cdots, p_n) = -f_2(-a_2, p_3, \cdots, p_n) = K_2' \tag{14}$$

where

$$K_2' = \frac{K_1^2}{K_2}.$$

Consequently, the Bott–Duffin synthesis procedure can be repeated for f_1 and f_2. The preceding result is summarized below as Theorem 3.

Theorem 3

Sufficient conditions for an n-variable positive real function $f(p_1, p_2, \cdots, p_n)$ to be completely realizable by a multivariable Bott–Duffin procedure with successive applications of the extended Richards' theorem are

1) $\deg_{p_i} f(p_1, p_2, \cdots, p_n) = 1, \qquad i = 1, 2, \cdots, n-1$ (9)

2) $f(a_1, p_2, p_3, \cdots, p_n) = -f(-a_1, p_2, p_3, \cdots, p_n) = K_1$
$f(0, a_2, p_3, \cdots, p_n) = -f(0, -a_2, p_3, \cdots, p_n) = K_2$
$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$
$f(0, \cdots, 0, a_{n-1}, p_n) = -f(0, \cdots, 0, -a_{n-1}, p_n) = K_{n-1}$ (15)

where a_i $(i = 1, 2, \cdots, n-1)$ are positive real numbers and K_j $(j = 1, 2, \cdots, n-1)$ are constants.

Higher degree functions than those satisfying (9) in Theorems 2 and 3 can also be realized by the multivariable Bott–Duffin technique. The following theorem is stated without proof for brevity. The details of the proof can be obtained in [8].

Theorem 4

Sufficient conditions that an n-variable positive real function $Z(p_1, p_2, \cdots, p_n)$ possesses a multivariable Bott–Duffin cycle are

1) there exist a set of numbers $(j\omega_1, p_{20}, \cdots, p_{n0})$ where ω_1 is real and $\operatorname{Re} p_{i0} > 0$ $(i = 2, 3, \cdots, n)$ such that $\operatorname{Re} Z(j\omega_1, p_{20}, \cdots, p_{n0}) = 0$, implying that $Z(j\omega_1, p_2, \cdots, p_n) = jX_1$ where X_1 may be either positive or negative real constant;
2) there exist a positive real constant a_1 such that

   a) $a_1 = \dfrac{\omega_1}{X_1} Z(a_1, p_2, \cdots, p_n)$ if X_1 is positive;

   b) $a_1 = \dfrac{-\omega_1 X_1}{Z(a_1, p_2, \cdots, p_n)}$ if X_1 is negative.

The fact that not all multivariable positive real functions can be realized by the Bott–Duffin technique follows because positive real functions like [7]

$$Z(p_1, p_2) = \frac{p_1 p_2 + 4p_1 + p_2}{9p_1 + p_2 + 1}$$

are unrealizable by the Bott–Duffin procedure.

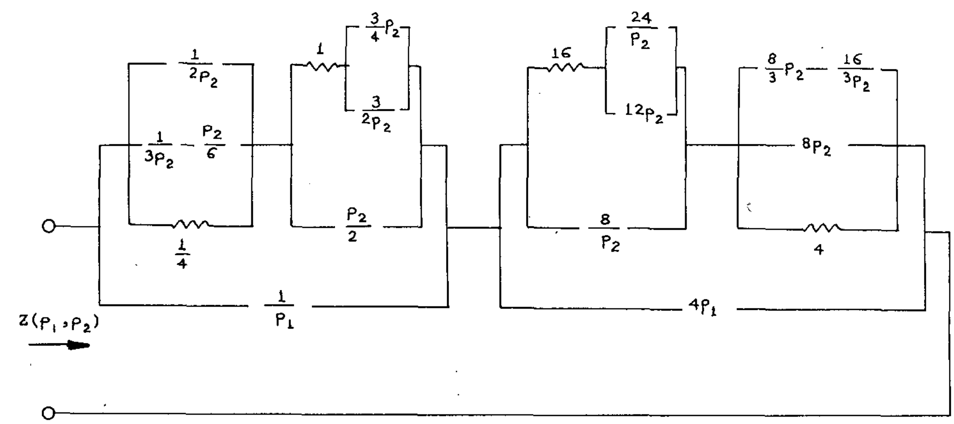

Fig. 1.

Example

Consider the realization of the following two-variable driving-point impedance function

$$Z(p_1, p_2) = \frac{4p_1p_2^2 + 4p_1p_2 + p_2^2 + p_2 + 16p_1 + 1}{p_1p_2^2 + p_1p_2 + p_2^2 + p_2 + p_1 + 4}.$$

It can be seen that

$$Z(\tfrac{1}{2}, p_2) = -Z(-\tfrac{1}{2}, p_2) = 2$$

and

$$\deg_{p_1} Z(p_1, p_2) = 1$$

which thus satisfy the conditions of Theorem 2. $Z(p_1, p_2)$ can be decomposed as

$$Z(p_1, p_2) = \frac{1}{\dfrac{p_2^2 + p_2 + 4}{p_2^2 + p_2 + 1} + p_1} + \frac{1}{\dfrac{p_2^2 + p_2 + 1}{4(p_2^2 + p_2 + 4)} + \dfrac{1}{4p_1}}$$

whence

$$f_1(p_2) = \frac{p_2^2 + p_2 + 1}{p_2^2 + p_2 + 4}; \qquad f_2(p_2) = \frac{4(p_2^2 + p_2 + 4)}{p_2^2 + p_2 + 1}.$$

Using the classical Bott–Duffin procedure, $f_1(p_2)$ and $f_2(p_2)$ can be realized. The complete realization of $Z(p_1, p_2)$ is shown in Fig. 1.

III. Conclusions

It has been shown that if multivariable positive real functions, which are in general difficult to realize, satisfy some conditions, then the classical Bott–Duffin procedure can be utilized to yield network realization of those multivariable positive real function of degree one in all but one variable. It is noted that Theorem 3 is more general than Theorem 2. However, the conditions in Theorem 2 might be easier to test for in many cases. Theorem 4 makes feasible realization of higher degree functions by the Bott–Duffin technique. It is seen, however, that unlike in the single variable case, the scope of multivariable positive real function synthesis using the Bott–Duffin technique is limited.

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Minimal Realizations of Nonminimum Phase Biquadratic *RC* Voltage Transfer Functions

Abstract—Complete solution is given to the problem of realizing, with the minimum possible number of elements, a biquadratic nonminimum phase *RC* voltage transfer function, having zeros anywhere in the *s* plane. All possible equivalent realizations are discussed, and it is shown that there exists a great flexibility in choosing the network elements.

I. Introduction and Scope of the Paper

In a recent communication, Advany and Reddy [1] presented balanced realizations of nonminimum phase *RC* biquadratic voltage transfer functions. These realizations are minimal except in the case of one zero on the negative real axis and the other on the positive real axis. For this particular case, Advani and Reddy were unable to obtain simple networks like those possible in the other cases and they followed the method of Lin and Siskind [2] with slight modification to get a seven-element realization.

The approach adopted in [1] is to guess the network realization, calculate its transfer function, and obtain element values by comparing this with the specified one. In this correspondence, we use a network synthetic approach to the problem and derive minimal realizations for all cases of zero configurations, including the one just mentioned. For each case of zero configuration, all possible equivalent realizations are discussed. Our approach also leads to a great flexibility in the choice of element values for the synthesized networks.

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