

ORIGINAL ARTICLE

Boundary conditions for hyperbolic systems of partial differential equations

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Abstract An easy-to-apply algorithm is proposed to determine the correct set(s) of boundary conditions for hyperbolic systems of partial differential equations. The proposed approach is based on the idea of the incoming/outgoing characteristics and is validated by considering two problems. The first one is the well-known Euler system of equations in gas dynamics and it proved to yield set(s) of boundary conditions consistent with the literature. The second test case corresponds to the system of equations governing the flow of viscoelastic liquids.

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Introduction and literature review

In most physical applications of systems of fully hyperbolic first-order partial differential equations (PDEs) the data include not only initial conditions (governing the so-called Cauchy problem) but also boundary conditions (leading to the so-called initial-boundary-value problem or IBVP for short). One of the crucial issues at a boundary is the determination of the correct number and kind of boundary conditions that must (or can) be imposed to yield a well-posed problem. This work presents a formalism for the treatment of boundary conditions for systems of hyperbolic equations. This treatment is intended to encompass all possible boundary conditions for

first-order hyperbolic systems in any number of dimensions. The central concept of this work is that hyperbolic systems of equations represent the propagation of waves and that at any boundary some of the waves are propagating into the computational domain while others are propagating out of it [1]. The outward propagating waves have their behavior defined entirely by the solution at and within the boundary, and no boundary conditions can be specified for them. The inward propagating waves depend on the fields exterior to the solution domain and therefore require boundary conditions to complete the specification of their behavior [2]. For a hyperbolic system of equations, considerations on characteristics show that one must be cautious about prescribing the solution on the boundary. In some particular cases, the boundary conditions can be found by physical considerations (such as a solid wall), but their derivation in the general case is not obvious. The problem of finding the “correct” set(s) of boundary conditions, i.e., those that lead to a well-posed problem, is difficult in general from both the theoretical and practical points of view (proof of well-posedness, choice of the physical variables that can be prescribed). The implementation of these boundary conditions

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is crucial in practice; however, it strongly depends on the problem at hand as shown in Godlewski and Raviart [2]. The theory developed by Kreiss [3] and others [4,5], known as uniform Kreiss condition (UKC), is one of the earliest works in this area. This theory relies on the analysis of “normal modes”, which are introduced by applying a Fourier transformation in the spatial direction normal to the boundary of interest and a Laplace transform in the time variable. The main idea in the derivation of necessary conditions on the boundary data so that the problem is well-posed is to exclude the cases that can lead to an ill-posed problem by looking for particular normal modes that cannot satisfy an energy estimate. The main disadvantage of this theory, as pointed out by Higdon [6], is that it is extremely complicated, and its physical interpretation is not immediately apparent. Another approach called the “vanishing viscosity” method was introduced by Benabdallah and Serre [7]. In this approach one should define a set of admissible boundary values for which a boundary entropy inequality holds. This approach is difficult to use by the lack of entropy flux pairs as pointed out by Dubois and Le Floch [8]. To overcome this difficulty, Dubois and Le Floch [8] proposed a second way of selecting admissible boundary conditions involving the resolution of Riemann problems. These two approaches coincide in some cases (scalar, linear systems). Olinger and Sundstrom [9] discussed some theoretical and practical aspects for IBVP in fluid mechanics. They began with a general discussion of well-posedness. Then the rigid wall and open boundary problems are very well treated. A different way of thinking and a much simpler approach is presented by Thompson [1], who proposed a simple and general algorithm to determine the correct boundary conditions based on the idea of the incoming/outgoing characteristics. The main disadvantages of his approach are

- (1) At any time t the boundary conditions contribute only to the determination of the time derivative of the dependent variable at the boundary, but never define the variable itself. For example, a boundary treatment which explicitly sets the normal velocity of a fluid to zero at a wall boundary is not allowed in his approach. Instead one would set the normal velocity to zero in the initial data and then specify boundary conditions which would force the time derivative of the normal velocity to be zero at all times.
- (2) A direct consequence of point (1) is the exclusion of cases in which a discontinuity exists between the initial data and the boundary conditions. In the proposed approach we avoid this disadvantage by not using the initial data in imposing the boundary conditions.

In the very recent work by Meier et al. [10], three methods are presented for modeling open boundary conditions. The first method, approximate Riemann boundary conditions (ARBCs), locally computes fluxes using an approximate Riemann technique to specify incoming wave strengths. In the second method, lacuna-based open boundary conditions (LOBCs), an exterior region is attached to the interior domain where hyperbolic effects are damped before reaching the exterior region boundary where the remaining parabolic effects are bounded using conventional boundary conditions. The third method, zero normal derivative boundary conditions (ZND BCs), enforces zero normal derivatives on each dependent variable at the open boundary. ZND BC is by far the easiest to

implement of the three open boundary conditions. However, for problems that are sensitive to boundary effects, ZND BC could be inadequate. In regard to the second method, ARBC, the boundary conditions are applied by specifying the flux, which means the system of equations must be in conservation form such that no source terms are present, which limits the range of the validity of the method. For the third method, LOCB, implementation of LOBC is complicated and problem-dependent.

The aim of the current work is to provide an easy-to-apply algorithm to determine the correct type and number of boundary conditions for first order hyperbolic systems of equations by providing a necessary condition between the characteristic variables and the primitive variables at the boundary of interest. The current work avoids the limitation of the ARBC method [10], i.e. the system of equation does not have to be in the conservation form. The current work is based on the idea of the incoming/outgoing characteristics but avoids the disadvantages of the Thompson approach [1].

One-dimensional systems in general form

Consider the general one-dimensional hyperbolic system,

$$\begin{cases} \frac{\partial \mathbf{w}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{w}}{\partial x} = \mathbf{0}, & 0 < x < 1, \quad t > 0, \\ \mathbf{w}(x, 0) = \mathbf{w}_0(x) \end{cases} \quad (1)$$

where $\mathbf{w} \in \mathbb{R}^p$.

The equations of the one-dimensional case may be put into a characteristic form in which the waves propagate in a single well-defined direction because only one direction is available [1], namely x in this problem.

One should start by diagonalizing the matrix \mathbf{A} . The matrix \mathbf{A} has p real eigenvalues a_i , $1 \leq i \leq p$ (since we are assuming the system to be purely hyperbolic) and a complete set of eigenvectors. Denote by $\mathbf{r}_1, \dots, \mathbf{r}_p$ (resp. $\mathbf{l}_1, \dots, \mathbf{l}_p$) a complete system of right eigenvectors of \mathbf{A} (resp. \mathbf{A}^T).

The matrices \mathbf{T} with columns $(\mathbf{r}_1, \dots, \mathbf{r}_p)$, and \mathbf{T}^{-1} with rows $(\mathbf{l}_1^T, \dots, \mathbf{l}_p^T)$ satisfy

$$\mathbf{T}^{-1} \mathbf{A} \mathbf{T} = \text{diag}(a_i) \equiv \Lambda \quad (2)$$

For ease of notation, we set $p' =$ number of nonpositive eigenvalues of \mathbf{A} ($a_i \leq 0$, $1 \leq i \leq p'$) and $q = p - p' =$ number of positive eigenvalues of \mathbf{A} ($a_i > 0$, $p' + 1 \leq i \leq p$) let the superscript I (respectively II) correspond to positive eigenvalues $a_i > 0$ (respectively nonpositive $a_i \leq 0$) and set

$$\mathbf{u}^I = (u_{p'+1}, \dots, u_p), \quad \mathbf{u}^{II} = (u_1, \dots, u_{p'}) \quad (3)$$

where \mathbf{u} is known as the vector of characteristic variables defined as

$$\mathbf{u} = \mathbf{T}^{-1} \mathbf{w} \text{ i.e. } u_k = \mathbf{l}_k^T \mathbf{w} \quad (4)$$

Also, \mathbf{u} is considered to be a solution of the decoupled system

$$\frac{\partial \mathbf{u}}{\partial t} + \Lambda \frac{\partial \mathbf{u}}{\partial x} = \mathbf{0} \quad (5)$$

In order to avoid the coupling between characteristic equations which may be caused by the presence of the tangential modes, the system of equations presented by Eq. (5) is assumed to be linear (or linearized). Consideration on characteristics shows that we have \mathbf{u}^I (respectively \mathbf{u}^{II}) incoming waves

(respectively outgoing waves) at $x = 0$ and \mathbf{u}'' (respectively \mathbf{u}') incoming waves (respectively outgoing waves) at $x = 1$ which means that this problem is well-posed if the boundary conditions for $\mathbf{u} = (\mathbf{u}', \mathbf{u}'')^T \in \mathbb{R}^{p-p'} \times \mathbb{R}^{p'}$ are:

$$\mathbf{u}'(0, t) = \mathbf{g}'(t), \quad (6)$$

$$\mathbf{u}''(1, t) = \mathbf{g}''(t), \quad (7)$$

where $\mathbf{g}'(t)$ is a given $(p - p')$ -component vector function and $\mathbf{g}''(t)$ is (p') -component vector function.

The question now is what should the boundary conditions be in terms of the original dependent variables \mathbf{w} or any other set of variables not in terms of the characteristic variables \mathbf{u} ? The main target of this paper is to give one possible answer to this question.

Multidimensional systems in general form

We deal with a general system of m quasi-linear first order PDEs for m functions $w^\alpha (\alpha = 1, \dots, m)$ of $n + 1$ independent variables $x^i, t (i = 1, \dots, n)$. We assume that, perhaps on physical grounds, we have privileged and distinguished the time variable t from its space counterparts x^i , such a system can be written in matrix notation as:

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{i=1}^n \mathbf{A}_i \frac{\partial \mathbf{w}}{\partial x^i} = \mathbf{b} \quad (8)$$

The coefficients \mathbf{A} , as well as the right hand side \mathbf{b} , are possibly functions of x^i, t and \mathbf{w} .

At the boundary of interest, we start by choosing the vector \mathbf{N} normal to the boundary at a point P (P lies on the boundary of interest) in space and time and pointing towards the interior of the domain. We will carry out the analysis in a non-rigorous way by restricting our problem in the vicinity of the point P to a single spatial dimension (namely, the normal to the boundary) and leaving the time variable unchanged. Let $y^i (i = 1, \dots, n)$ be a new spatial Cartesian coordinate system with the origin at P and such that the coordinate axis y^1 is aligned with \mathbf{N} . Naturally, the remaining axes will be in the hypersurface tangent to the boundary at P . The relation (translation plus a rotation) between the two (Cartesian) coordinate systems is given by an expression of the form:

$$y^i = c^i + R_j^i x^j \quad (9)$$

where c^i is a constant vector and $\{R_j^i\}$ is an orthogonal matrix. Notice that the first column of this matrix must coincide, by construction, with the components of \mathbf{N} in the old coordinate system, namely:

$$R_j^1 = N_j \quad (10)$$

We can now calculate the derivative

$$\frac{\partial w^\alpha}{\partial x^i} = \sum_{j=1}^n \frac{\partial w^\alpha}{\partial y^j} R_j^i \quad (11)$$

Whence the original system of Eq. (7) or (8) can be rewritten in terms of the new coordinates as:

$$\frac{\partial \{w\}}{\partial t} + \sum_{i=1}^n \sum_{j=1}^n [A]^i \frac{\partial \{w\}}{\partial y^j} R_j^i = \{b\} \quad (12)$$

The summation convention is used for all the diagonally repeated indices. By virtue of (10) Eq. (12) can be rewritten as:

$$\frac{\partial \{w\}}{\partial t} + \sum_{i=1}^n [A]^i N_i \frac{\partial \{w\}}{\partial y^1} = \{b\} - \sum_{i=1}^n \sum_{j=2}^n [A]^i \frac{\partial \{w\}}{\partial y^j} R_j^i \quad (13)$$

where the summation convention was suspended with respect to the index j .

It is only now that we implement an approximation. We assume, in fact, that in a small neighborhood of P the variation of the functions w^α in the direction normal to the boundary can be calculated as if the derivatives in the other coordinate directions were somehow known. In other words, to advance in the plane formed by y^1 and t , we regard (13) as system of m quasi-linear first order PDEs in just two independent variables. This means that the multidimensional system (7) or (8) may be treated in the same way as the system (1) in regards to the boundary conditions analysis by considering one direction at a time as explained in the previous section. The well-known paper by Thompson [1] reaches a similar conclusion: derivatives in directions transverse to the boundary may be evaluated just as in the interior of the domain.

It is worthwhile mentioning that, in general, tangential modes, which can determine coupling between characteristic equations, cannot be ignored, thus restricting the applicability of the proposed method to the cases where transverse derivatives can be safely carried along passively [1]. In other words we are assuming that the tangential modes play a minor role in defining stability criteria.

Methodology

The equivalent set of boundary conditions

This section introduces the proposed approach and explains one way to practically implement it. In the next sub-section, the theory behind the proposed algorithm is explained. Then in the following subsection, the proposed approach is validated.

Theoretical analysis

Consider the general system of Eq. (7) for the characteristic analysis for the x direction, the other directions being similar. According to [1], all terms not involving x derivatives of \mathbf{w} are carried along passively and do not contribute in any substantive fashion to the analysis; therefore we may lump them together and write

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{w}}{\partial x} + \mathbf{C} = \mathbf{0} \quad (14)$$

where \mathbf{C} is a term that contains all the terms not involving x derivatives of \mathbf{w} . The matrix \mathbf{A} could be diagonalized using Eq. (2). According to the theory of characteristics, discussed above, we need to prescribe q (the number of the positive eigenvalues of \mathbf{A}) boundary conditions i.e. $\mathbf{u}^q(0, y, t) = \mathbf{g}^q(y, t)$. With no loss of generality and for the sake of easiness, we consider the vector of unknowns \mathbf{w} to be of length four. Assuming that we have calculated the eigenvalues of the matrix \mathbf{A} , let $\mathbf{u} \equiv (u_1, u_2, u_3, u_4)$ be the characteristic variables, with the first three, namely, $\mathbf{u}^q = (u_1, u_2, u_3)$, to be assigned on the boundary of interest. If we want to replace $\mathbf{u}^q = (u_1, u_2, u_3)$ with \mathbf{w}^q (where \mathbf{w}^q may be any combination of the original variables, with the same number of the

characteristic variables to be prescribed, e.g. $\mathbf{w}^q \equiv (w_1, w_2, w_3)$, $\mathbf{w}^q \equiv (w_1, w_2, w_4)$, or $\mathbf{w}^q \equiv (w_2, w_3, w_4)$, etc.), we start by forming the following four (four here is the number of the dependant variables) combinations,

$$\begin{aligned} w_1 &= w_1(u_1, u_2, u_3, u_4), \\ w_2 &= w_2(u_1, u_2, u_3, u_4), \\ w_3 &= w_3(u_1, u_2, u_3, u_4), \\ w_4 &= w_4(u_1, u_2, u_3, u_4). \end{aligned} \quad (15)$$

Then we need to satisfy the condition that no functional, F combination of \mathbf{w}^q produces u_4 . The mathematical representation to this statement is:

$$F(w_1, w_2, w_3, w_4) = u_4 \quad (16)$$

This functional must not exist. The total derivative of (16) yields

$$dF = \frac{\partial F}{\partial w_1} dw_1 + \frac{\partial F}{\partial w_2} dw_2 + \frac{\partial F}{\partial w_3} dw_3 + \frac{\partial F}{\partial w_4} dw_4 = du_4 \quad (17)$$

Using (15) in (17) yields

$$\begin{aligned} & \left[\frac{\partial F}{\partial w_1} \frac{\partial w_1}{\partial u_1} + \frac{\partial F}{\partial w_2} \frac{\partial w_2}{\partial u_1} + \frac{\partial F}{\partial w_3} \frac{\partial w_3}{\partial u_1} + \frac{\partial F}{\partial w_4} \frac{\partial w_4}{\partial u_1} \right] du_1 \\ & + \left[\frac{\partial F}{\partial w_1} \frac{\partial w_1}{\partial u_2} + \frac{\partial F}{\partial w_2} \frac{\partial w_2}{\partial u_2} + \frac{\partial F}{\partial w_3} \frac{\partial w_3}{\partial u_2} + \frac{\partial F}{\partial w_4} \frac{\partial w_4}{\partial u_2} \right] du_2 \\ & + \left[\frac{\partial F}{\partial w_1} \frac{\partial w_1}{\partial u_3} + \frac{\partial F}{\partial w_2} \frac{\partial w_2}{\partial u_3} + \frac{\partial F}{\partial w_3} \frac{\partial w_3}{\partial u_3} + \frac{\partial F}{\partial w_4} \frac{\partial w_4}{\partial u_3} \right] du_3 \\ & + \left[\frac{\partial F}{\partial w_1} \frac{\partial w_1}{\partial u_4} + \frac{\partial F}{\partial w_2} \frac{\partial w_2}{\partial u_4} + \frac{\partial F}{\partial w_3} \frac{\partial w_3}{\partial u_4} + \frac{\partial F}{\partial w_4} \frac{\partial w_4}{\partial u_4} \right] du_4 \\ & = du_4 \end{aligned} \quad (18)$$

Since $du_1 \dots du_3$ are arbitrary, Eq. (18) is not simply an equation but rather represents an identity, which means that all bracketed terms vanish simultaneously, namely

$$\begin{bmatrix} \frac{\partial w_1}{\partial u_1} & \frac{\partial w_2}{\partial u_1} & \frac{\partial w_3}{\partial u_1} & \frac{\partial w_4}{\partial u_1} \\ \frac{\partial w_1}{\partial u_2} & \frac{\partial w_2}{\partial u_2} & \frac{\partial w_3}{\partial u_2} & \frac{\partial w_4}{\partial u_2} \\ \frac{\partial w_1}{\partial u_3} & \frac{\partial w_2}{\partial u_3} & \frac{\partial w_3}{\partial u_3} & \frac{\partial w_4}{\partial u_3} \\ \frac{\partial w_1}{\partial u_4} & \frac{\partial w_2}{\partial u_4} & \frac{\partial w_3}{\partial u_4} & \frac{\partial w_4}{\partial u_4} \end{bmatrix} \begin{bmatrix} \frac{\partial F}{\partial w_1} \\ \frac{\partial F}{\partial w_2} \\ \frac{\partial F}{\partial w_3} \\ \frac{\partial F}{\partial w_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (19)$$

Eq. (19) may be solved for the function F . To make sure that no such function exists i.e. to avoid the satisfaction of (16), it is sufficient to have a nonzero (partial) Jacobian (since the right hand side is zero), the last bracketed term does not appear in (19) since we require $dF = 0$, consequently $du_4 = dF = 0$ from Eq. (17).

$$J = \frac{\partial \mathbf{w}}{\partial \mathbf{u}^q} = \frac{\partial(w_1, w_2, w_3, w_4)}{\partial(u_1, u_2, u_3)} \neq 0 \quad (20)$$

Now we can choose for this boundary any three combinations \mathbf{w}^q satisfying (20).

Eq. (20) is a necessary condition for the boundary conditions to be consistent with the theory of characteristics. A similar condition, in a more complicated way, is proposed by Higdon [6]. A separate work is needed to check whether it is sufficient for well-posedness or not. An energy analysis such as that discussed by Hesthaven and Gottlieb [11], could be used to check for well-posedness.

Results

Validation of the proposed algorithm

Before applying the proposed approach to one of the benchmark problems, the Euler equations, we summarize the proposed algorithm in a flow chart.

Flow chart to determine the appropriate boundary conditions

Fig. 1 shows a flow chart that summarizes the proposed algorithm and put it in a simpler way to understand and implement it without the need to understand the theoretical analysis behind it.

Boundary conditions for the Euler equations

In this sub-section, we validate the proposed algorithm described in the previous sub-section. note that the proposed approach requires only the computation of the matrix \mathbf{T} and the determinants of sub-matrices which could be done for any system of equations. The well known Euler system of equations for the inviscid flows in one-dimensional form is

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{w}}{\partial x} = \mathbf{0}, \quad 0 < x < 1 \quad (21)$$

where

$$\mathbf{w} = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} u_0 & \rho_0 & 0 \\ 0 & u_0 & 1/\rho_0 \\ 0 & \rho_0 c_0^2 & u_0 \end{bmatrix}, \quad c_0 : \text{the speed of sound.}$$

Step 1: get the eigenvalues for the Jacobian matrix \mathbf{A} ,

$$\lambda_1 = u_0 - c_0, \lambda_2 = u_0, \lambda_3 = u_0 + c_0$$

Step 2: get the eigenvectors associated to the eigenvalues,

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ -c_0/\rho_0 \\ c_0^2 \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}_3 = \begin{pmatrix} 1 \\ c_0/\rho_0 \\ c_0^2 \end{pmatrix}$$

Step 3: get the matrix \mathbf{T} ,

$$\mathbf{T} = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3] = \begin{bmatrix} 1 & 1 & 1 \\ -\frac{c_0}{\rho_0} & 0 & \frac{c_0}{\rho_0} \\ c_0^2 & 0 & c_0^2 \end{bmatrix}$$

Step 4: determine the sign of the eigenvalues.

Case 1: subsonic inflow, $q = 2$ positive eigenvalues, namely λ_2 and λ_3 , so we need to impose the corresponding characteristic variables, namely u_2 and u_3 as boundary conditions. To get all the possible set(s) of boundary conditions in terms of the original variables \mathbf{w} , one needs to check the Jacobian defined by (20).

$$J = \frac{\partial \mathbf{w}}{\partial \mathbf{u}^q} = \frac{\partial(w_1, w_2, w_3)}{\partial(u_2, u_3)} \equiv \frac{\partial(\rho, u, p)}{\partial(u_2, u_3)}$$

Recall that $\mathbf{w} = \mathbf{T}\mathbf{u}$, which means that the elements of J could be copied simply from the matrix \mathbf{T} . So in this case

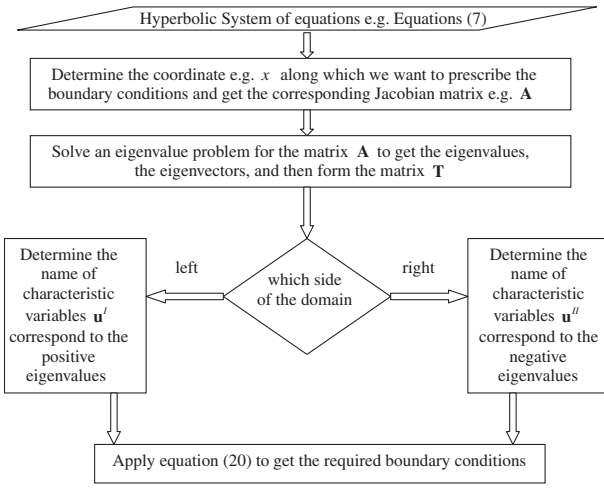


Fig. 1 Flow chart to determine the appropriate boundary conditions.

$$J = \begin{vmatrix} T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{vmatrix} = \begin{vmatrix} \rho & u & p \\ 1 & 0 & 0 \\ 1 & \frac{c_0}{\rho_0} & c_0^2 \end{vmatrix} \begin{matrix} u_2 \\ u_3 \end{matrix}$$

Note that all the required information about the possible set(s) of boundary conditions is included in J . One way to get information from J is to form any 2×2 (2 here is the number of characteristic variables to be specified at the boundary) matrix and check its determinant, zero determinant means an ill-posed problem while non-zero determinant means it is an acceptable choice.

e.g.

- The pairs (ρ, u) and (ρ, p) produce non-zero determinant, which means that one of them could be used at the inlet as boundary conditions, which is consistent with the literature. Using one of these two pairs means that its values at the boundary are user-specified while the rest of the dependant variables are determined from the interior of the domain.
- The pair (u, p) produces zero determinant, which means it is not acceptable to be used at the inlet as boundary conditions as it will lead to an ill-posed problem, which is consistent with the literature as well.

Case 2: supersonic inflow, $q = 3$ positive eigenvalues, three boundary conditions are required, which means the whole state must be prescribed. In this case all the dependant variables are user-specified at the boundary and nothing is computed using the interior of the domain.

Case 3: subsonic outflow, $q = 1$ negative eigenvalue, namely λ_1 , one condition is required. In terms of the characteristic variables, we need to prescribe u_1 . To get the corresponding original variable(s), one needs to check the Jacobian

$$J = \frac{\partial w_i}{\partial u_1}, \quad i = 1, \dots, 3 = \begin{vmatrix} T_{11} & T_{21} & T_{31} \\ \rho & u & p \\ 1 & -\frac{c_0}{\rho_0} & c_0^2 \end{vmatrix} u_1$$

Again, one way to get information from J is to form any 1×1 (1 here is the number of characteristic variables) matrix (scalar) and check its determinant (value). By inspection, there are non-zero elements which means we can prescribe any of the primitive variables at the exit.

Case 4: supersonic outflow, $q = 0$ negative eigenvalue, no conditions.

Boundary conditions for viscoelastic liquids

A viscoelastic liquid is a fluid that exhibits a physical behavior intermediate between that of a viscous liquid and an elastic solid. For this reason, both the mathematical formulation and the experimental techniques used to describe the response of viscoelastic liquids are substantially different from their viscous liquid counterparts. In particular, the numerical implementation of the governing system of equations contains important qualitative differences, such as the character of the equations, the choice of the independent variables and the enforcing of boundary conditions.

The determination of the correct set(s) of boundary conditions for viscoelastic liquids is/are considered to be one of the major problems in numerical simulation as explained by Joseph [12]. In this section we are applying the proposed approach to get the possible set(s) of boundary conditions for the governing system of equations for viscoelastic liquids. Then the resulting set(s) is/are used in the numerical simulation to show the validity of the proposed approach. The governing system of equations for viscoelastic liquids is given by (for more details see Guaily and Epstein [13]):

$$\mathbf{A}_t \frac{\partial \mathbf{q}}{\partial t} + \mathbf{A}_x \frac{\partial \mathbf{q}}{\partial x} + \mathbf{A}_y \frac{\partial \mathbf{q}}{\partial y} = \mathbf{r} \quad (22)$$

where $\mathbf{q} = [\rho \quad u \quad p \quad S \quad Q \quad T]^T$ is the vector of unknowns. The matrix \mathbf{A}_t is the identity matrix and

$$\mathbf{A}_x = \begin{bmatrix} u_0 & \rho_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_0 & 0 & 1/\rho_0 & -1/\rho_0 & 0 & 0 \\ 0 & 0 & u_0 & 0 & 0 & -1/\rho_0 & 0 \\ 0 & \gamma p_0 & 0 & u_0 & 0 & 0 & 0 \\ 0 & -2S_0 - 2/(R_e W_e) & 0 & 0 & u_0 & 0 & 0 \\ 0 & -Q_0 & -S_0 - 1/(R_e W_e) & 0 & 0 & u_0 & 0 \\ 0 & 0 & -2Q_0 & 0 & 0 & 0 & u_0 \end{bmatrix}$$

$$\mathbf{A}_y = \begin{bmatrix} 0 & 0 & \rho_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\rho_0 & 0 \\ 0 & 0 & 0 & 1/\rho_0 & 0 & 0 & -1/\rho_0 \\ 0 & 0 & \gamma p_0 & 0 & 0 & 0 & 0 \\ 0 & -2Q_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -T_0 - 1/(R_e W_e) & -Q_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2T_0 - 2/(R_e W_e) & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{S_0}{W_e} & -\frac{Q_0}{W_e} & -\frac{T_0}{W_e} \end{bmatrix}^T$$

ρ is the density, u the velocity component in the axial direction, the velocity component in the normal direction, S the stress component in the axial direction, Q the shear stress, T the stress component in the normal direction, $R_e = \frac{\rho_0 c_0 L}{\mu_0}$ the Reynolds number, and $W_e = \frac{\lambda_0}{(L/c_0)}$ is the Weissenberg number.

And L, μ_0, C_o, λ_o are a characteristic length, the viscosity, the free stream speed of sound, and the relaxation time respectively.

Step 1: get the eigenvalues for the Jacobian matrix \mathbf{A}_x ,

$$\lambda_1 = u_0 + \sqrt{\frac{k}{\rho_0}}, \quad \lambda_2 = u_0 - \sqrt{\frac{k}{\rho_0}}, \quad \lambda_3 = u_0 + \sqrt{\frac{k+\gamma p_0}{\rho_0}}, \quad \lambda_4 = u_0 - \sqrt{\frac{k+\gamma p_0}{\rho_0}},$$

$$\lambda_5 = u_0, \quad \lambda_6 = u_0, \quad \lambda_7 = u_0, \quad k = S_0 + \frac{1}{R_e W_e}$$

Step 2: get the eigenvectors and the matrix \mathbf{T} , Remember that the eigenvectors should be in the same order as the eigenvalues.

$$\mathbf{T} = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3 \quad \mathbf{r}_4 \quad \mathbf{r}_5 \quad \mathbf{r}_6 \quad \mathbf{r}_7] \quad (23)$$

We are presenting \mathbf{T}^T because it represents the Jacobian matrix for the vector of unknowns \mathbf{q} with respect to the characteristic variables

$$\mathbf{T}^T = \begin{bmatrix} 0 & 0 & \frac{-\sqrt{k\rho_0}}{2\rho_0 Q_0} & 0 & 0 & \frac{k}{2Q_0} & 1 \\ 0 & 0 & \frac{\sqrt{k\rho_0}}{2\rho_0 Q_0} & 0 & 0 & \frac{k}{2Q_0} & 1 \\ \frac{(\gamma p_0 \rho_0 + k \rho_0)}{2Q_0^2} & \frac{(\gamma p_0 + k)\sqrt{\gamma p_0 \rho_0 + 2k\rho_0}}{2\rho_0 Q_0^2} & \frac{-\sqrt{\gamma p_0 \rho_0 + 2k\rho_0}}{2\rho_0 Q_0} & \frac{-\gamma p_0(k+\gamma p_0)}{2Q_0^2} & \frac{k(k+\gamma p_0)}{Q_0^2} & \frac{(2k+\gamma p_0)}{2Q_0} & 1 \\ \frac{(\gamma p_0 \rho_0 + k \rho_0)}{2Q_0^2} & \frac{(\gamma p_0 + k)\sqrt{\gamma p_0 \rho_0 + 2k\rho_0}}{2\rho_0 Q_0^2} & \frac{\sqrt{\gamma p_0 \rho_0 + 2k\rho_0}}{2\rho_0 Q_0} & \frac{-\gamma p_0(k+\gamma p_0)}{2Q_0^2} & \frac{k(k+\gamma p_0)}{Q_0^2} & \frac{(2k+\gamma p_0)}{2Q_0} & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: determine the sign of the eigenvalues, Consider the flow of viscoelastic liquid in a channel. See Fig. 2 for the geometry and the grid (for more details about the problem, see [13]). At the left end of the channel we have five positive eigenvalues, namely $\lambda_1, \lambda_3, \lambda_5, \lambda_6$, and λ_7 and two negative eigenvalues, namely λ_2 and λ_4 at the right end.

Step 4: boundary conditions in terms of the primitive variables,

- *The left end*

Since we have five positive eigenvalues, $\lambda_1, \lambda_3, \lambda_5, \lambda_6$, and λ_7 , (five incoming waves); we need to prescribe five boundary conditions at the inlet corresponding to the characteristic variables $\mathbf{u}^q = (u_1, u_3, u_5, u_6, u_7)$.

To get all the possible set(s) of boundary conditions in terms of the original variables \mathbf{q} and to see the choices that may lead to an ill-posed problem, we need to apply Eq. (20). Recall that the Jacobian defined by Eq. (20) is simply a part of the matrix T^T considering the appropriate rows only.

$$J = \frac{\partial \mathbf{q}}{\partial \mathbf{u}^q} = \frac{\partial(q_1, q_2, q_3, q_4, q_5, q_6, q_7)}{\partial(u_1, u_3, u_5, u_6, u_7)}$$

$$J = \begin{array}{c|ccc} & q_1 & q_2 & q_3 \\ \hline u_1 & 0 & 0 & \frac{-\sqrt{k\rho_0}}{2\rho_0 Q_0} \\ u_3 & \frac{(\gamma p_0 \rho_0 + k \rho_0)}{2Q_0^2} & \frac{-(\gamma p_0 + k)\sqrt{\gamma p_0 \rho_0 + 2k\rho_0}}{2\rho_0 Q_0^2} & \frac{-\sqrt{\gamma p_0 \rho_0 + 2k\rho_0}}{2\rho_0 Q_0} \\ u_5 & 1 & 0 & 0 \\ u_6 & 0 & 0 & 0 \\ u_7 & 0 & 0 & 0 \end{array} \quad \begin{array}{c|ccc} & q_4 & q_5 & q_6 & q_7 \\ \hline & 0 & 0 & \frac{k}{2Q_0} & 1 \\ & \frac{\gamma p_0(k+\gamma p_0)}{2Q_0^2} & \frac{k(k+\gamma p_0)}{Q_0^2} & \frac{(2k+\gamma p_0)}{2Q_0} & 1 \\ & 0 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 1 \end{array}$$

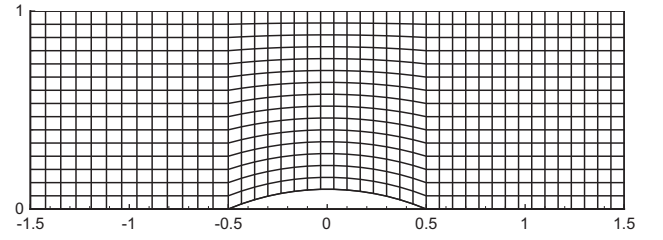


Fig. 2 Channel flow with a bump, geometry and grid.

Again, one way to get information from this Jacobian is to construct any 5×5 (again, five here is the number of positive eigenvalues at the boundary) matrix, and then check the determinant of this matrix; if it is zero, then this choice will lead to an ill-posed problem. Otherwise, it is an acceptable choice. In Table 1: the left column shows a few sets of the boundary conditions that may be prescribed over the left boundary while the right column shows sets of boundary conditions that leads to an ill-posed problem.

- *The right end*

Since we have two negative eigenvalues, λ_2 and λ_4 (two incoming waves); we need to prescribe two boundary conditions at the outlet corresponding to the characteristic variables u_2, u_4 .

Again we will present the Jacobian defined by (20) of the seven primitive variables (ρ, u, p, S, Q, T) , namely, $(q_1, q_2, q_3, q_4, q_5, q_6, q_7)$ so we could know by inspection the consequences of having different sets of boundary conditions.

$$J = \frac{\partial \mathbf{q}}{\partial \mathbf{u}^q} = \frac{\partial(q_1, q_2, q_3, q_4, q_5, q_6, q_7)}{\partial(u_2, u_4)}$$

$$= \begin{array}{c|cccccc} & q_1 & q_2 & q_3 & q_4 & q_5 & q_6 & q_7 \\ \hline u_2 & 0 & 0 & \frac{\sqrt{k\rho_0}}{2\rho_0 Q_0} & 0 & 0 & \frac{k}{2Q_0} & 1 \\ u_4 & \frac{(\gamma p_0 \rho_0 + k \rho_0)}{2Q_0^2} & \frac{(\gamma p_0 + k)\sqrt{\gamma p_0 \rho_0 + 2k\rho_0}}{2\rho_0 Q_0^2} & \frac{\sqrt{\gamma p_0 \rho_0 + 2k\rho_0}}{2\rho_0 Q_0} & \frac{-\gamma p_0(k+\gamma p_0)}{2Q_0^2} & \frac{k(k+\gamma p_0)}{Q_0^2} & \frac{(2k+\gamma p_0)}{2Q_0} & 1 \end{array}$$

Constructing any 2×2 matrix, and then check the determinant; if it is zero, then this choice will lead to an ill-posed problem otherwise it is an acceptable choice. In Table 2: the left column shows a few sets of the boundary conditions that may be prescribed over the right boundary while the right column shows sets of boundary conditions that lead to an ill-posed problem.

Table 1 Different sets of boundary conditions for the left boundary.

$J \neq 0$, Acceptable choice	$J = 0$, Ill-posed problem
$(\rho, u, , S, T)$, namely $(q_1, q_2, q_3, q_5, q_7)$	$(u, , S, Q, T)$, namely $(q_2, q_3, q_5, q_6, q_7)$
$(\rho, u, , p, T)$, namely $(q_1, q_2, q_3, q_4, q_7)$	$(\rho, u, , p, S)$, namely $(q_1, q_2, q_3, q_4, q_5)$
(ρ, u, p, S, T) , namely $(q_1, q_2, q_4, q_5, q_7)$	$(\rho, u, , S, Q)$, namely $(q_1, q_2, q_3, q_5, q_6)$
$(\rho, , p, S, T)$, namely $(q_1, q_3, q_4, q_5, q_7)$	$(u, , p, S, Q)$, namely $(q_2, q_3, q_4, q_5, q_6)$

Table 2 Different sets of boundary conditions for the right boundary.

$J \neq 0$, Acceptable choice	$J = 0$, Ill-posed problem
(p, Q) , namely (q_4, q_6)	(ρ, p) , namely (q_1, q_4)
(S, Q) , namely (q_5, q_6)	(S, Q) , namely (q_5, q_6)
$(, Q)$, namely (q_2, q_6)	$(, p)$, namely (q_2, q_4)

Discussion

Numerical test

Numerical experiments, using a channel with a bump, Fig. 2, are carried out to observe the effect of well-posedness and ill-posedness on the residual of each dependent variable. A

hybrid finite element/finite difference technique is used to solve the governing system of equation. For more details regarding the numerical algorithm, the physical description and results, see [13].

• *Successful test case*

To run the simulations; the first choice in Table 1, namely $(\rho, u, , S, T)$ from the left side, is used as a boundary condition on the left end with the corresponding choice from Table 2, namely (p, Q) , at the right end, is used

The exact values used for this specific case are

$$\rho = 1, \quad u = 4U_\infty y(1 - y), \quad v = 0, \quad S = 32 \frac{W_e}{R_e} U_\infty^2 (1 - 2y)^2, \quad T = 0$$

At the exit,

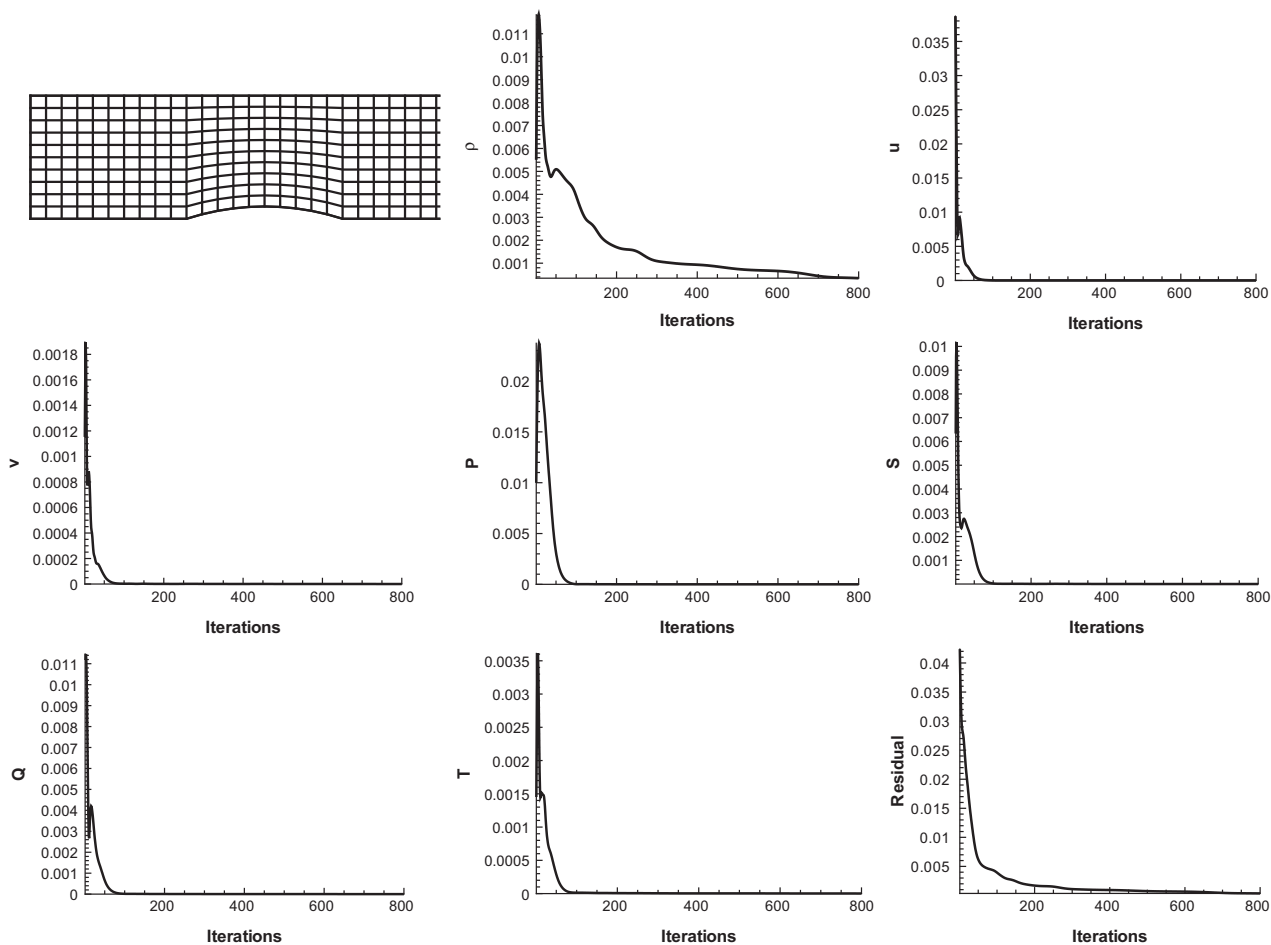


Fig. 3 The residual for all the variables (successful case).

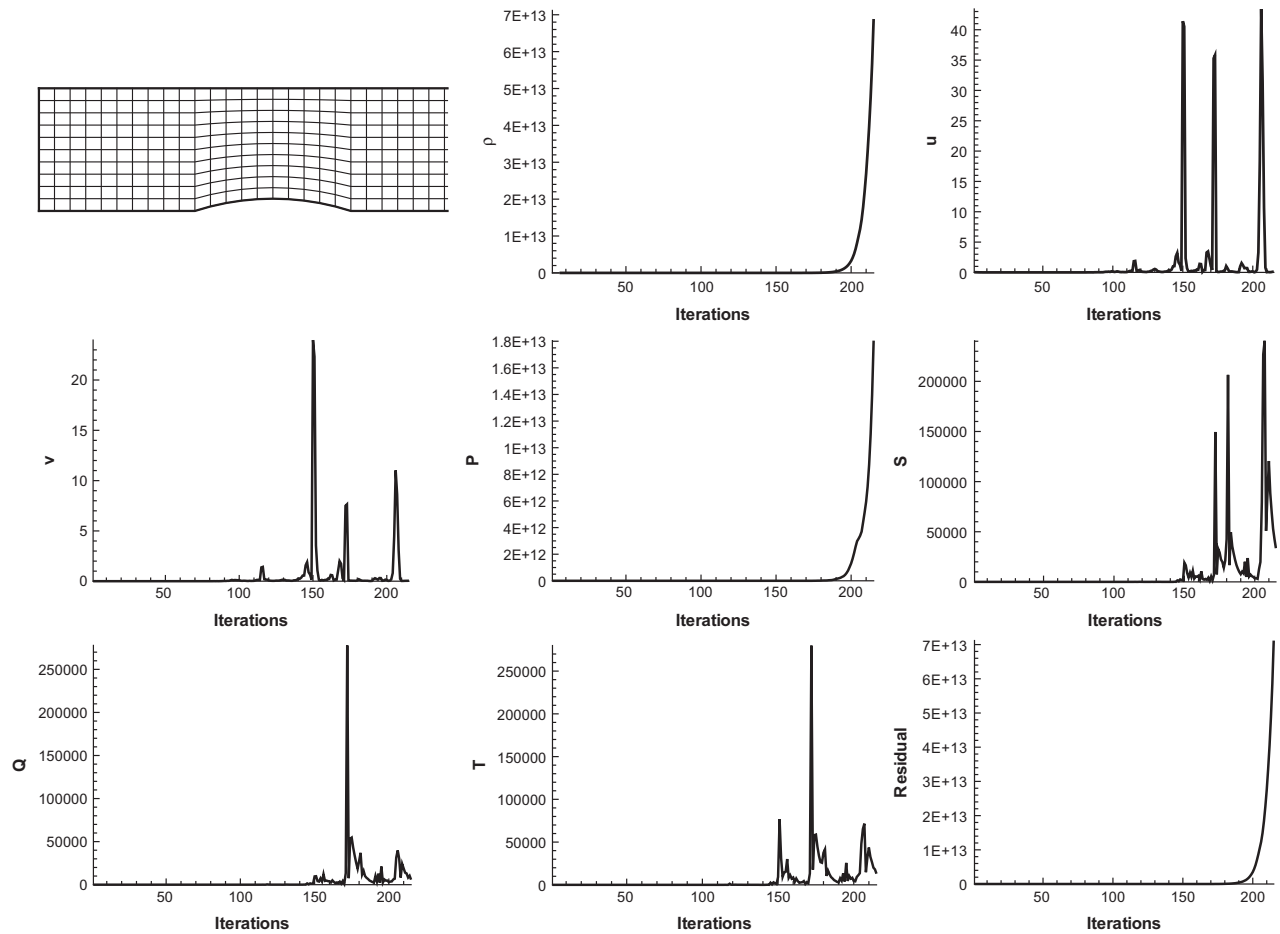


Fig. 4 The residual for all the variables (failed case).

$$p = 1/\gamma, \quad Q = 4 \frac{U_\infty}{R_e} (1 - 2y)$$

The viscoelastic flow computations are performed with ($\Delta t = 0.15$, $\gamma = 7.15$, $U_\infty = 0.2$, $R_e = 1.0$, $W_e = 0.1$).

Fig. 3 shows the residual for all the variables. As seen in the figure, all the dependant variables converge, which assures the correctness of the proposed algorithm.

- *Failed test case*

To run the simulations; the last choice in Table 1, namely (u, v, p, S, Q) from the right side, is used as a boundary condition on the left end with (ρ, T) at the right end.

The exact values used for this specific case are

$$\begin{aligned} u &= 4U_\infty y(1 - y), \quad v = 0, \quad p = 1/\gamma, \quad S \\ &= 32 \frac{W_e}{R_e} U_\infty^2 (1 - 2y)^2, \quad Q = 4 \frac{U_\infty}{R_e} (1 - 2y) \end{aligned}$$

At the exit,

$$\rho = 1, \quad T = 0$$

Fig. 4 shows the residual for all the variables. As seen in the figure, all the dependant variables are diverging or oscillating which is a sign of ill-posedness which, again, assures the correctness of the proposed algorithm.

Conclusion and future work

A necessary condition, Eq. (20), for the boundary conditions for hyperbolic systems of partial differential equations is derived to be consistent with the theory of characteristics. The theory behind the new approach is presented in detail. The new approach is easy to apply and to understand and has been applied successfully to two problems. In future work, a separate study is needed to check whether condition (20) is sufficient for well-posedness or not.

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References

- [1] Thompson kW. Time-dependent boundary conditions for hyperbolic systems II. *J Comput Phys* 1990;89(2):439–61.
- [2] Godlewski E, Raviart PA. Numerical approximation of the hyperbolic systems of conservation laws. New York: Applied mathematical Sciences Springer-Verlag; 1996, p. 417–60.
- [3] Kreiss HO. Initial boundary value problems for hyperbolic systems. *Comm Pure Appl Math* 1970;23:277–98.

- [4] Majda A, Osher S. Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary. *Comm Pure Appl Math* 1975;28:607–75.
- [5] Ralston JV. Note on a paper of Kreiss. *Comm Pure Appl Math* 1971;24:759–62.
- [6] Higdon RL. Initial-boundary value problem for linear hyperbolic systems. *SIAM Rev* 1986;28(2):177–217.
- [7] Benabdallah A, Serre D. Problèmes aux limites pour des systèmes hyperboliques nonlinéaires de deux équations à une dimension d'espace. *C R Acad Sci Paris Sér I Math* 1987;305(15):677–80.
- [8] Dubois F, Le Floch P. Boundary conditions for nonlinear hyperbolic systems of conservation laws. *J Diff Eqs* 1988;71:93–122.
- [9] Olinger J, Sundstrom A. Theoretical and practical aspects of some initial Boundary value problems in fluid mechanics. *SIAM Appl Math* 1978;35:419–46.
- [10] Meier ET, Glasser AH, Lukin VS, Shumlak U. Modeling open boundaries in dissipative MHD simulation. *J Comput Phys* 2012;231:2963–76.
- [11] Hesthaven JS, Gottlieb D. A stable penalty method for the compressible Navier–Stokes equations: I. Open boundary conditions. *J Sci Comput* 1996;17:579–612.
- [12] Joseph D. Fluid dynamics of viscoelastic liquids. New York: Applied mathematical Sciences Springer-Verlag; 1990, p. 127–38.
- [13] Guaily A, Epstein M. Unified hyperbolic model for viscoelastic liquids. *Mech Res Comm* 2010;37:158–63.