Hyperbolic Model for the Classical Navier-Stokes Equations

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A new formulation of the classical Navier-Stokes equations is presented, which overcomes the equations' main disadvantage: being a mixed parabolic-hyperbolic system. The new model is achieved by adopting the compressible codeformational time derivative in the stress-strain constitutive relation, resulting in a consistency with the principle of material frame indifference. The main advantage of the new formulation is that the resulting system of equations is purely hyperbolic. The proposed formulation is used to model the benchmark problem of the shock/boundary layer/expansion fan interaction with an apparent degree of success.

Keywords: Navier-Stokes, stress relaxation, codeformational time derivative, viscous, shock reflection

INTRODUCTION

The main disadvantage of the conventional Navier-Stokes equations is that they produce nonphysical solutions to the problem of propagating transverse perturbations, as shown by Wilhelm and Hong.[1] Historically, this problem is attacked by two approaches: modifying the constitutive equation, or modifying the inertia term in the linear momentum equation. The first approach is suggested by Carrassi and Morro[2] by adding a time-relaxation term to the constitutive equation, relating the extra stress tensor to the rate of deformation tensor. The work of Carrassi and Morro[2] is based on the idea previously proposed by Cattaneo[3] regarding modifying the Fourier law of heat conduction in order to obtain heat waves travelling with finite speeds. Wilhelm and Hong[1] extended the work of Carrassi and Morro[2] by adding an extra term to the constitutive equation accounting for convection effects, through which they succeeded in obtaining physical solutions in the form of a discontinuous wave with a finite wave speed due to viscous stress relaxation. The drawback with the work of Wilhelm and Hong[1] and Carrassi and Morro[2] is that they used the usual partial and material time derivative, respectively, which violate the principle of material frame indifference when used with tensors. Other techniques have been used, e.g. Lebon and Cloot[4] used the framework of extended irreversible thermodynamics (EIT) to derive evolution equations for the heat flux as well as the extra-stress tensor. They used their model to study the effects of dissipative phenomena, with emphasis on nonlocal effects, on propagation and absorption of sound waves in a rarefied monoatomic gas. Another technique, suggested by Ruggeri,[5] is based on the idea of preserving the symmetrical structure of the classical Fourier-Navier-Stokes theory. The resulting model possesses the main properties of EIT as the one proposed by Lebon and Cloot.[4] In Brenier et al.,[6] a hyperbolic singular perturbation of the classical incompressible Navier-Stokes equations is reached after a relaxation of the Euler equations and a rescaling of the variables. Following the work of Carrassi and Morro[2] and Carbonaro and Rosso,[7] Schöwe[8] proves the global existence of small solutions and asymptotic results and shows that the relaxed system is close to the classical Navier-Stokes equations in the sense that for short periods of time the solutions converge to form the solution of the incompressible Navier-Stokes equations. The second approach is suggested by Kuznetsov[9] who proposed another modification to the conventional Navier-Stokes equations, but from a different point of view. In his work, Kuznetsov[9] considered that the source of infinite wave speeds lies in the classical way used to derive the linear momentum equation, and so he delayed the inertia term, resulting in a purely hyperbolic system with finite wave speeds. We refer to the work of Polyanin and Zhurov[10] for a recent literature survey of hyperbolic models for the incompressible Navier-Stokes equations. For general parabolic systems other than the Navier-Stokes equations we refer to the following works: Naldi et al.,[11] who use a local relaxation approximation which approximates the original system with a small dissipative correction; and Selezov,[12] who presents and briefly characterizes some hyperbolic models obtained as extensions of previously-known parabolic models. Although the time derivatives added by Wilhelm and Hong[1] and Carrassi and Morro[2] to the constitutive equation succeeded in solving the problem of infinite wave speeds, the resulting modified constitutive equation is no longer consistent with the principle of material frame indifference. The principle of material frame indifference, which is one of five principles composing the constitutive theory,[13] is tied to the physical idea that stress is associated exclusively with deformations, and hence does not depend on the motion of the observer.[14] An important use of this principle is made by Speziale,[14] as he found that the kinetic theory is in support of the principle of material frame indifference with regard to the superimposed translational accelerations of the gas. Thus, the invariance group of constitutive equations should be at least the extended Galilean group. The present work extends the work of Wilhelm and Hong[1] and Carrassi and Morro[2] by modifying the constitutive equation for the compressible Newtonian fluids to be consistent with the principle of material frame indifference, resulting in a purely hyperbolic system with finite wave speeds and in the limit it reduces to the classical Navier-Stokes equations.

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GOVERNING EQUATIONS

Balance Laws

For non-polar fluids, conservation of mass, momentum, and energy, in index notation, with no mass or heat addition, yield the following:

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0 \tag{1}$$

$$\frac{\partial v_i}{\partial t} + v_j v_{i,j} + \frac{1}{\rho}\left(p_{,i} - \tau_{ij,j}\right) = 0 \tag{2}$$

$$\frac{\partial p}{\partial t} + (p v_i)_{,i} + (\gamma - 1)\left(p v_{i,i} - \tau_{ij} d_{ij}\right) = 0 \tag{3}$$

The perfect gas equation of state is used in deriving Equation (3). ρ is the spatial density, v_i are the velocity components, t is time, τ_{ij} are the components of extra stress tensor, and p is the isotropic pressure. In the above equations, the comma notation for partial differentiation is used. $\tau_{ij} d_{ij}$ is the viscous dissipation term and the components of the rate of deformation tensor, d_{ij}, are as follows:

$$d_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \tag{4}$$

Constitutive Relation

The constitutive "linear" relation for compressible Newtonian fluids relating the extra stress tensor to the rate of deformation tensor is as follows:

$$\tau_{ij} = 2\eta_s d_{ij} + \eta_d d_{kk} \delta_{ij} \tag{5}$$

where η_s is the shear viscosity, η_d is the dilatational viscosity, and δ_{ij} is the Kronecker-delta. One way to relate the dilatational viscosity to the shear viscosity is to use the well-known Stokes assumption, $\eta_d + 2/3\eta_s = 0$.[15] The validity of the Stokes assumption is beyond the scope of this work, but one must be cautious about this assumption, as it is not valid in general as pointed out by Gad-El-Hak[15] or should be abandoned as pointed out by Rajagopal.[16]

Equation (5) is based on the concept of linear relationship, similar to Fourier's law of heat conduction. Usually Equation (5) is plugged into Equation (6) to yield the classical compressible Navier-Stokes equations, yielding a mixed parabolic-hyperbolic system. Being a mixed hyperbolic-parabolic system has many drawbacks, the most important of which are: (i) the existence of waves travelling at infinite speeds; and (ii) that the correct type and number of boundary conditions that yield a well-posed problem are not very well understood for these mixed-type systems. Note that for an inviscid fluid, $\eta_s = 0$ and $\eta_d = 0$, i.e. the Euler system of equations, all the problematic issues with the classical Navier-Stokes equations disappear.

Carrassi and Morro[2] suggest adding an artificial term representing a time-lag effect or viscous inertia, as follows:

$$\lambda \frac{\partial \tau_{ij}}{\partial t} + \tau_{ij} = 2\eta_s d_{ij} + \eta_d d_{kk} \delta_{ij} \tag{6}$$

where λ is a time-lag constant and its value is kept to a minimum to minimize the ability of the fluid to stress-relaxation. Adding this term overcomes the problem of having infinite wave speeds which yield nonphysical solutions to the problem of propagating transverse perturbations. Wilhelm and Hong[1] extended the work of Carrassi and Morro[2] by adding another term representing the convection effects, as follows:

$$\lambda \frac{\partial \tau_{ij}}{\partial t} + v_k \tau_{ij,k} + \tau_{ij} = 2\eta_s d_{ij} + \eta_d d_{kk} \delta_{ij} \tag{7}$$

Since the term added by Carrassi and Morro[2] vanishes in the steady-state case. With this simple modification, Wilhelm and Hong[1] succeeded in obtaining physical solutions in the form of a discontinuous wave with a finite wave speed due to viscous stress relaxation. The time derivative used in Equations (6, 7) violates the principle of material frame indifference,[17] which means that if one tries to use these equations to experimentally find the viscosity of a fluid, the attempt will result in as many values for the viscosity as the number of frames of references used in each experiment, although we are dealing with the same fluid. To remedy this flaw, one should use one of the standard codeformational time derivatives that is consistent with the material frame indifference principle defined in the classic paper by Oldroyd.[18] For compressible relative tensors of weight W, the compressible codeformational time derivative is as follows:[18]

$$\frac{d^{cc}[\,]_{ij}}{dt} = \frac{\partial [\,]_{ij}}{\partial t} + v_k [\,]_{ij,k} - v_{i,k}[\,]_{kj} - v_{j,k}[\,]_{ki} + W v_{k,k}[\,]_{ij} \tag{8}$$

where $W = 1$ for the extra stress tensor.[19] Note that this is the only codeformational time derivative that accounts for volumetric (dilatational) changes.

Using the codeformational time derivatives (8) instead of the partial or total time derivatives yields the final form of the modified constitutive relation as follows:

$$\lambda\left(\frac{\partial \tau_{ij}}{\partial t} + v_k \tau_{ij,k} - v_{i,k}\tau_{kj} - v_{j,k}\tau_{ki} + v_{k,k}\tau_{ij}\right) + \tau_{ij} = 2\eta_s d_{ij} + \eta_d d_{kk}\delta_{ij} \tag{9}$$

The modified constitutive relation (9) overcomes the main shortcoming of the Newtonian model as well as the drawbacks of the models proposed by Wilhelm and Hong[1] and Carrassi and Morro.[2] When (9) is coupled with the conservation laws, it renders a purely hyperbolic system of equations, i.e. with signals travelling with finite wave speeds as proven in following section. Note that the classical compressible Newtonian assumption is recovered when $\lambda = 0$.

NON-DIMENSIONALIZATION AND MATRIX FORMULATION

The non-dimensionalization scheme is as follows:

$$v_i^* = \frac{v_i}{c_\infty}, p^* = \frac{p}{\rho_\infty c_\infty^2} = \frac{p}{\gamma p_\infty}, \tau_{ij}^* = \frac{\tau_{ij}}{\rho_\infty c_\infty^2}, t^* = \frac{t c_\infty}{D}$$

where $c_\infty^2 = \dfrac{\gamma p_\infty}{\rho_\infty}$ is the speed of sound at free stream conditions.

In two-dimensional flow, the velocity vector is $\mathbf{v} = (u, v)$, and the extra stress tensor $\boldsymbol{\tau} = \begin{bmatrix} S & Q \\ Q & \sigma \end{bmatrix}$, where (u, v) are the velocity components in the axial and normal directions, respectively, and $(S,\ Q,\ \sigma)$ are the axial, shear, and normal extra-stress components, respectively. The non-dimensional system of equations represented by Equations (1–3) and (9) could be written in matrix form as follows:

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{A}_x \frac{\partial \mathbf{q}}{\partial x} + \mathbf{A}_y \frac{\partial \mathbf{q}}{\partial y} = \mathbf{r} \tag{10}$$

where $\mathbf{q} = [\rho \quad u \quad v \quad p \quad S \quad Q \quad \sigma]^T$ is the column of unknowns. The Jacobian matrices ($\mathbf{A}_x$, $\mathbf{A}_y$) and the right-hand side r are easy to derive and will not be presented to save space.

CLASSIFICATION OF THE SYSTEM OF EQUATIONS

Equation (10) represents a quasi-linear system of first-order partial differential equations. To treat such a system numerically, we must first classify it mathematically.[20] The system of equations (10) can be classified according to the eigenvalues of the matrix $\nu_1 \mathbf{A}_x + \nu_2 \mathbf{A}_y$, where ν_1 and ν_2 are arbitrary scalars.[20] The rigorous condition for hyperbolicity[20] requires the matrix $\nu_1 \mathbf{A}_x + \nu_2 \mathbf{A}_y$ to have real eigenvalues for every ν_1 and ν_2. Equivalently, according to the work of Edwards and Beris,[21] the criterion for hyperbolicity for Equation (10) reduces to the requirement that the matrix $\mathbf{A}_x$ or $\mathbf{A}_y$ has a full set of real eigenvalues. This condition, for arbitrary velocity and stress tensor fields, is both necessary and sufficient for the system of equations (10) to be fully hyperbolic. For our system of equations the eigenvalues without the viscous dissipation term, just for simplicity, for $\mathbf{A}_x$ are as follows:

$$\lambda_1 = u, \lambda_2 = u, \lambda_3 = u,$$

$$\lambda_4 = u + \sqrt{\frac{S}{\rho} + \frac{M_\infty^2}{\rho R_e W_e}}, \lambda_5 = u - \sqrt{\frac{S}{\rho} + \frac{M_\infty^2}{\rho R_e W_e}},$$

$$\lambda_6 = u + \sqrt{\frac{(\gamma p + 2S)}{\rho} + \frac{4M_\infty^2}{3\rho R_e W_e}},$$

$$\lambda_7 = u - \sqrt{\frac{(\gamma p + 2S)}{\rho} + \frac{4M_\infty^2}{3\rho R_e W_e}},$$

where $M_\infty = \frac{U_\infty}{c_\infty}$ is the free stream Mach number, $R_e = \frac{\rho_\infty U_\infty D}{\eta}$ is the Reynolds number, and $W_e = \frac{\lambda U_\infty}{D}$ is the Weissenberg number. One can easily show that all of the above eigenvalues are always real, and consequently that the system of Equation (10) is always purely hyperbolic. Including the viscous dissipation term will just make the eigenvalues lengthier.

Two important limits are considered:

1) The limit $W_\infty = 0$, where the classical formulation is recovered with its major problem of having waves with infinite speeds, namely the last four eigenvalues.
2) The incompressible limit, $M_\infty \approx 0$, where all the eigenvalues are still real and so the hyperbolic nature of the proposed formulation is maintained.

ADVANTAGES OF THE CURRENT FORMULATION

Being a totally hyperbolic system, as proven in the previous section, is of great importance for many reasons, the most important of which are the following:

(i) Being of a single, not mixed, type greatly simplifies the mathematical treatment both analytically and numerically. For example, no special treatment is needed in switching between different types of numerical schemes in the same numerical algorithm to account for the parabolic part of the system.
(ii) The boundary conditions can be determined without ambiguity, which may not be the case for other mixed types of systems.
(iii) All the waves are travelling at finite speeds, as proven in the previous section, which is more consistent with the physics.
(iv) Since it is much easier to deal with hyperbolic systems, the proposed system could be used in theoretical studies related to the well-posed nature of the classical formulation and taking the limit $W_e \approx 0$.
(v) The appearance of the stresses as primarily unknowns allows for the possibility of imposing boundary conditions in terms of the stresses. It is worth noting that this property is not available in the classical formulation or in many other proposed equivalent first-order systems.[22–24]

SOLUTION METHOD AND BOUNDARY CONDITIONS

Numerical Technique

A hybrid least squares finite element/finite difference scheme coupled with Newton-Raphson's linearization scheme[25] is used to solve the governing system of Equations (10).

Newton-Raphson's linearization

The nonlinear terms may be solved iteratively using Newton-Raphson's method, by setting the following:

$$\mathbf{q}^{n+1} = \mathbf{q}^n + \Delta\mathbf{q} \tag{11}$$

where $\Delta\mathbf{q}$ is the difference between the solution of two successive iterations $\mathbf{q}^n$ and $\mathbf{q}^{n+1}$. Neglecting the higher order terms. Equation (10) can be rewritten as follows:

$$\mathbf{L}\Delta\mathbf{q}^{n+1} = -\mathbf{f} \tag{12}$$

where:

$$\begin{aligned} \mathbf{L} &= \mathbf{A}_x^n \frac{\partial}{\partial x} + \mathbf{A}_y^n \frac{\partial}{\partial y} + (\frac{\mathbf{I}}{\Delta t} + \mathbf{A}^n) \\ \mathbf{f} &= (\mathbf{A}_x^n \frac{\partial \mathbf{q}^n}{\partial x} + \mathbf{A}_y^n \frac{\partial \mathbf{q}^n}{\partial y}) \end{aligned} \tag{13}$$

where $\mathbf{I}$ is the identity matrix.

Least-squares finite element/finite difference scheme

Consider the least squares weak form:

$$\iint_\Omega (\mathbf{LN})^T (\mathbf{L}\Delta\mathbf{q}^{n+1} + \mathbf{f}) d\Omega = 0 \tag{14}$$

Introducing the finite element approximation as follows:

$$\Delta\mathbf{q}^{n+1} \approx \Delta\mathbf{q}_h^{n+1} = \sum_{j=1}^{n_e} \mathbf{N}_j \Delta\mathbf{q}_j^{n+1} \tag{15}$$

where n_e is the number of nodes per element and $\mathbf{N}_j (j = 1, ...n_e)$ are the basis functions. Introducing the finite element approximation

(15) into Equation (14) results in the linear algebraic system of equations:

$$[\mathbf{K}]\{\Delta\mathbf{q}\} = -\{\mathbf{R}\} \tag{16}$$

where $\mathbf{K}_{ij}^e = \iint_{\Omega^e} (\mathbf{LN}_i)^T (\mathbf{LN}_j) d\Omega^e$, $\mathbf{r}_i^e = \iint_{\Omega^e} (\mathbf{LN}_i)^T \mathbf{f} d\Omega$

All integrations are evaluated numerically using Gauss-Legendre quadrature.

BOUNDARY CONDITIONS

One of the advantages of the proposed formulation is reflected in the ability to determine the correct type and number of boundary conditions to be imposed on a boundary since we are dealing with a purely hyperbolic system of equations. For a hyperbolic system of equations, the correct number and type of boundary conditions to be imposed on a boundary are determined by the theory of characteristics according to the incoming/outgoing characteristics. For the mathematical details we refer to one of these works: Thompson,[26] Dubois and Floch,[27] Kreiss,[28] Guaily and Epstein.[29]

NUMERICAL EXAMPLES

Opposed Wedge Flow (OWF)

The problem of shock/expansion fan/boundary layer interaction is investigated. An oblique shock is generated by a wedge which causes a deflection of the flow by an angle $\phi = 10^{\circ}$. The upstream Mach number is $M = 2.0$, the Reynolds number is $R_e = 10^7$, and the Weissenberg number is $W_e = 10^{-4}$ to minimize the effect of the added terms. The computational grid is 60×20 as shown in Figure 1. The computational domain is decomposed into three subdomains: the entrance region $L_1 = 0.2$ with 8 bilinear elements, the wedge region $L_2 = 0.6$ with 40 bilinear elements, and the exit region $L_3 = 0.2$ with 12 bilinear elements in the axial direction. Since we have a supersonic inlet and so all the eigenvalues are positive, all the variables have to be imposed at the inlet according to the work of Guaily and Epstein.[29] The inlet boundary conditions are $(\rho,\ u,\ v,\ p,\ S,\ Q,\ \sigma) = (1.0,\ 2.0,\ 0.0,\ 0.7143,\ 0.,\ 0.,\ .0)$ while the upper boundary is a symmetry line with $(v, Q) = (0.,\ 0.)$. The lower boundary is a wall with the no-slip condition. The exit boundary is a supersonic exit and so is unaffected by any boundary conditions since all the eigenvalues are positive. For details on the how the boundary conditions are determined, we refer to Guaily and Epstein[29] as they present a detailed and straightforward algorithm to determine the proper boundary conditions that depend directly on the eigenvalues.

Figure 2 shows the pressure isocontours for the supersonic internal viscous double wedge flow. The shock starts at the wedge corner to guide the supersonic flow along the wedge, then reflects from the symmetry line. The reflected shock exits the domain along with the expansion fan formed at the wedge end to redirect the flow. The figure shows this shock reflection/expansion fan structure for two different values for the time steps. As expected, as the time step decreases the shock is sharpened.

Similar to Figure 2, Figure 3 shows the Mach number contours for different time steps. Here the shock/boundary layer and the expansion fan/boundary layer interactions are clear. Figure 4 shows that the flow waits until reaching the shock formed at the

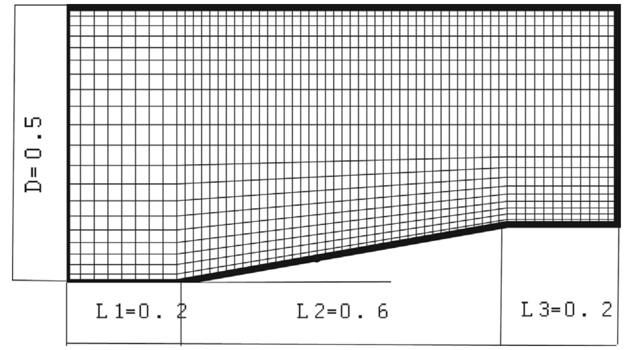

Figure 1. Geometry and grid for the OWF problem.

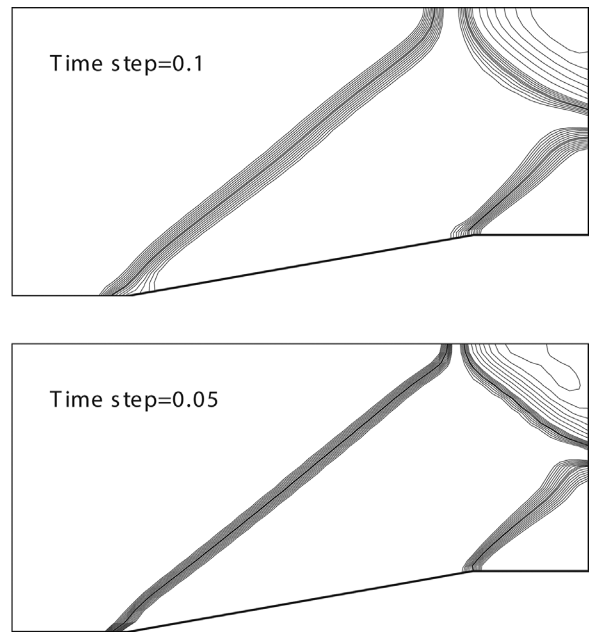

Figure 2. Pressure contours for the OWF problem.

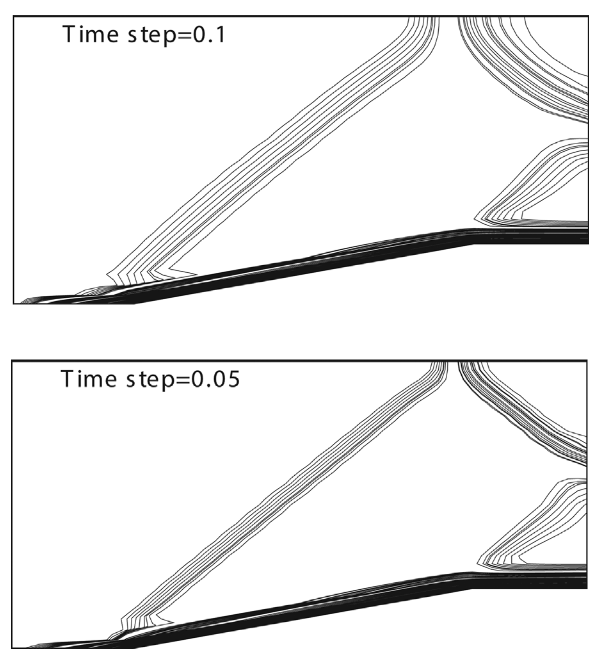

Figure 3. Mach number contours for the OWF problem.

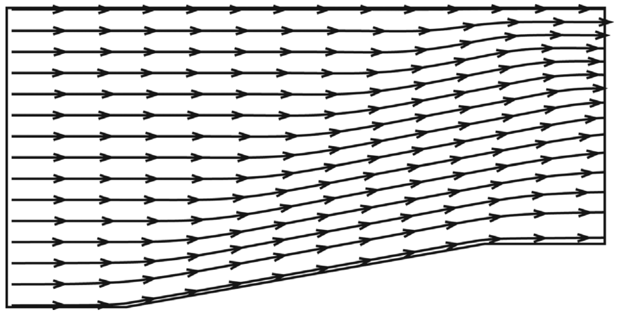

Figure 4. Redirecting the flow by shock wave, then by expansion fan, for the OWF problem.

wedge entrance to be turned. Then it waits again for the expansion fan to be returned.

The Mach number at the axis of symmetry is shown in Figure 5. The Mach number continues to be constant at the entrance value until it reaches the shock zone, where it decreases. The effect of the time step on the shock resolution is shown in Figure 5, where the time step changes from 0.1 to 0.05. As expected, oscillations near the shock start growing with the smaller time step and the shocks become more smeared by increasing it.

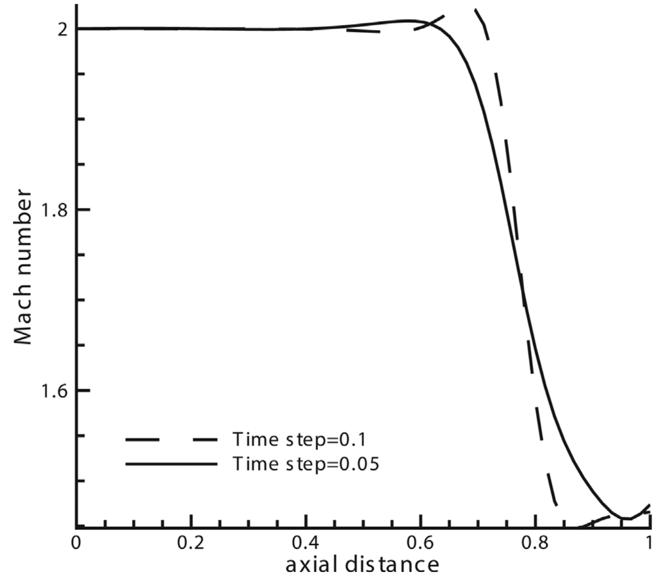

Figure 5. Mach number distribution at the centreline for different time steps for the OWF problem.

Viscous Shock Reflection (VSR)

The second example is shown is Figure 6: a compressible, viscous flow along a wall where the reflection of an oblique shock occurs. The computational domain is a 4.1×1.0 rectangle, which is discretized uniformly into 60×20 bilinear elements. Consider flow at an inlet Mach number of $M = 2.9$, $R_e = 10^7$, $W_e = 10^{-4}$ and specify the following boundary conditions:

- At the inlet, since we have a supersonic inlet, all the eigenvalues are positive and so according to the work of Guaily and Epstein[29] all the variables have to be imposed at the inlet, i.e. $(\rho, u, v, p, S, Q, \sigma) = (1.0, 2.9, 0.0, 0.7143, 0., 0., 0.)$.
- To simulate the incident oblique shock we specify the following at the top boundary $(\rho, u, v, p, S, Q, \sigma) = (1.7, 2.6193, -0.5063, 1.5282, 0., 0., 0.)$.
- At the lower boundary the no-slip boundary condition is imposed and the exit boundary is left free of any boundary conditions—a supersonic exit, since all the eigenvalues are positive.

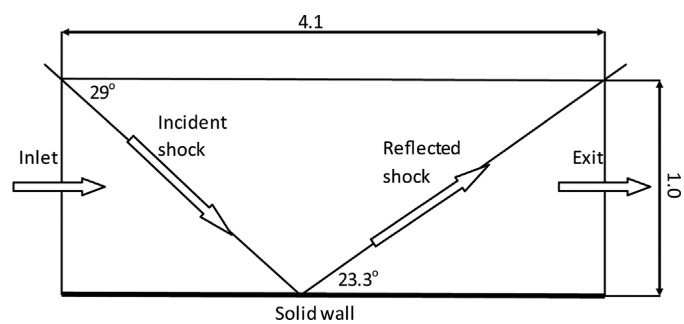

Figure 6. Physical domain for the VSR problem.

Figure 7 shows the Mach number contours for different time steps. The shock/boundary layer interaction is well captured in Figure 7 as well as the effect of the time step on the shock resolution, when the time step changes from 0.15 to 0.05. As expected, oscillations near the shock start growing after the time step is reduced and the shock becomes more smeared by increasing. The reflected shock exits the domain from the top right corner which means that the shock reflection angle is captured very well, as shown in Figure 7. Figure 8 shows the pressure distribution for the current work against the work of Taghaddosi et al.[30] Note that the results of Taghaddosi et al.[30] are for inviscid flow. As shown in Figure 8, the pressure increases after passing the incident shock then increases again after the reflected shock. The shock is smeared since the grid used is coarse due to the limited computational resources.

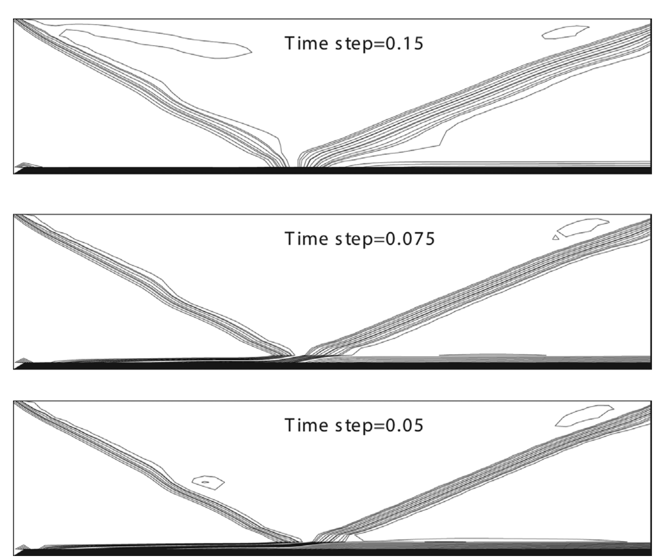

Figure 7. Mach number contours for the VSR problem.

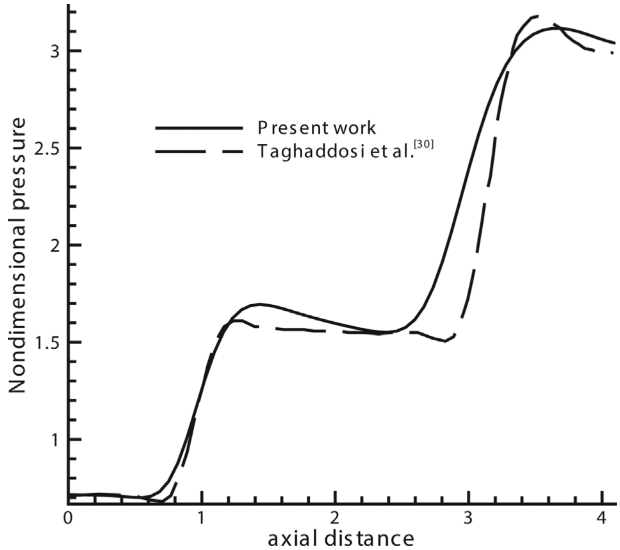

Figure 8. Comparison of the pressure distribution at y = 0.5 for the VSR problem.

CONCLUSIONS

The classical mixed parabolic-hyperbolic Navier-Stokes equations are modified. The new formulation overcomes many of the drawbacks of the classical formulation. The proposed system is consistent with the principle of material frame indifference and is purely hyperbolic with all the waves travelling at finite speeds. The boundary conditions can be determined without ambiguity, which may not be the case for the classical mixed type. The appearance of the stresses as primarily unknowns allows for the possibility of imposing boundary conditions in terms of the stresses. The new formulation is solved numerically using the finite element method and the results are obtained for the opposed wedge problem and the viscous shock reflection problem with an apparent degree of success.

NOMENCLATURE

ρ	fluid density (kg/m^3)
v_i	velocity components (m/s)
t	time (s)
τ_{ij}	components of extra stress tensor (Pa/m)
$\mathbf{p}$	isotropic pressure (Pa/m)
$\mathbf{d}_{ij}$	components of the rate of deformation tensor (s^{-1})
η_s	shear viscosity (Pa/s)
η_d	dilatational viscosity (Pa/s)
δ_{ij}	components of the Kronecker-delta
λ	time-lag coefficient (s)
$\mathbf{q}$	column of unknowns
$(\mathbf{A}_x, \mathbf{A}_y)$	Jacobain matrices
(S, Q, σ)	axial, shear, and normal extra-stress components respectively

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