1. INTRODUCTION

Fractional calculus is the branch of mathematics that deals with noninteger orders. The conventional integer derivatives and integrals taught to students in their basic courses of mathematics are special cases of the more general situation where a derivative or integral can be, for example, of the order 0.5. Despite the existence of the fractional calculus contentious ago; it was not commonly involved in any applications until the early 1960s (Oldham and Spanier, 1974; Podlubny, 1999a). It has made lots of contributions in biology and bioimpedance (Freeborn et al., 2017; AboBakr et al., 2017), control (Dimeas et al., 2017; Podlubny, 1999b; Azar et al., 2017a,b), PV modeling (AbdelAty et al., 2016), chaotic systems (Tolba et al., 2017; Soliman et al., 2017; Radwan et al., 2011), signal processing (Tseng, 2007), analog filter design (Said et al., 2016a; Radwan et al., 2009; Freeborn et al., 2012; Soltan et al., 2014, 2015; Radwan and Fouda, 2013), oscillator design (Said et al., 2014a,b, 2015a, 2016b, 2017a,b), and electric engineering (Tsirimokou, 2017; Biswas et al., 2008; Valsa and Vlach, 2013; Nakagawa and Sorimachi, 1992; Saito and Michio, 1993; Michio et al., 1999)

An oscillator is one of the basic building blocks required by many electronic systems such as radars, electronic computers, biological systems, instrumentation, and other electronic devices such as function generators (Sturley, 1955; Kuperman and Zanette, 2009).

The two-port network is basically a network with four terminals such that two ports are arranged (input and output ports). The purpose of a two-port network is to extract the relationships between the input/output currents and...
voltages as a matrix to characterize and analyze any linear two-port electrical systems. The knowledge of the two-port network parameters enables users to treat it as a black box placed within a larger network. It helps dissolve large circuits into smaller ones and deal with each one separately. More fundamentals and new designs were introduced over the last few decades in many applications by using two-port network (Nilsson, 2008).

This chapter aims to study the interdisciplinary nature between the mathematical fundamentals of fractional calculus and the concept of a two-port network in the design of oscillators with their analyses. It presents a prototype of fractional-order two-port network oscillators. The prototype involves two categories: one includes a general two-port network with three impedances and the other with two impedances. The first category includes three different topologies based on general two-port network with input, output, and feedback impedances. It is expected that some designs could not work in the integer case. However, oscillation occurs with the use of fractional-order elements. For the second category, two topologies are presented based on a two-port network with either input or output impedance and feedback impedance. The transmission matrix representation will be used in modeling a general two-port network. With these general parameters, a design parameter is chosen to control the condition of oscillation. Then, the oscillation frequency and phase difference between the two oscillatory outputs are deduced as a general equation function of the transmission matrix elements, the fractional-order parameters, and the extra impedances added to the prototype. Different active two-port networks will be studied with their numerical, Spice simulations, and experimental results. The structure of this chapter is depicted in Fig. 1.

1.1 Fractional-Order Oscillator Theory

Recently, the generalized theorems for the fractional-order oscillators were introduced in Radwan et al. (2008a,b). Some well-known RC oscillators were generalized into the fractional-order domain (Radwan et al., 2008a,b). For example, the classical Wien-bridge sinusoidal oscillator, Colpitts oscillator,
phase-shift oscillator, and the negative resistor RC oscillator were also studied in Radwan et al. (2008b), where conventional capacitors were replaced by fractional-order ones.

The theory states that any system described by Eq. (1b) can sustain oscillations if there exists a solution for \( \omega \) that satisfies Eq. (2) (Radwan et al., 2008a,b).

\[
\begin{bmatrix}
D^\alpha x_1 \\
D^\beta x_2
\end{bmatrix} =
\begin{bmatrix}
-b & c \\
d & -a
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
\]

(1a)

\[s^{\alpha+\beta} + as^\alpha + bs^\beta + |A| = 0,\]

(1b)

where \( |A| \) is the determinant of the matrix.

\[
\omega^{\alpha+\beta} \cos \left( \frac{(\beta + \alpha)\pi}{2} \right) + b\omega^\beta \cos \left( \frac{\beta\pi}{2} \right) + a\omega^\alpha \cos \left( \frac{\alpha\pi}{2} \right) + |A| = 0, \quad (2a)
\]

\[
\omega^\beta \sin \left( \frac{(\beta + \alpha)\pi}{2} \right) + b\omega^{\beta-\alpha} \sin \left( \frac{\beta\pi}{2} \right) + a\sin \left( \frac{\alpha\pi}{2} \right) = 0. \quad (2b)
\]

The phase difference between the two output waveforms is calculated as follows (Radwan et al., 2008a,b):

\[
\varphi = \begin{cases} 
\tan^{-1} \left( \frac{\omega^\alpha \sin \left( \frac{\alpha\pi}{2} \right)}{\omega^\beta \cos \left( \frac{\alpha\pi}{2} \right) + b} \right), & c > 0 \\
\tan^{-1} \left( \frac{\omega^\alpha \sin \left( \frac{\alpha\pi}{2} \right)}{\omega^\beta \cos \left( \frac{\alpha\pi}{2} \right) + b} \right) - \pi, & c < 0
\end{cases}
\]

(3)

The theorem describes oscillators with two fractance devices of fractional orders \( \alpha \) and \( \beta \). The added fractional-order parameters increase the design flexibility to add more constraints such as controlled phase difference, which is hard to achieve in the integer-order systems.

### 1.2 Two-Port Network

The chapter addresses another concept, which is the two-port network (Nilsson, 2008). The two-port network structure depicted in Fig. 2 is a valuable technique in circuit theory as most linear electrical systems can be viewed as a two-port network. It is preferred in analyzing circuits that include transistors such as amplifiers, oscillators, filters, and other systems such as transformers (Nilsson, 2008). It is widely employed in many electrical and communication applications.

![Two-port network](image-url)
Transformation of LC ladder filters using a current feedback operational amplifier as an active block was investigated based on the two-port network concept in Said et al. (2011a). Recently, common mode currents in complex variable-speed drive systems were analyzed using the two-port networks concept (Jettanasen et al., 2008). It was used also to describe two interactions of rhomboidal strain amplifiers (Schultz and Ueda, 2013).

The relationships between the port currents and voltages can be written in a matrix form. The matrix parameters completely describe the network behavior in terms of the voltage and current at each port. These parameters enable the users to treat any network as a black box placed within a larger network. It helps in dividing large circuits into smaller ones and separately dealing with each one.

There exist different matrices to describe the relationship between the voltages and currents such as impedance matrix $Z$, admittance matrix $Y$, hybrid matrix $H$, inverse hybrid matrix $G$, and transmission matrix $A$. The choice of each matrix representation depends on the connection of the two ports. Adding the parameters of the impedance matrix is equivalent to a series connection of two-port networks. Similarly, the parallel connection is equivalent to adding the admittance matrix while the cascaded connection is equivalent to the multiplication of the transmission parameters of each connected network. It is easy to convert from one representation to another, as illustrated in Table 1 (Nilsson, 2008). The transmission parameters, which are well suited for networks with feedback, are used through this research and can be described by:

$$
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix} =
[A]
\begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix},
$$

where $V_1$ and $I_1$ are the input port voltage and current while $V_2$ and $I_2$ are the output port voltage and current, respectively. The two-port network concept allows deriving the expressions, once independent of the internal connection and the complexity of the active device model. Networks can be connected in series, parallel, or cascaded, which facilitates the analysis, especially within larger networks.

The two-port transmission matrix parameters were the key to obtaining the general characteristic equation of a cross-coupled oscillator circuit topology as in Elwakil (2010). In Elwakil and Al-Radhawi (2011), all possibilities that yield a second-order oscillator with a maximum of three capacitors, or two capacitors and one inductor, were found with their analysis using the two-port concept. The general characteristic equations for a two-port network with three and two impedances were derived in Elwakil (2009). It classified the oscillators into three categories: common A, B, and C, according to which terminal is grounded. Two-port network transmission parameters were used in Elwakil and Maundy (2010) to derive expressions for the voltage/current gains and the input/output impedances of common amplifier topologies. The derived expressions are valid both for BJT and MOS-based amplifiers.
### TABLE 1 Two-Port Parameter Conversion

<table>
<thead>
<tr>
<th>Desired Network</th>
<th>Given Network</th>
</tr>
</thead>
<tbody>
<tr>
<td>[A]</td>
<td>[Z]</td>
</tr>
<tr>
<td>$a_{11}$ $a_{12}$</td>
<td>$z_{11}$ $z_{21}$ $\Delta z$ $z_{22}$</td>
</tr>
<tr>
<td>$a_{21}$ $a_{22}$</td>
<td>$-\frac{V_{22}}{Y_{21}}$ $-\frac{1}{Y_{21}}$ $-\frac{1}{y_{11}}$ $-\frac{1}{y_{21}}$ $\frac{-\Delta h}{h_{21}}$ $\frac{-h_{11}}{h_{21}}$</td>
</tr>
<tr>
<td>[Z]</td>
<td>[Y]</td>
</tr>
<tr>
<td>$\frac{a_{11}}{z_{21}}$ $\frac{a_{22}}{z_{21}}$</td>
<td>$\frac{V_{22}}{\Delta y}$ $-\frac{1}{\Delta y}$ $\frac{-Y_{12}}{\Delta y}$ $\frac{-Y_{11}}{\Delta y}$ $\frac{z_{11}}{z_{21}}$ $\frac{z_{22}}{z_{21}}$</td>
</tr>
<tr>
<td>$\frac{a_{22}}{z_{22}}$</td>
<td>$\frac{-\Delta A}{\Delta z}$ $\frac{z_{11}}{\Delta z}$ $\frac{z_{12}}{\Delta z}$</td>
</tr>
<tr>
<td>[Y]</td>
<td>[H]</td>
</tr>
<tr>
<td>$\frac{a_{12}}{y_{12}}$ $\frac{a_{22}}{y_{12}}$</td>
<td>$\frac{1}{Y_{11}}$ $Y_{11}$ $\frac{-Y_{12}}{Y_{21}}$ $\frac{-Y_{11}}{Y_{21}}$ $\frac{h_{11}}{h_{11}}$ $\frac{-h_{12}}{h_{11}}$</td>
</tr>
<tr>
<td>$\frac{a_{22}}{y_{22}}$</td>
<td>$\frac{-\Delta z}{\Delta z}$ $\frac{z_{12}}{\Delta z}$ $\frac{z_{22}}{\Delta z}$</td>
</tr>
<tr>
<td>[H]</td>
<td></td>
</tr>
<tr>
<td>$\frac{a_{12}}{h_{12}}$ $\frac{a_{22}}{h_{22}}$</td>
<td>$\frac{1}{Y_{11}}$ $Y_{11}$ $\frac{-Y_{12}}{Y_{21}}$ $\frac{-Y_{11}}{Y_{21}}$ $\frac{h_{11}}{h_{21}}$ $\frac{-h_{12}}{h_{21}}$</td>
</tr>
</tbody>
</table>

### 2. Prototype of Two-Port Network Three-Impedance Fractional-Order Oscillators

In this section, three classifications of the fractional-order two-port network oscillators based on three impedances are summarized (Said et al., 2014c,d, 2015b, 2016d). The general characteristic equation for each topology is introduced where one topology (Common B) will be studied in detail. For some applications, it is preferred to design-specific phase difference between the two oscillatory outputs as well as the oscillation frequency. This can be achieved easily in the fractional-order domain due to the extra degrees of freedom that enhance the design flexibility.

#### 2.1 General Oscillator Structures

Fig. 3 depicts three different topologies based on general two-port networks with input ($Z_1$), output ($Z_2$), and feedback ($Z_3$) impedances (Elwakil, 2009; Said et al., 2016c). In this chapter, the three impedances are chosen to be two fractional-order capacitors and a resistor. The two-port terminals are labeled as A, B, and C; the name common A, common B, and common C depends on the grounded terminal (Elwakil, 2009), which gives the three topologies shown in Fig. 3.

From Fig. 3A, $I_2 = I_{Z3} - I_{Z2}$, so $I_2$ can be written as:

$$I_2 = \frac{V_1}{Z_3} - V_2 \left( \frac{1}{Z_3} + \frac{1}{Z_2} \right).$$

(5)
FIG. 3 Oscillators topologies: (A) common B topology, (B) common A topology, and (C) common C topology.

With the transmission matrix and Eq. (5), the following can be deduced:

\[
\frac{V_1}{V_2} = \frac{a_{11} + a_{12} \left( \frac{1}{Z_3} + \frac{1}{Z_2} \right)}{1 + \frac{a_{12}}{Z_3}}. \tag{6}
\]

Also, \( I_1 + I_{Z1} + I_{Z3} = 0 \), so; \( I_1 \) can be written as:

\[
I_1 = \frac{V_2}{Z_3} - V_1 \left( \frac{1}{Z_3} + \frac{1}{Z_1} \right). \tag{7}
\]

Considering Eq. (7) and the transmission matrix, another relation between \((V_1, V_2)\) can be written as follows:

\[
\frac{V_1}{V_2} = \frac{a_{12} + |A|}{a_{12} \left( \frac{1}{Z_3} + \frac{1}{Z_1} \right) + a_{22}}. \tag{8}
\]

By eliminating \((V_1, V_2)\), the characteristic equation of this oscillator is obtained generally in terms of the transmission matrix of an unknown two-port network and the three impedances as follows (Said et al., 2016c):

\[
a_{21}Z_1Z_2Z_3 + a_{11}Z_2Z_3 + a_{22}Z_1Z_3 + (a_{11} + a_{22} - 1 - |A|)Z_1Z_2 + a_{12}(Z_1 + Z_2 + Z_3) = 0. \tag{9}
\]
The characteristic equations for Common A and Common C topologies are represented by Eqs. (10), (11), respectively, where the two-port network is assumed not to add any current at node B (Said et al., 2016c).

\[ a_{11}Z_2Z_3 + a_{12}(Z_1 + Z_2 + Z_3) + Z_1Z_3 = 0 \]  \hspace{1cm} (10)

\[ \frac{|A|}{a_{22}}Z_2Z_3 + \frac{a_{12}}{a_{22}}(Z_1 + Z_2 + Z_3) + Z_1Z_3 = 0 \]  \hspace{1cm} (11)

where \( |A| = a_{11}a_{22} - a_{12}a_{21} \) is the determinant of the transmission matrix. This chapter focuses only on Common B topology. Various impedance combinations can be employed to obtain valid oscillation parameters.

### 2.2 Design Procedure for Selected Impedances

In this section, the three impedances are selected to be two fractional-order capacitors and a resistor. General design equations are deduced in terms of the transmission matrix parameters of a general network. The general oscillation frequency, condition, and phase difference between the two oscillatory outputs are also investigated. Table 2 shows the employed impedance combination with the characteristic equation for Common B topology. The characteristic equation for each oscillator is used to generate the oscillation frequency and condition.

From Table 2, case 3 is related to case 1 because: \( (a_{11}, a_{22}, Z_1, Z_2)_{\text{case 1}} = (a_{22}, a_{11}, Z_2, Z_1)_{\text{case 3}} \) (Said et al., 2016c).

For the first case to achieve oscillation, the following two conditions must be satisfied:

<table>
<thead>
<tr>
<th>No.</th>
<th>Impedances</th>
<th>Characteristic Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{s^\alpha C_1} )</td>
<td>( R )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{s^\alpha C_1} )</td>
<td>( \frac{1}{s^\beta C_2} )</td>
</tr>
<tr>
<td>3</td>
<td>( R )</td>
<td>( \frac{1}{s^\alpha C_1} )</td>
</tr>
</tbody>
</table>
\[ \omega^{\alpha + \beta} \cos \left( \frac{(\beta + \alpha)\pi}{2} \right) + \left( \frac{a_{21}}{C_1} + \frac{1}{C_1 R} + \frac{(a_{11} - 1)(1 - a_{22})}{C_1 a_{12}} \right) \omega^\beta \cos \left( \frac{\beta \pi}{2} \right) \]

\[ + \left( \frac{1}{C_2 R} + \frac{a_{11}}{C_2 a_{12}} \right) \omega^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + \frac{a_{22}}{C_1 C_2 R a_{12}} + \frac{a_{21}}{C_1 C_2 a_{12}} = 0, \quad (12a) \]

\[ \omega^\beta \sin \left( \frac{(\beta + \alpha)\pi}{2} \right) + \left( \frac{a_{21}}{C_1} + \frac{1}{C_1 R} + \frac{(a_{11} - 1)(1 - a_{22})}{C_1 a_{12}} \right) \omega^{\beta - \alpha} \sin \left( \frac{\beta \pi}{2} \right) \]

\[ + \left( \frac{1}{C_2 R} + \frac{a_{11}}{C_2 a_{12}} \right) \sin \left( \frac{\alpha \pi}{2} \right) = 0. \quad (12b) \]

If \( R \) is chosen as a design parameter to control the oscillation behavior, then

\[ R = \frac{-1}{\omega^{\beta + \alpha} \cos \left( \frac{(\beta + \alpha)\pi}{2} \right) + \frac{a_{22}}{C_1 a_{12}} + \frac{a_{21} a_{22} + (a_{11} - 1)(1 - a_{22})}{C_1^2 C_2 a_{12}} \omega^\beta \cos \left( \frac{\beta \pi}{2} \right) + \frac{a_{22}}{C_1 C_2 a_{12}} \omega^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + \frac{a_{21}}{C_2 a_{12}} \omega^{\beta - \alpha} \sin \left( \frac{\beta \pi}{2} \right) + \frac{a_{22}}{C_1 a_{12}} \omega^{\beta - \alpha} \sin \left( \frac{\beta \pi}{2} \right)}{\frac{1}{C_1} \omega^\beta \sin \left( \frac{(\beta + \alpha)\pi}{2} \right) + \frac{a_{22} a_{21} + (a_{11} - 1)(1 - a_{22})}{C_1^2 a_{12}} \omega^{\beta - \alpha} \sin \left( \frac{\beta \pi}{2} \right) + \frac{a_{21}}{C_2 a_{12}} \omega^{\beta - \alpha} \sin \left( \frac{\beta \pi}{2} \right)}. \quad (13a) \]

\[ R = \frac{-1}{\frac{1}{C_1} \omega^{\beta - \alpha} \sin \left( \frac{\beta \pi}{2} \right) + \frac{a_{22} a_{21} + (a_{11} - 1)(1 - a_{22})}{C_1^2 a_{12}} \omega^{\beta - \alpha} \sin \left( \frac{\beta \pi}{2} \right) + \frac{a_{21}}{C_2 a_{12}} \omega^{\beta - \alpha} \sin \left( \frac{\beta \pi}{2} \right)}{\frac{1}{C_1} \omega^\beta \sin \left( \frac{(\beta + \alpha)\pi}{2} \right) + \frac{a_{22}}{C_1 a_{12}} + \frac{a_{21} a_{22} + (a_{11} - 1)(1 - a_{22})}{C_1^2 C_2 a_{12}} \omega^\beta \cos \left( \frac{\beta \pi}{2} \right) + \frac{a_{22}}{C_1 C_2 a_{12}} \omega^\alpha \cos \left( \frac{\alpha \pi}{2} \right) + \frac{a_{21}}{C_2 a_{12}} \omega^{\beta - \alpha} \sin \left( \frac{\beta \pi}{2} \right)}. \quad (13b) \]

By eliminating \( R \), the equation that governs the oscillation frequency can be written as follows:

\[ C_2 a_{12} \omega^{2 \beta} \sin \left( \frac{\alpha \pi}{2} \right) + (|A| + 1 - a_{22}) \omega^\beta \sin \left( \frac{(\alpha - \beta)\pi}{2} \right) \]

\[ + a_{22} \omega^\alpha \sin \left( \frac{(\beta + \alpha)\pi}{2} \right) + C_1 a_{12} \omega^{\beta + \alpha} \sin \left( \frac{\beta \pi}{2} \right) + \frac{|A|}{C_2 a_{12}} \sin \left( \frac{\alpha \pi}{2} \right) \]

\[ + \frac{a_{22}(a_{11} + a_{22} - 1 - |A|) - a_{12} a_{21}}{C_1 a_{12}} \omega^{\beta - \alpha} \sin \left( \frac{\beta \pi}{2} \right) = 0. \quad (14) \]

Table 3 shows the general oscillation frequency and condition for the rest of the cases using a similar procedure as in case 1.

### 2.2.1 Numerical Discussion

The first network depicted in Table 4 is a non-ideal gyrator network that contains two elements out of the four elements. Because \( a_{11} = 0 \) and \( a_{22} = 0 \); cases 1 and 3 are identical (Said et al., 2016c). For case 2 to oscillate, \((\alpha + \beta) > 2\), which contradicts the oscillation frequency equation, so it is not a valid oscillator unless negative resistance is used to eliminate the contradiction (Said et al., 2016c). Table 5 depicts the oscillation parameters with different values of \( R_1 \) for the case \( \alpha = \beta \). Increasing \( R_1 \) leads to a decrease in the operating range of the fractional orders for this case, as shown in Table 5. It also shows frequency versus phase, which allows designing for specific phases.

The second network depicted in Table 4 is based on Op-Amp, which contains the four elements of the transmission matrix (Said et al., 2016c). Cases 1 and 2
must have $(\alpha + \beta) > 2$ to achieve oscillations, while case 3 has no conditions (Said et al., 2016c). Table 5 shows oscillation parameter surfaces (frequency and condition) versus the $\alpha - \beta$ plane for case 3.

The third network depicted in Table 4 is based on a second-generation current conveyor (CCII) (Said et al., 2016c; Madian et al., 2006). Case 1 must have $(\alpha + \beta) > 2$ to achieve oscillations while case 2 will not oscillate in cases where $\alpha = \beta$. For case 3, increasing $k$ leads to lowering the obtained frequency range with fixed $\beta = 0.4$ and changing $\alpha$, as illustrated in Table 5. According to Table 5, two phases are achieved for the same frequency and the same $\beta$ with different $\alpha$ for $k = 1.1$.

### 2.2.2 Simulation Results

Fig. 4 shows different structures for case 3 oscillators using the three studied networks. The simulations were done using Spice with an AD844 macro model to simulate the CCII and Op-Amp. Equal C design is chosen to be simulated over all cases with $C_1 = C_2 = 1.2 \times 10^{-6}$ and $R$ is calculated for each simulated case.
### TABLE 4 Examples for Different Two-Port Networks

<table>
<thead>
<tr>
<th>Circuit</th>
<th>Transmission Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gyrator network</strong></td>
<td>![Circuit Diagram]</td>
</tr>
<tr>
<td>[ \begin{bmatrix} I_1 \ V_1 \end{bmatrix} ] + \begin{bmatrix} V_2 \ I_2 \end{bmatrix} + \begin{bmatrix} V_1 \ I_1 \end{bmatrix} = \begin{bmatrix} 0 &amp; \frac{1}{R_1} \ \frac{1}{R_2} &amp; 0 \end{bmatrix} \begin{bmatrix} V_2 \ -I_2 \end{bmatrix} ]</td>
<td></td>
</tr>
</tbody>
</table>

| **Op-Amp network** | ![Circuit Diagram] |
| \[ \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} k & kR_3 \\ \frac{k-1}{R_4} & \frac{k-1R_1}{R_4} \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}, \quad k = \frac{R_1}{R_1+R_2} \] |

| **CCII network** | ![Circuit Diagram] |
| \[ \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ -\frac{1}{R_2} & -k \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}, \quad k = 1 + \frac{R_1}{R_2} \] |

For the gyrator network oscillator depicted in Fig. 4A, the simulation parameters are selected to be \( R_1 = 0.2 \, \text{k}\Omega, R_2 = 1 \, \text{k}\Omega, \) and \( \alpha = \beta = 0.7. \) Fig. 4B shows the simulated output waveforms with oscillation frequency = 7.2 kHz.

For the Op-Amp-based network oscillator depicted in Fig. 4C, the simulation parameters are selected to be \( R_1 = R_2 = R_3 = R_4 = 1 \, \text{k}\Omega, \beta = 0.8, \) and \( \alpha = 0.7. \) Fig. 4D shows the simulated output waveforms with frequency = 819 Hz.

For the CCII network oscillator depicted in Fig. 4E, the simulation parameters are selected to be \( k = 1.1 \) with \( \beta = 0.4 \) and \( \alpha = 0.8. \) Fig. 4F shows the simulated output waveforms measured at the CCII output and input terminals with frequency = 16 kHz.

### 3. PROTOTYPE OF TWO-PORT NETWORK TWO-IMPEDANCE FRACTIONAL-ORDER OSCILLATORS

This section presents a generalized design procedure for two derived topologies of fractional-order oscillators (Said et al., 2016d).
### TABLE 5 Numerical Simulations for Different Cases

<table>
<thead>
<tr>
<th>Gyrator network</th>
<th>Op-Amp network</th>
<th>CCII network</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = \beta$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_2 = 200 \Omega$</td>
<td>$R_2 = 2k \Omega$</td>
<td></td>
</tr>
</tbody>
</table>

3.1 Oscillator Structure

Fig. 5 shows two topologies based on a general two-port network and two impedances. The presented topologies are derived from Common B topology (Said et al., 2016d). They contain a feedback impedance with either input or output impedances. The two impedances are selected to be two fractional-order elements (capacitors and inductors). Four combinations for each topology are extracted corresponding to eight possible oscillators. For Fig. 5A (topology D), $I_1 = I_{Z2}$, so $I_1$ can be written as $I_1 = ((V_2 - V_1)/Z_2$ and $I_1 + I_{Z1} + I_2 = 0$, that is, $I_2 = -(V_2/Z_1) + ((V_1 - V_2))/Z_2)$. According to these equations and the transmission matrix representation, the relation between $V_1$ and $V_2$ can be expressed by:

\[
\frac{V_1}{V_2} = \frac{a_{21} + a_{11}a_{22}}{a_{12} + \frac{1}{Z_2}},
\]

\[
\frac{V_1}{V_2} = \frac{a_{11} + a_{12} \left(\frac{1}{Z_2} + \frac{1}{Z_1}\right)}{1 + \frac{a_{12}}{Z_2}}.
\]

By eliminating $\frac{V_1}{V_2}$, the general characteristic equation of topology D can be written as follows (Said et al., 2016d):

\[
((a_{11} - 1)(1 - a_{22}) + a_{12}a_{21})Z_1 + a_{22}Z_2 + a_{21}Z_1Z_2 + a_{12} = 0.
\]
FIG. 4 Common B topology oscillators case 3 with (A) Gyrator-based, (B) Spice simulations, (C) Op-Amp-based, (D) Spice simulations, (E) CCII-based, and (F) Spice simulations.

FIG. 5 Derived family of oscillators: (A) topology D and (B) topology E.

With similar analysis, the general characteristic equation of topology E can be written as (Said et al., 2016d):

\[ ((a_{11})(a_{22} - 1) + a_{12}a_{21})Z_1 + a_{11}Z_2 + a_{21}Z_1Z_2 + a_{12} = 0. \]  

(17)
By observation, the two characteristic equations are practically the same with exchanging \(a_{11}\) with \(a_{22}\), so throughout the chapter topology D is studied and the other topology can be obtained through the exchange. Eqs. (16), (17) can be obtained from Eq. (9) by taking the limits of \(Z_1\) or \(Z_2\) tends to infinity.

### 3.2 Discussion and Examples

Table 6 shows the impedance combination with the characteristic equation of the Topology D oscillator.

To obtain the oscillation frequency and condition for each case, the theory of the fractional-order oscillator is applied on the four general cases and a design parameter is chosen to control the oscillation behavior. For example, \(a_{12}\), the general element of the two-port network, is chosen throughout this section to be the design parameter.

For the first case:

\[
a_{12} = -\frac{(1 - a_{22})(a_{11} - 1)C_2\omega^\beta \cos \left(\frac{\beta \pi}{2}\right) + a_{22}C_1\omega^\alpha \cos \left(\frac{\alpha \pi}{2}\right) + a_{21}}{C_1C_2\omega^\alpha + \frac{\beta \pi}{2} \cos \left(\frac{\beta + \alpha \pi}{2}\right) + a_{21}C_2\omega^\beta \cos \left(\frac{\beta \pi}{2}\right)},
\]

(18a)

\[
a_{12} = -\frac{(1 - a_{22})(a_{11} - 1)C_2\omega^\beta \sin \left(\frac{\beta \pi}{2}\right) + a_{22}C_1\omega^\alpha \sin \left(\frac{\alpha \pi}{2}\right)}{C_1C_2\omega^\alpha + \frac{\beta \pi}{2} \sin \left(\frac{\beta + \alpha \pi}{2}\right) + a_{21}C_2\omega^\beta \sin \left(\frac{\beta \pi}{2}\right)}.
\]

(18b)

By eliminating \(a_{12}\), the equation governing the oscillation frequency is written as follows:

<table>
<thead>
<tr>
<th>TABLE 6 General Characteristic Equation for Topology D</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Impedances</strong></td>
</tr>
<tr>
<td>No.</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>
(1 - a_{22})(a_{11} - 1)C_2\omega^{\alpha + \beta} \sin\left(\frac{\alpha \pi}{2}\right) + a_{22}C_1\omega^{2\alpha} \sin\left(\frac{\beta \pi}{2}\right)
+ a_{21}\omega^\alpha \left(a_{22} \sin\left(\frac{\beta - \alpha \pi}{2}\right) + \sin\left(\frac{(\beta + \alpha) \pi}{2}\right)\right) + \frac{a_{21}^2}{C_1} \sin\left(\frac{\beta \pi}{2}\right) = 0.

(19)

The phase difference between the two oscillatory outputs for this combination is described by:

\[\varphi = \begin{cases} 
tan^{-1}\left(\frac{a_{12}C_1\omega^\alpha \sin\left(\frac{\alpha \pi}{2}\right)}{a_{12}C_1\omega^\alpha \cos\left(\frac{\alpha \pi}{2}\right) + a_{11} + a_{22} - |A| - 1}\right) & \text{if } \frac{a_{22} - 1}{a_{12}C_1} > 0 \\
\tan^{-1}\left(\frac{a_{12}C_1\omega^\alpha \sin\left(\frac{\alpha \pi}{2}\right)}{a_{12}C_1\omega^\alpha \cos\left(\frac{\alpha \pi}{2}\right) + a_{11} + a_{22} - |A| - 1}\right) - \pi & \text{if } \frac{a_{22} - 1}{a_{12}C_1} < 0
\end{cases}.

(20)

Similarly, the general oscillation parameters and phase difference for the remaining three combinations are illustrated in Table 7. The oscillation frequency and condition of topology E could be deduced through the exchanging illustrated before \(a_{22}\) is replaced by \(a_{11}\). To calculate the phase difference for topology E, exchange \(a_{22}\) with \(a_{11}\) and 1 with \(|A|\) in topology D equations.

In this section, two different networks are presented to verify the theoretical concept where \(a_{12}\) is chosen to be both dependent and independent on other circuit parameters.

### 3.2.1 \(a_{12}\) Independent of Other Matrix Parameters

The first network is the gyrator network depicted in Table 4. With \(a_{11} = 0\) and \(a_{22} = 0\), the two presented topologies would give identical frequency responses with a phase difference equal to \(\pi\) (Said et al., 2016d). Only cases 1 and 4 would produce oscillations (Said et al., 2016d). With \(a_{12}\) independent of \(a_{21}\). Table 7 can be used directly to deduce the oscillation parameters. Equal \(C\) and equal \(L\) design are employed throughout the following numerical discussion illustrated in Fig. 6 for cases 1 and 4 for the two presented topologies. It shows the surfaces of the oscillation frequency and condition versus the \(\alpha - \beta\) plane. The obtained frequency band is from small hertz to hundreds of megahertz, according to the selection of the fractional orders.

### 3.2.2 \(a_{12}\) Dependent on Other Matrix Parameters

The second investigated network is shown in Fig. 7 with a transmission matrix described by (Said et al., 2016d):

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix}
k & kR_3 \\
kR_3 & kR_3/kR_2(1+k)
\end{bmatrix} \begin{bmatrix}
V_2 \\
-I_2
\end{bmatrix}.
\]

(21)

where \(k = \frac{R_1}{R_2}\), and it could be greater than or less than 1 according to the resistance configurations. \(a_{12}\) is dependent on \(R_3\) and \(k\), so. Table 7 cannot be directly used to deduce the oscillation frequency equation. The dependency on other transmission matrix parameters must be eliminated first. For example, if \(R_3\) is selected to be the condition controller, substituting Eq. (21) into Eq. (18)
### TABLE 7 General Oscillation Parameters and Phase Difference for Topology D

<table>
<thead>
<tr>
<th>No.</th>
<th>Oscillation Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a_{12}$</td>
</tr>
<tr>
<td></td>
<td>$\omega_{osc}$</td>
</tr>
<tr>
<td></td>
<td>$\phi$</td>
</tr>
<tr>
<td></td>
<td>$a_{12} \omega a^\alpha + \beta \cos \left( \frac{(\alpha + \beta) \pi}{2} \right) + \frac{a_{21} \omega a^\beta \cos \left( \frac{\beta \pi}{2} \right)}{1 + a_{21} \omega a^\alpha}$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{a_{22} C_1 \omega a^\alpha \sin \left( \frac{(\alpha + \beta) \pi}{2} \right) + (1 - a_{22}) (a_{11} - 1) C_1 \omega a^\alpha \sin \left( \frac{\beta \pi}{2} \right)}{C_1 \omega a^\alpha \sin \left( \frac{\beta \pi}{2} \right)}$</td>
</tr>
<tr>
<td></td>
<td>$= \frac{a_{12}}{a_{22} C_1}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{a_{12}}{a_{22} C_1} &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$\frac{a_{12}}{a_{22} C_1} &lt; 0$</td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a_{12}$</td>
</tr>
<tr>
<td></td>
<td>$\omega_{osc}$</td>
</tr>
<tr>
<td></td>
<td>$\phi$</td>
</tr>
<tr>
<td></td>
<td>$a_{12} \omega a^\alpha + \beta \sin \left( \frac{\beta \pi}{2} \right) + \frac{a_{21} \omega a^\beta \sin \left( \frac{\beta \pi}{2} \right)}{1 + a_{21} \omega a^\alpha}$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{a_{22} C_1 \omega a^\alpha \sin \left( \frac{(\alpha + \beta) \pi}{2} \right) + (1 - a_{22}) (a_{11} - 1) C_1 \omega a^\alpha \sin \left( \frac{\beta \pi}{2} \right)}{C_1 \omega a^\alpha \sin \left( \frac{\beta \pi}{2} \right)}$</td>
</tr>
<tr>
<td></td>
<td>$= \frac{a_{12}}{a_{22} C_1}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{a_{12}}{a_{22} C_1} &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$\frac{a_{12}}{a_{22} C_1} &lt; 0$</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a_{12}$</td>
</tr>
<tr>
<td></td>
<td>$\omega_{osc}$</td>
</tr>
<tr>
<td></td>
<td>$\phi$</td>
</tr>
<tr>
<td></td>
<td>$a_{12} \omega a^\alpha + \beta \sin \left( \frac{\beta \pi}{2} \right) + \frac{a_{21} \omega a^\beta \sin \left( \frac{\beta \pi}{2} \right)}{1 + a_{21} \omega a^\alpha}$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{a_{22} C_1 \omega a^\alpha \sin \left( \frac{(\alpha + \beta) \pi}{2} \right) + (1 - a_{22}) (a_{11} - 1) C_1 \omega a^\alpha \sin \left( \frac{\beta \pi}{2} \right)}{C_1 \omega a^\alpha \sin \left( \frac{\beta \pi}{2} \right)}$</td>
</tr>
<tr>
<td></td>
<td>$= \frac{a_{12}}{a_{22} C_1}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{a_{12}}{a_{22} C_1} &gt; 0$</td>
</tr>
<tr>
<td></td>
<td>$\frac{a_{12}}{a_{22} C_1} &lt; 0$</td>
</tr>
</tbody>
</table>
is a must to eliminate the dependency. Then the following could be deduced for topology D case 1:

\[ kR_3 = -\left(1 - \frac{k}{R_2(1+k)}\right) (k - 1)C_2\omega^\beta \cos\left(\frac{\beta\pi}{2}\right) + \frac{k}{R_2(1+k)} C_1\omega^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + \frac{k}{R_2(1+k)} C_1C_2\omega^{\alpha+\beta} \cos\left(\frac{(\beta+\alpha)\pi}{2}\right) \]

(22a)

\[ kR_3 = -\left(1 - \frac{k}{R_2(1+k)}\right) (k - 1)C_2\omega^\beta \sin\left(\frac{\beta\pi}{2}\right) + \frac{k}{R_2(1+k)} C_1\omega^\alpha \sin\left(\frac{\alpha\pi}{2}\right) + \frac{k}{R_2(1+k)} C_1C_2\omega^{\alpha+\beta} \sin\left(\frac{(\beta+\alpha)\pi}{2}\right) \]

(22b)

By eliminating \( R_3 \) from Eq. (22); the equation that governs oscillation frequency could be deduced, as illustrated in Table 8 for topology D. A similar procedure can be done for topology E. The phase difference between the two oscillatory outputs can be calculated based on the general cases presented in Table 7. For case 1, the two topologies produce oscillations if \( k < 1 \) (Said et al., 2016d), as illustrated in Fig. 8A. It shows the oscillation frequency, condition, and
TABLE 8 Oscillation Parameters for Topology D With Op-Amp Network

<table>
<thead>
<tr>
<th>( R_3 )</th>
<th>( \omega_{\text{osc}} )</th>
<th>( \Phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{-kC_1C_2ω^2 \sin \left( \frac{βπ}{2} \right)}{kC_1C_2ω^2 + β \sin \left( \frac{(β+α)π}{2} \right) + \frac{kC_1L_1}{(R_2^2(1+κ))^2} \sin \frac{απ}{2} + \frac{kC_2L_2}{(R_2^2(1+κ))^2} \sin \frac{βπ}{2}} )</td>
<td>( \tan^{-1} \left( \frac{kC_1C_2ω^2 \sin \left( \frac{απ}{2} \right)}{kC_1C_2ω^2 + β \sin \left( \frac{(β+α)π}{2} \right) + \frac{kC_1L_1}{(R_2^2(1+κ))^2} \sin \frac{απ}{2} + \frac{kC_2L_2}{(R_2^2(1+κ))^2} \sin \frac{βπ}{2}} \right) = \pi )</td>
</tr>
</tbody>
</table>

| 2 | \( \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 + β \sin \left( \frac{βπ}{2}\right) + \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 \sin \left( \frac{(β+α)π}{2} \right) + \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 \sin \left( \frac{απ}{2} \right) + \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 \sin \left( \frac{βπ}{2} \right) \) | \( \begin{align*} \Phi &= \tan^{-1} \left( \frac{\omega^2 \sin \left( \frac{απ}{2} \right)}{\omega^2 \cos \left( \frac{απ}{2} \right) + \frac{k}{(k^2-1)R_2^2 + kr R_2^2L_1} \sin \left( \frac{βπ}{2} \right)} \right) > 0 \\
\Phi &= \tan^{-1} \left( \frac{\omega^2 \sin \left( \frac{απ}{2} \right)}{\omega^2 \cos \left( \frac{απ}{2} \right) + \frac{k}{(k^2-1)R_2^2 + kr R_2^2L_1} \sin \left( \frac{βπ}{2} \right)} \right) < 0 \end{align*} \) |

| 3 | \( \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 + β \sin \left( \frac{βπ}{2}\right) + \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 \sin \left( \frac{(β+α)π}{2} \right) + \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 \sin \left( \frac{απ}{2} \right) + \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 \sin \left( \frac{βπ}{2} \right) \) | \( \begin{align*} \Phi &= \tan^{-1} \left( \frac{\omega^2 \sin \left( \frac{απ}{2} \right)}{\omega^2 \cos \left( \frac{απ}{2} \right) + \frac{k}{(k^2-1)R_2^2 + kr R_2^2L_1} \sin \left( \frac{βπ}{2} \right)} \right) > 0 \\
\Phi &= \tan^{-1} \left( \frac{\omega^2 \sin \left( \frac{απ}{2} \right)}{\omega^2 \cos \left( \frac{απ}{2} \right) + \frac{k}{(k^2-1)R_2^2 + kr R_2^2L_1} \sin \left( \frac{βπ}{2} \right)} \right) < 0 \end{align*} \) |

| 4 | \( \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 + β \sin \left( \frac{βπ}{2}\right) + \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 \sin \left( \frac{(β+α)π}{2} \right) + \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 \sin \left( \frac{απ}{2} \right) + \frac{kC_1L_1}{R_2^2(1+κ)^2} \omega^2 \sin \left( \frac{βπ}{2} \right) \) | \( \begin{align*} \Phi &= \tan^{-1} \left( \frac{\omega^2 \sin \left( \frac{απ}{2} \right)}{\omega^2 \cos \left( \frac{απ}{2} \right) + \frac{k}{(k^2-1)R_2^2 + kr R_2^2L_1} \sin \left( \frac{βπ}{2} \right)} \right) > 0 \\
\Phi &= \tan^{-1} \left( \frac{\omega^2 \sin \left( \frac{απ}{2} \right)}{\omega^2 \cos \left( \frac{απ}{2} \right) + \frac{k}{(k^2-1)R_2^2 + kr R_2^2L_1} \sin \left( \frac{βπ}{2} \right)} \right) < 0 \end{align*} \) |
phase for the special cases $\beta = 1$, where the obtained frequency range is proportional to $k$.

For cases 2 and 3, topology D achieves oscillation with conditions such that $k < 1$, and $(\alpha + \beta) > 2$, while topology E achieves oscillation for $k > 1$ and $(\alpha + \beta) > 2$ (Said et al., 2016d). For case 2, $\beta$ must be greater than $\alpha$ for a working condition of oscillation and the reverse for case 3 (Said et al., 2016d).

For case 4, both topologies achieve oscillation with $k < 1$. Fig. 8B illustrates the oscillation frequency, condition, and phase for the special cases $\beta = 1$ for different values of $k$.

### 3.3 Simulation Results

For the gyrator network depicted in Fig. 9A, case 1 from topology E is chosen to be simulated with fractional orders, $\alpha = 0.7$ and $\beta = 0.8$. The simulation parameters are $C_2 = C_1 = 1.2 \times 10^{-6}, R_2 = 1 \, \text{k}\Omega$ and the design parameter $R_1$ is calculated to be 620 $\Omega$. The output waveforms are depicted in Fig. 9B with oscillation frequency equal 1.8 kHz.

For the Op-Amp-based network depicted in Fig. 9C, case 4 from topology D is chosen to be simulated with fractional orders $\alpha = 1$, and $\beta = 0.8$. The simulation parameters are $L_2 = L_1 = 21 \times 10^{-3}, R_2 = 1 \, \text{k}\Omega, k = 0.25$, and the design parameter $R_3$ is calculated according to be 3.5 $\text{k}\Omega$. The output waveforms are illustrated in Fig. 9D with oscillation frequency equal 60 kHz.

Fig. 10A and C shows topology D case 1, assembled on the NI ELVIS II series kit for the integer case, and the case with $\alpha = 0.8$ and $\beta = 1$, respectively.
FIG. 9  (A) Gyrator-based oscillator, (B) Spice simulations, (C) Op-Amp-based oscillator, and (D) Spice simulations.

FIG. 10  Op-Amp case 1 topology D (A) integer case, (B) scope output voltage waveforms, (C) case $\alpha = 0.8, \beta = 1$, and (D) scope output voltage waveforms.
The kit is from a national instrument and it is also used for measuring the output voltages. The TL082 Op-Amp chip is used for the circuit implementations with a potentiometer to adjust the condition of oscillation. The experimented cases have the following parameters: \( C_2 = C_1 = 1 \times 10^{-8}, k = 0.5, R_2 = 1 \, \text{k}\Omega \).

For the integer case, the value of \( R_3 \) is adjusted through the potentiometer to be 750 \( \Omega \). The scope output oscillation waveforms are demonstrated in Fig. 10B and the kit works at a sample rate of 5 MS/s.

For the second experimented case, \( \alpha = 0.8 \) and \( \beta = 1 \), the value of \( R_3 \) is adjusted through the potentiometer to be 1.27 k\( \Omega \). The scope output oscillation waveforms are demonstrated in Fig. 10D and the kit works at a sample rate of 12.5 MS/s.

4. CONCLUSION

A prototype of a fractional-order family of oscillators was introduced with three different topologies based on general two-port networks with input, output, and feedback impedances. The transmission matrix representation has been used in modeling a general two-port network (black box). With these general parameters, a design parameter was chosen to control the condition of oscillation. The oscillation frequency and the phase difference between the two oscillatory outputs were deduced as a general equation function of the transmission matrix elements, the fractional-order parameters. This concept has been verified using many two-port networks based on Op-Amps and a CCII.

Another prototype of fractional-order oscillators based on a two-port network and two impedances was derived from the original topology. Two topologies were investigated with either input or output impedance and feedback impedance. The two impedances were chosen to be fractional elements, which give four combinations for each topology. The general oscillation frequency, condition, and phase difference between the two oscillatory outputs are deduced in terms of the transmission matrix parameter of a general two-port network. As a case study, two different networks were presented, which are an Op-Amp-based circuit and a nonideal gyrator circuit. Experimental results were provided that validated the theoretical findings.

REFERENCES


