Three Fractional-Order-Capacitors-Based Oscillators with Controllable Phase and Frequency

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This paper presents a generalization of six well-known quadrature third-order oscillators into the fractional-order domain. The generalization process involves replacement of three integer-order capacitors with fractional-order ones. The employment of fractional-order capacitors allows a complete tunability of oscillator frequency and phase. The presented oscillators are implemented with three active building blocks which are op-amp, current feedback operational
amplifier (CFOA) and second generation current conveyor (CCII). The general state matrix, oscillation frequency and condition are deduced in terms of the fractional-order parameters. The extra degree of freedom provided by the fractional-order elements increases the design flexibility. Eight special cases including the integer case are illustrated with their numerical discussions. Three different phases are produced with fixed sum of \(2^{k/25}\) which can be completely controlled by fractional-order elements. A general design procedure is introduced to design an oscillator with a specific phase and frequency. Two general design cases are discussed based on exploiting the degrees of freedom introduced by the fractional order to obtain the required design. Spice circuit simulations with experimental results for some special cases are presented to validate the theoretical findings.

**Keywords:** Fractional-order circuits; op-amp; CFOA; CCII; oscillators.

1. **Introduction**

Fractional calculus is the branch of mathematics concerning differentiations and integrations of noninteger orders. In the late sixties, the fractional calculus grabbed the attention of many researchers because of its ability to accurately describe the real dynamical behavior of systems rather than the integer calculus. One of the most frequently used definitions for the general fractional derivatives is the Caputo definition which can be expressed as follows:

\[
D^\alpha_{t_0} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} f^{(m)}(u)(t-u)^{m-\alpha-1} du,
\]

where \(m\) is an integer such that \((m - 1) < \alpha < m\). The fractional derivative gives extra degree of freedom which is the order of the derivative. It increases the flexibility and controllability of any design, moreover it adds new fundamentals. The conventional integer derivatives and integrals are considered special cases of the more general situation introduced by fractional calculus.

The fractional differential equation aroused a considerable interest due to its existence in many applications such as viscoelasticity, electrical circuits, electroanalytical chemistry, fractional multiples, neuron modeling, mechanics, encryption and biological sciences. Recently, many of the numerical techniques were presented to solve the fractional-order differential equations such as the FracPECE subroutine. This method depends on prediction and correction which provide more accurate solution.

The fractional-order capacitors which have the impedances \(Z(s) = ks^{-\alpha}\) are a new concept introduced by the fractional calculus. Although the fractional-order capacitors are not available in markets so far, there exist lots of uprising researches towards their implementation. The RC tree depicted in Fig. 1(a) is used to approximate the fractional capacitor (FC) of order 0.5. This technique was developed in Refs. 21 and 22 for approximating any general \(\alpha\) using the parallel RC ladder structure shown in Fig. 1(b). The fractional-order capacitor was also realized by developing fractal-structures on silicon, while it was fabricated by providing a
thin coating of PMMA film on the electrode surface of a capacitive-type probe as presented in Ref. 24. Another network model of the fractional-order capacitor, built of passive resistors and capacitors, was introduced in Ref. 25 which does not need complicated optimization steps.

A topology based on operational transconductance amplifiers suitable for emulating fractional-order capacitors and inductors using current excitation was presented in Ref. 29. Electronic tuning of the order, capacitance/inductance and bandwidth of operation by modification of only the bias current was proved to be possible.

The aim of this work is to generalize the design of six well-known quadrature third-order oscillators presented in Ref. 30 into the fractional-order domain. The extra degree of freedom provided by the fractional-order parameters allows the control of both phase and frequency. The generalization is done through replacing the integer-order capacitors with fractional-order ones. The general state matrix, characteristic equation, design equations and transfer function (TF) are deduced. Eight special cases are discussed with their numerical analysis. For some applications, it is preferred to design oscillator with a specific phase and frequency. This paper introduces two cases for designing an oscillator with a specific phase and frequency where the fractional order could be tuned to achieve them.

The paper is organized as follows: Section 2 presents the fractional-order oscillation theory with three fractional elements, and different active devices employed in the oscillators. Also, it illustrates oscillators’ structure and numerical discussion. Stability analysis for the system poles is also investigated. Design of oscillator with a specific frequency and phase through a numerical model is presented. Section 3 presents the Spice circuit simulation and experimental results which verify the reliability of the fractional-order oscillator and finally Sec. 4 draws the conclusion.
2. Fractional-Order Oscillators

2.1. Theory of oscillation

Recently, the theorem and basic design procedure of fractional-order sinusoidal oscillators have been introduced mathematically, for $n$-fractance devices.\textsuperscript{10,11} For the special case $n=3$, three fractional elements, the theory states that a linear fractional-order system of the form described by

$$\begin{bmatrix}
D^{\alpha}v_1 \\
D^{\beta}v_2 \\
D^{\gamma}v_3
\end{bmatrix}
= \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= \begin{bmatrix}
A_1v_1 \\
A_2v_2 \\
A_3v_3
\end{bmatrix}$$

sustains sinusoidal oscillations if there exists a value for $\omega$ satisfying the following equation:

$\omega^{\alpha+\beta+\gamma}\cos\left(\frac{(\beta+\alpha+\gamma)\pi}{2}\right)
+ \omega^\beta|A_{\beta}|\cos\left(\frac{\beta\pi}{2}\right)
+ \omega^\alpha|A_{\alpha}|\cos\left(\frac{\alpha\pi}{2}\right)$

$+ \omega^\gamma|A_{\gamma}|\cos\left(\frac{\gamma\pi}{2}\right) - a_{11}\omega^{\beta+\gamma}\cos\left(\frac{(\gamma+\beta)\pi}{2}\right)
- a_{22}\omega^{\alpha+\gamma}\cos\left(\frac{(\alpha+\gamma)\pi}{2}\right)
- a_{33}\omega^{\alpha+\beta}\cos\left(\frac{(\alpha+\beta)\pi}{2}\right) - |A| = 0,$

(3a)

$\omega^{\alpha+\beta+\gamma}\sin\left(\frac{(\beta+\alpha+\gamma)\pi}{2}\right)
+ \omega^\beta|A_{\beta}|\sin\left(\frac{\beta\pi}{2}\right)
+ \omega^\alpha|A_{\alpha}|\sin\left(\frac{\alpha\pi}{2}\right) - a_{11}\omega^{\beta+\gamma}\sin\left(\frac{(\gamma+\beta)\pi}{2}\right)
- a_{22}\omega^{\alpha+\gamma}\sin\left(\frac{(\alpha+\gamma)\pi}{2}\right)
- a_{33}\omega^{\alpha+\beta}\sin\left(\frac{(\alpha+\beta)\pi}{2}\right) = 0,$

(3b)

where $|A_{\alpha}| = a_{22}a_{33} - a_{23}a_{32}$, $|A_{\beta}| = a_{11}a_{33} - a_{13}a_{31}$ and $|A_{\gamma}| = a_{11}a_{22} - a_{12}a_{21}$.

The TF between the states in s-domain is important to find the phase difference between them.

2.2. Different employed active devices

The current feedback operational amplifier (CFOA) was presented\textsuperscript{31} to replace the voltage-mode op-amp. It is a four-port network characterized by the following matrix:

$$\begin{bmatrix}
I_Y \\
V_X \\
I_Z \\
V_O
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
V_Y \\
I_X \\
V_Z \\
I_O
\end{bmatrix},$$

(4)

where $V_X, V_Y, V_Z, V_O, I_X, I_Y, I_Z$ and $I_O$ are the voltages and the currents of $X$, $Y$, $Z$ and $O$-terminals, respectively. The symbol diagram of CFOA is shown in Fig. 2(a).
Another current-mode block is the second generation current conveyor (CCII−) depicted in Fig. 2(b) which was firstly introduced by Sedra.\textsuperscript{32} It is a three-port network with terminal characteristics described by the following matrix:

\[
\begin{bmatrix}
  I_Y \\
  V_X \\
  I_Z \\
\end{bmatrix} =
\begin{bmatrix}
  0 & 0 & 0 \\
  1 & 0 & 0 \\
  0 & -1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  V_Y \\
  I_X \\
  V_Z \\
\end{bmatrix},
\]  

(5)

where \(V_X, V_Y, V_Z, I_X, I_Y\) and \(I_Z\) are the voltages and the currents of \(X\)-, \(Y\)- and \(Z\)-terminals, respectively. The \(Y\)-terminal has high input impedance suitable for input voltage while the \(X\)-terminal has low input impedance suitable for input current. The CFOA and CCII− blocks were used in the implementation of different applications such as filters and oscillator circuits.\textsuperscript{33–36}

2.3. Oscillator design and discussions

The six oscillators depicted in Fig. 3 have the same functionality with different realizations. They are the generalization of the integer-order oscillator presented in Ref. 30, where the conventional ones are third-order quadrature oscillators. The
generalized fractional-order circuits are obtained by replacing the integer-order capacitors with fractional-order ones. The employed fractional capacitors give the designer full control through the order of the capacitor. The employed active building blocks in these oscillators are op-amps, CFOAs and CCIIs, respectively.

The state matrix describing these oscillators is written as

\[
\begin{bmatrix}
D^\alpha v_1 \\
D^\beta v_2 \\
D^\gamma v_3
\end{bmatrix} = \begin{bmatrix}
\frac{-1}{C_1 R_1} & 0 & -1 \\
-1 & \frac{-1}{C_2 R_2} & 0 \\
0 & -1 & \frac{-1}{C_3 R_4}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}. \tag{6}
\]

The characteristic equation that describes the above state matrix is as follows:

\[
s^{\alpha+\beta+\gamma} + \frac{s^{\beta+\gamma}}{C_1 R_1} + \frac{s^{\alpha+\gamma}}{C_2 R_2} + \frac{s^\gamma}{C_1 C_2 R_2 R_4} + \frac{1}{C_1 C_2 C_3 R_3 R_4 R_5} = 0. \tag{7}
\]
Applying the theory of fractional-order oscillators on the above system, there must be a value of $\omega$ that satisfies the following two equations:

\[
\begin{align*}
\omega^{\alpha+\beta+\gamma} \cos \left( \frac{\beta + \alpha + \gamma}{2} \pi \right) &+ \frac{\omega^{\beta+\gamma} \cos \left( \frac{\gamma + \beta}{2} \pi \right)}{C_1 R_1} + \frac{\omega^{\alpha+\gamma} \cos \left( \frac{\alpha + \gamma}{2} \pi \right)}{C_2 R_2} \\
+ \frac{\omega^{\alpha+\beta} \cos \left( \frac{\alpha + \beta}{2} \pi \right)}{C_1 C_2 R_2 R_1} &+ \frac{1}{C_1 C_2 C_3 R_3 R_4 R_5} = 0, \\
\omega^{\alpha+\beta} \sin \left( \frac{\beta + \alpha + \gamma}{2} \pi \right) &+ \frac{\omega^{\beta+\gamma} \sin \left( \frac{\gamma + \beta}{2} \pi \right)}{C_1 R_1} + \frac{\omega^{\alpha+\gamma} \sin \left( \frac{\alpha + \gamma}{2} \pi \right)}{C_2 R_2} \\
+ \frac{\sin \left( \frac{\alpha + \beta}{2} \pi \right)}{C_1 C_2 R_2 R_1} = 0.
\end{align*}
\]

(8a)

The oscillation frequency and condition can be independently controlled by solving (8b) first to get $\omega$ then substituting back in (8a) to calculate the condition of oscillation. It is clear that $R_1, R_2, C_1, C_2$ and the fractional-order parameters affect the radian frequency of oscillation. $R_3, R_4$ or $R_5$ can be chosen to be the design parameter for these oscillators. Note that the value of $R_4$ is selected throughout this work to control the oscillation condition.

To obtain the phase difference between the oscillatory output voltages, it is required to calculate the TF between each two outputs then get the phase difference between them. Table 1 summarizes the TFs between the oscillatory output voltages. From Table 1, the phase is not affected by $R_3, R_4$ or $R_5$. It is completely controlled by the radian frequency, $R_1, R_2, C_1, C_2$ and the fractional-order parameters. The conventional oscillator only produces the quadrature phase with no control over the other two phases. However, this is changed using the fractional-order parameters. The phase difference between the other two oscillatory voltages could be completely manipulated through the fractional order. The relation between three oscillatory phases is constant as demonstrated in Fig. 4(a), where $\phi_{21} + \phi_{13} + \phi_{32} = 2\pi$.

According to Table 1, the quadrature phase (the conventional case) is achieved at

<table>
<thead>
<tr>
<th>TF</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{21} = \frac{s^{\beta}}{v_1}$</td>
<td>$\phi_{21}$</td>
</tr>
<tr>
<td>$T_{32} = \frac{s^{\beta}}{v_1}$</td>
<td>$\phi_{32}$</td>
</tr>
<tr>
<td>$T_{13} = \frac{s^{\beta}}{v_1}$</td>
<td>$\phi_{13}$</td>
</tr>
</tbody>
</table>

Table 1. The TF and phases between oscillator’s states.
\( C_{13} \neq 1 \), from \( C_{30} \). However, when \( C_{13} \neq 1 \), the phase changes linearly with \( C_{13} \) as shown in Fig. 4(b) regardless of any other parameter.

### 2.4. Special cases

Investigation of eight special cases is presented in Table 2 including the conventional case which is now considered as a special case from each one of them. The investigated cases are equal order with \( C_{11} = C_{12} = C_{13} \), (case 1) and mixed structure when one of the capacitors is integer and the other two are fractional ones, (cases 2–7). Consider \( C_1 = C_2 = C \) and \( R_1 = R_2 = R \) for all the studied cases summarized in Table 2. This will make cases 4, 5 give identical responses as cases 6, 7, respectively.

When \( C_{11} = C_{12} = C_{13} \) (case 1), the oscillation frequency equation has a closed-form solution as follows:

\[
\omega^\alpha = \left( \left( -\sin (\alpha \pi) \pm \sin \left( \frac{\alpha \pi}{2} \right) \right) / \left( CR \sin \left( \frac{3\alpha \pi}{2} \right) \right) \right),
\]

where there exists a solution for this case only in the range when \( \left( \frac{2}{3} \right) < \alpha \leq \left( \frac{4}{3} \right) \) as illustrated in Fig. 5(a). When \( \alpha = \beta = \gamma \) (case 1), the phase changes linearly with \( \alpha \) as depicted in Fig. 5(b).

Similarly, for case 2 \(( \gamma = 1, \alpha = \beta)\), there is a frequency of oscillation for specific range of \( \alpha \). From Table 2, the oscillation frequency equation has a closed-form solution described as follows:

\[
\omega^\alpha = \left( \left( -\cos \left( \frac{\alpha \pi}{2} \right) \pm \sin \left( \frac{\alpha \pi}{2} \right) \right) / \left( CR \cos (\alpha \pi) \right) \right),
\]

where there is a solution only for \( 0.5 < \alpha \leq 1.5 \) as shown in Fig. 6(a). The phase is always a constant value and the fractional-order parameter \( \alpha \) does not affect the phase which is due to the fixed sum of the three phases as shown in Fig. 5(b). Only the oscillation frequency makes advantage of the fractional-order parameters for this case. The frequency range reaches higher values when \( \gamma = 1, \alpha \neq \beta \) (case 3) rather than the other special cases as illustrated in Fig. 6(b). It shows two surfaces of oscillation frequency and condition versus \( \alpha \beta \)-plane. Also \( \phi_{32} = \pi/2 \) for this case, while the other two phases are tunable such that \( \phi_{21} + \phi_{13} = 1.5\pi \).
Table 2. Special cases discussion.

<table>
<thead>
<tr>
<th>Case #</th>
<th>Oscillation parameters ((R_i, \omega_{m}))</th>
</tr>
</thead>
</table>
| 1 \(\alpha = \beta = \gamma \neq 1\) | \[R_i = \frac{1}{C_1 C_2 C_3 R_4 \left( \omega^{2\alpha} \cos \left( \frac{3\alpha \pi}{2} \right) + \left( \frac{1}{C_1 R_1} + \frac{1}{C_2 R_2} \right) \omega^{2\beta} \cos \left( \alpha \pi \right) + \frac{\omega^{\alpha} \cos \left( \frac{\alpha \pi}{2} \right)}{C_1 C_2 R_2 R_1} \right)^{-1}} \]
| \(\omega^\alpha = \frac{-\left( \frac{1}{C_1 R_1} + \frac{1}{C_2 R_2} \right) \sin \left( \alpha \pi \right) \pm \sqrt{\left( \frac{1}{C_1 R_1} + \frac{1}{C_2 R_2} \right) \sin \left( \alpha \pi \right)^2 - \frac{4 \sin \left( \frac{3\alpha \pi}{2} \right) \sin \left( \frac{\alpha \pi}{2} \right)}{C_1 C_2 R_2 R_1}}}{2 \sin \left( \frac{3\alpha \pi}{2} \right)} \] |
| 2 \(\gamma = 1\) \(\alpha = \beta\) | \[R_i = \frac{1}{C_1 C_2 C_3 R_4 \left( \omega^{2\beta+1} \sin \left( \alpha \pi \right) + \left( \frac{1}{C_1 R_1} + \frac{1}{C_2 R_2} \right) \omega^{\beta+1} \sin \left( \frac{\alpha \pi}{2} \right) \right)^{-1}} \]
| \(\omega^\alpha = \frac{-\left( \frac{1}{C_1 R_1} + \frac{1}{C_2 R_2} \right) \cos \left( \frac{\alpha \pi}{2} \right) \pm \sqrt{\left( \frac{1}{C_1 R_1} + \frac{1}{C_2 R_2} \right) \cos \left( \frac{\alpha \pi}{2} \right)^2 - \frac{4 \cos \left( \alpha \pi \right)}{C_1 C_2 R_2 R_1}}}{2 \cos \left( \alpha \pi \right)} \] |
| 3 \(\alpha \neq \beta\) | \[R_i = \frac{1}{C_1 C_2 C_3 R_4 \left( \omega^{\alpha+\beta+1} \sin \left( \frac{(\beta + \alpha) \pi}{2} \right) + \frac{\omega^{\beta+1} \sin \left( \frac{\beta \pi}{2} \right)}{C_1 R_1} + \frac{\omega^{\alpha+1} \sin \left( \frac{\alpha \pi}{2} \right)}{C_2 R_2} \right)^{-1}} \]
| \(\omega^{\alpha+\beta+1} \sin \left( \frac{(\beta + \alpha) \pi}{2} \right) + \frac{\omega^{\beta+1} \sin \left( \frac{\beta \pi}{2} \right)}{C_1 R_1} + \frac{\omega^{\alpha+1} \sin \left( \frac{\alpha \pi}{2} \right)}{C_2 R_2} + \frac{1}{C_1 C_2 R_2 R_1} = 0 \] |
| 4 \(\beta = 1\) \(\alpha = \gamma\) | \[R_i = \frac{1}{C_1 C_2 C_3 R_4 \left( \omega^{2\alpha+1} \sin \left( \alpha \pi \right) + \frac{\omega^{\alpha+1} \sin \left( \frac{\alpha \pi}{2} \right)}{C_1 R_1} - \frac{\omega^{2\beta} \cos \left( \alpha \pi \right)}{C_2 R_2} - \frac{\omega^{\alpha} \cos \left( \frac{\alpha \pi}{2} \right)}{C_1 C_2 R_2 R_1} \right)^{-1}} \]
| \(\omega^{\alpha+1} \cos \left( \alpha \pi \right) + \frac{\omega \cos \left( \frac{\alpha \pi}{2} \right)}{C_1 R_1} + \omega^\alpha \sin \left( \alpha \pi \right) + \frac{\sin \left( \frac{\alpha \pi}{2} \right)}{C_1 C_2 R_2 R_1} = 0 \] |
Table 2. (Continued)

<table>
<thead>
<tr>
<th>Case #</th>
<th>Oscillation parameters ((R_4, \omega_v))</th>
</tr>
</thead>
</table>
| 5      | \(\phi \neq \gamma\) | \(R_4 = \frac{1}{C_1 C_2 C_3 R_4 R_5} \left( \omega^{\phi+\gamma+1} \sin \left( \frac{(\alpha + \gamma)\pi}{2} \right) + \frac{\omega^{\phi+\gamma} \sin \left( \frac{\gamma\pi}{2} \right)}{C_1 R_1} - \frac{\omega^{\phi+\gamma} \cos \left( \frac{(\alpha + \gamma)\pi}{2} \right)}{C_2 R_2} - \frac{\omega^\gamma \cos \left( \frac{\gamma\pi}{2} \right)}{C_1 C_2 R_2 R_1} \right) \)  
\(\omega^{\phi+1} \cos \left( \frac{(\gamma + \alpha)\pi}{2} \right) + \frac{\omega \cos \left( \frac{\gamma\pi}{2} \right)}{C_1 R_1} + \frac{\omega^\alpha \sin \left( \frac{(\alpha + \gamma)\pi}{2} \right)}{C_2 R_2} + \frac{\omega^\gamma \sin \left( \frac{\gamma\pi}{2} \right)}{C_1 C_2 R_2 R_1} = 0\)  
| 6      | \(\phi = 1\) \(\beta = \gamma\) | \(R_4 = \frac{1}{C_1 C_2 C_3 R_4 R_5} \left( \omega^{1+\beta+\gamma} \sin \left( \beta\pi \right) - \frac{\omega^{2\beta} \cos \left( \beta\pi \right)}{C_1 R_1} + \frac{\omega^{1+\beta} \sin \left( \frac{\beta\pi}{2} \right)}{C_2 R_2} - \frac{\omega^\beta \cos \left( \frac{\beta\pi}{2} \right)}{C_1 C_2 R_2 R_1} \right) \)  
\(\omega^{1+\beta} \cos \left( \beta\pi \right) + \frac{\omega^\beta \sin \left( \beta\pi \right)}{C_1 R_1} + \frac{\omega \cos \left( \frac{\beta\pi}{2} \right)}{C_2 R_2} + \frac{\omega^\gamma \sin \left( \frac{\gamma\pi}{2} \right)}{C_1 C_2 R_2 R_1} = 0\)  
| 7      | \(\beta \neq \gamma\) | \(R_4 = \frac{1}{C_1 C_2 C_3 R_4 R_5} \left( \omega^{1+\beta+\gamma} \sin \left( \frac{(\gamma + \beta)\pi}{2} \right) - \frac{\omega^{2+\beta} \cos \left( \frac{(\gamma + \beta)\pi}{2} \right)}{C_1 R_1} + \frac{\omega^{1+\beta} \sin \left( \frac{(\gamma + \beta)\pi}{2} \right)}{C_2 R_2} - \frac{\omega^\beta \cos \left( \frac{(\gamma + \beta)\pi}{2} \right)}{C_1 C_2 R_2 R_1} \right) \)  
\(\omega^{1+\beta} \cos \left( \frac{(\gamma + \beta)\pi}{2} \right) + \frac{\omega^\beta \sin \left( \frac{(\gamma + \beta)\pi}{2} \right)}{C_1 R_1} + \frac{\omega \cos \left( \frac{(\gamma + \beta)\pi}{2} \right)}{C_2 R_2} + \frac{\omega^\gamma \sin \left( \frac{(\gamma + \beta)\pi}{2} \right)}{C_1 C_2 R_2 R_1} = 0\)  
| 8      | \(\alpha = \beta = \gamma = 1\) (Integer case) | \(R_4 = \frac{R_2 R_1}{C_3 R_2 R_3 \left( \frac{1}{C_1 R_1} + \frac{1}{C_2 R_2} \right)}, \quad \omega = \sqrt{\frac{1}{C_1 C_2 R_2 R_1}}\)  

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For all the investigated cases, for fixed value of \( C_1 \), the oscillation frequency decays as the fractional order increases while the condition of oscillation \( R_4 \) increases with the fractional-order parameters. The oscillation frequency range is inversely proportional to \( C_1 \) while \( R_4 \) is directly proportional to it. The employment of fractional-order capacitors achieves a wide frequency range from small Hz to thousands of MHz. The effect of \( C_1 \) on the oscillation parameters is similar to the effect of \( R_1 \).

For case 4 (\( \beta = \alpha = \gamma \)), a peak arises in \( \phi_{13} \) curve for \( \alpha < 1 \) then the curve’s behavior changes after the inflection point when \( \alpha = 1 \) as depicted in Fig. 7(a). A color contour map for case 5 (\( \beta = 1, \alpha \neq \gamma \)) is depicted in Fig. 7(b) where a wider range of phase is achieved from \( \phi_{13} \). The phases for cases 6, 7 are similar to cases 4, 5, respectively, with exchanging \( \phi_{21} \) and \( \phi_{13} \).

The fractional-order parameters add extra degrees of freedom which enhance the flexibility of any design. The conventional case provides only one frequency at a time.

Fig. 5. (a) for case 1(\( \alpha = \beta = \gamma \neq 1 \)), the oscillation frequency and condition; and (b) \( \phi_{21} \) and \( \phi_{13} \) for cases 1 and 2.

Fig. 6. Oscillation frequencies and conditions for (a) case 2 (\( \gamma = 1, \alpha = \beta \)) and (b) case 3(\( \gamma = 1, \alpha \neq \beta \)).

Fig. 7. \( \phi_{21} \) and \( \phi_{13} \) versus (a) \( \alpha \) for case 4 and (b) \( \alpha \gamma \)-plane for case 5.
with one quadrature phase and fixed two other phases of $0.75\pi$, so there is no control over phase. Yet that changed with the employment of the fractional order as illustrated in Tables 1 and 2 and Figs. 5–7 where for the same configurations, a wide range of phases is achieved while changing the fractional orders. It depends on the required phase and the orders of the capacitors are chosen.

### 2.5. Stability analysis

Study of the stability of fractional-order systems was investigated in Ref. 37. The study involved a system with characteristic equation as

$$ F(s) = \sum_{n=0}^{k} a_n s^{\frac{n}{m}} = 0, \quad (9) $$

where $n$ and $m$ are integers. The above characteristic equation can be written in the $W$-plane ($W = s^\frac{1}{m}$) as follows:

$$ F(W) = \sum_{n=0}^{k} a_n W^n = 0. \quad (10) $$

According to Ref. 37, the physical $s$-plane transforms into the region $|\phi_m| < \pi/m$ and the region where $|\phi_m| > \pi/m$ is not a physical region which implies that any root in this area of the $W$-plane does not have a corresponding root in the $s$-plane as illustrated in Fig. 8(a). The $\pm j\omega$-axes in the $s$-plane are mapped onto the lines $|\phi_m| = \pi/(2m)$. For the system to oscillate, at least one of the roots lies on the dotted lines $|\phi_m| = \pi/(2m)$ while all the other roots are in the stable region. The system will be unstable if there is at least one root in the region $|\phi_m| < \pi/(2m)$ as illustrated in Fig. 8(a). Recently, the numbers of poles inside the physical and nonphysical planes have been introduced as a function of the system coefficients as well as the fractional-order parameters.

For the system described by (7), conventionally $\alpha = \beta = \gamma = 1$, there are always three poles. However, for the fractional-order system, the number of poles depends on the fractional orders ($\alpha, \beta, \gamma$) and the system parameters. Figure 8(b) shows the system poles in both $s$-plane and $W$-plane for the system described by (7) for different values of $R_4 \in [30 \Omega - 10 k\Omega]$ for the case $\alpha = \gamma = 0.7, \beta = 1$. For low values of $R_4$ the system is unstable, while increasing $R_4$ gradually makes the system pass through the oscillatory state to the stable region. Figure 8(c) tracks the system poles when one order is integer and the other two orders are fractional $\in [0.6–2]$ while fixing all other parameters.

### 2.6. Design for a specific phase

The previous discussion is based on calculating the oscillation frequency and phases for known fractional orders. In this sub section, a design procedure is introduced to
obtain the fractional order for a specific frequency and phase. The value of $\gamma$ can be calculated directly from the phase $\phi_{32}$ where

$$\gamma = \frac{2}{\pi} (\pi - \phi_{32}).$$  \hspace{1cm} (11)

Since the sum of phases should equal to $2\pi$, only one more phase can be controlled. Therefore, the relation between the other fractional orders $\alpha$ and $\beta$ as a function of the other phases is decided independently of the oscillation frequency. Two cases are discussed to have controllable phases as well as the oscillation frequency.

2.6.1. Case 1: $\alpha \neq \beta$

When $\alpha \neq \beta$, in this case the relation will be nonlinear and should be calculated by the nonlinear numerical solution. Given $R_2C_2$, then the fractional order $\beta$ can be
calculated based on the controllable phase $\phi_{21}$ and the oscillation frequency $\omega_{osc}$ independently by solving the following nonlinear equation:

$$\omega_{osc}^\beta \sin \left( \phi_{21} + \beta \frac{\pi}{2} \right) + \frac{1}{R_2 C_2} \sin (\phi_{21}) = 0, \quad \left( \pi - \beta \frac{\pi}{2} \right) < \phi_{21} < \pi.$$  \hfill (12)

Figure 9 depicts how the fractional order $\beta$ can be tuned according to the required oscillation frequency and phase. It shows the surface of $\beta$ versus $\phi_{21} \omega_{osc}$-plane with two different values of $R_2 C_2$.

Similarly, the value of $\alpha$ can be obtained from the nonlinear relation:

$$\omega_{osc}^\alpha \sin \left( \phi_{13} + \alpha \frac{\pi}{2} \right) + \frac{1}{R_1 C_1} \sin (\phi_{13}) = 0, \quad \left( \pi - \alpha \frac{\pi}{2} \right) < \phi_{13} < \pi.$$  \hfill (13)

2.6.2. Case 2: $\alpha = \beta$

The second method considers the special case $\alpha = \beta$. Equations (12) and (13) will be written as

$$C_2 R_2 \omega_{osc}^\alpha = -\frac{\sin (\phi_{21})}{\sin (\phi_{21} + \frac{\alpha \pi}{2})}, \quad \left( \pi - \frac{\alpha \pi}{2} \right) < \phi_{21} < \pi,$$  \hfill (14a)

$$C_1 R_1 \omega_{osc}^\alpha = -\frac{\sin (\phi_{13})}{\sin (\phi_{13} + \frac{\alpha \pi}{2})}, \quad \left( \pi - \frac{\alpha \pi}{2} \right) < \phi_{13} < \pi.$$  \hfill (14b)

Then,

$$\frac{C_1 R_1}{C_2 R_2} = \frac{\sin (\phi_{13}) \sin (\phi_{21} + \frac{\alpha \pi}{2})}{\sin (\phi_{21}) \sin (\phi_{13} + \frac{\alpha \pi}{2})} = \frac{1}{k}.$$  \hfill (15)

Therefore, the value of fractional order $\alpha$ can be obtained analytically by

$$\alpha = \frac{2}{\pi} \tan^{-1} \left( \frac{(k-1) \sin (\phi_{13}) \sin (\phi_{21})}{\cos (\phi_{13}) \sin (\phi_{21}) - k \sin (\phi_{13}) \cos (\phi_{12})} \right),$$

$$\alpha > \max(\pi - \phi_{21}, \pi - \phi_{13}).$$  \hfill (16)
Fig. 10. Design regions versus $\phi_{21}, \phi_{13}$-plane for $k < 1$.

Fig. 11. (a) The fractional order $\alpha$, (b) the fractional-order $\gamma$, (c) the value of $R_1$ for specific oscillation frequency and (d) $R_3 R_4 R_5 C_3$. 
The values of $R_1 C_1$ and $R_2 C_2$ can be calculated from

$$R_1 C_1 = -\frac{\sin(\phi_{13})}{\omega_{\text{osc}} \sin(\phi_{13} + \frac{\pi}{2})}, \quad R_2 C_2 = k R_1 C_1.$$ (17)

Under equal $C$ design $R_2$ and $R_1$ have the same sign with factor $k$ between them. For $k < 1$, Fig. 10 illustrates three regions according to $R_1$’s sign so it is up to the designer to select the region suitable for the required design. The region where $\alpha < 1$, $R_1$ is positive, discrete passive components are needed for both resistances and FCs. This region is bounded down by the vertical asymptote (with respect to $R_1$) at $\phi_{21} = \phi_{13}$ and bounded down by $\tan(\phi_{21})\cot(\phi_{13}) = k$ as depicted in Fig. 10. As $k$ gets closer to 1, the region area shrinks till it becomes only a straight line at $k = 1$, where $\phi_{21} = \phi_{13}$ is independent of the fractional order $\alpha$ as mentioned in the discussion section.

The region where $\alpha < 1$ and $R_1 < 0$, negative resistances are needed to implement both $R_1$ and $R_2$ while using discrete passive component for the FCs. This region starts from above the line $\phi_{21} = \phi_{13}$ as shown in Fig. 10.

The region where $\alpha > 1$ and $R_1 < 0$, active components are needed to realize both the resistors and the FCs. This region is bounded from above by the curve $\tan(\phi_{21})\cot(\phi_{13}) = k$ as illustrated in Fig. 10.

For $k > 1$, the curve $\tan(\phi_{21})\cot(\phi_{13}) = k$ is mirrored around the line $\phi_{21} = \phi_{13}$. For example, let $k = 0.2$, Figs. 11(a) and 11(b) show the values of the fractional orders $\alpha$ and $\gamma$, respectively, versus the $\phi_{21}\phi_{13}$-plane. The special cases when $\alpha = 1$ and $\gamma = 1$ are highlighted. Figures 11(c) and 11(d) show $R_1$ and $R_3 R_4 R_5 C_3$ surfaces versus $\phi_{13}\phi_{21}$ plane for a specific design when $(\omega_{\text{osc}}, C_1) = (10^4, 1.2 \times 10^{-6})$.

3. Simulation and Experimental Results

Since the six circuits have the same functionality, one circuit is chosen to be simulated from each implementation. AD844 macromodel is used to simulate both CCII− and CFOA. For the CCII− based oscillator, the case with $\gamma = 1$, $\alpha = \beta = 0.8$ is simulated with Spice. The simulation parameters are $C_1 = C_2 = C_3 = 1.2 \times 10^{-6}$ F/(rad/s)$^{1-\alpha}$ and $R_1 = R_2 = R_3 = R_5 = 1 \, k\Omega$; $R_4$, $f_0$ and $\phi_{21} = \phi_{13}$ are calculated from Tables 1 and 2 to be $24 \, \Omega$, 1.24 kHz and 135. Figure 12(a) shows simulations’ output waveforms for this oscillator.

For the CFOA-based oscillator, the case with $\beta = 1$, $\alpha = \gamma = 0.7$ is also simulated with Spice. The simulation parameters are chosen to be $C_1 = C_2 = C_3 = 1.2 \times 10^{-6}$ F/(rad/s)$^{1-\alpha}$ and $R_1 = R_2 = 1 \, k\Omega$; $R_3 = R_5 = 0.1 \, k\Omega$. Also $R_4$, $f_0$, $\phi_{21}$ and $\phi_{13}$ are taken to be $3.8 \, k\Omega$, 2.2 kHz, 93.5 and 149.5. Figure 12(b) shows simulations’ output waveforms for this oscillator. For the op-amp-based oscillators, two cases are verified experimentally. The circuits are built on NI ELVIS II series, the kit from National Instruments which is connected to the computer through USB cable to measure the output voltages through the scope. The op-amp is implemented with TL082 as shown in Figs. 13(a) and 14(a).
The employed fractional orders in the first experimented case are $\alpha = 1$, $\beta = 0.8$ and $\gamma = 1$ with circuit components: $R_1 = R_2 = R_3 = R_5 = 10 \, k\Omega$, $C_1 = C_3 = 1 \times 10^{-9}$ and $C_2 = 1 \times 10^{-8}$ $F/(rad/sec)^{(1-\alpha)}$. The value of $R_4$ is calculated according to Table 2 and adjusted to the circuit through the potentiometer to be $2.95 \, k\Omega$ as shown in Fig. 13(a). The frequency of oscillation is calculated to be $19.43 \, kHz$ with the percentage of error as $1.18\%$ over the measured frequency. $\phi_{21}$ and $\phi_{13}$ are calculated to be 141 and 129. The scope shows two outputs at a time as demonstrated in Figs. 13(b)–13(d). The quadrature phase is obviously achieved between $V_2$ and $V_3$ as illustrated in Fig. 13(c). The measured output waveforms are redrawn by Matlab using the Scope data as illustrated in Fig. 13(e).

The second experimented case has the following fractional orders: $\alpha = 1$, $\beta = \gamma = 0.8$, with $R_1 = R_2 = R_3 = R_5 = 10 \, k\Omega$ and $C_1 = C_2 = C_3 = 1 \times 10^{-8}$. The circuit implementation is demonstrated in Fig. 14(a) with also two op-amps. $R_4, f_0, \phi_{21}$ and $\phi_{13}$ are calculated to be $1.84 \, k\Omega$, $9.74 \, kHz$, $152.5$ and $99.5$). The Scope
Fig. 13. (a) The case with, $\alpha = 1$, $\beta \neq \gamma$ circuit implementation; Scope output between (b) $V_1$ and $V_2$, $V_3$ and $V_2$ and (d) $V_4$ and $V_5$; and (e) The output waveforms of the Scope data voltages using Matlab.
Fig. 14. (a) The case with $\alpha = 1$, $\beta = \gamma = 0.8$ circuit implementation; Scope output between (b) $V_1$ and $V_2$, (c) $V_3$ and $V_2$ and (d) $V_1$ and $V_3$; and (e) The output waveforms of the Scope data voltages using Matlab.
outputs are demonstrated in Fig. 14(b)–14(d). The measured output waveforms are redrawn by Matlab using the Scope data as illustrated in Fig. 14(e).

4. Conclusion

In this paper, a generalization of six well-known third-order quadrature oscillators into the fractional-order domain was presented. General characteristic equation, transfer function and design equations were presented in terms of the fractional-order parameters. Eight special cases were investigated numerically. The manipulation with the fractional-order parameters proved to add control on phase and frequency. A design procedure for complete control on phase and frequency through the nonlinear model of equations was introduced. Spice circuit simulation and experimental results were introduced to verify the theoretical findings.

References


