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Schwarzschild—De Sitter Thin Shell Wormholes Supported by a Generalized Cosmic Chaplygin Gas

A. Eid[1,2*]

[1]*Department of of Physics, College of Science, Al-Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia*

[2]*Department of Astronomy, Faculty of Science, Cairo University, Giza, Egypt*

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Abstract—The dynamics of spherically symmetric thin-shell wormholes (TSW), supported by a generalized cosmic Chaplygin gas in Schwarzschild- de Sitter space-time, is studied using the cut-and-paste technique (the Darmois—Israel formalism). A mechanical stability analysis of spherically symmetric thin-shell wormhole is is carried out. by using the standard potential method. The existence of stable static solutions depends on the value of some parameters.

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1. INTRODUCTION

Traversable Lorentzian wormholes [1] are solutions of the equations of gravitation connecting two regions (of the same universe or two separate universes [2]) by a throat. A well-studied class of wormholes is that of thin-shell ones, constructed by cutting and pasting two regions [3] to form a geodesically complete new one with a shell placed on the joining surface (hypersurface). Bandyopadhyay and Chakraborty [4] studied the modified Chaplygin gas traversable wormholes. Eiroa [5] studied thin-shell wormholes with a generalized Chaplygin gas with the equation of state (EoS) $p\sigma^\alpha = -A$, $0 < \alpha \le 1$. Sharif and Azam [6] discussed thin-shell wormholes in a modified Chaplygin gas (with the EoS $p = A\sigma - B/\sigma^\alpha$, $A > 0$, $B > 0$ and $0 < \alpha \le 1$).

The stability of static wormholes was investigated either using a particular EoS or under linearized radial perturbations around a static solution. Poisson and Visser [7] discussed linear perturbations of a thin-shell wormhole (TSW) by pasting together two copies of the Schwarzschild solution. Lobo and Crawford [8] studied linearized stability of TSWs with a cosmological constant (in Schwarzschild—de Sitter space-time). Eiroa and Simeon [9] analyzed the stability of a Chaplygin gas TSW. Recently, Sharif and Azam [10] investigated Reissner Nordstrom TSWs with a generalized cosmic Chaplygin gas. Also, Azam [11] discussed Born—Infeld TSWs supported by a generalized cosmic Chaplygin gas. Eiroa [12], Rahaman et al. [13] and Usmani et al. [14] discussed the stability being concentrated on the parameter β^2, normally interpreted as the speed of sound.

In this paper, we are going to discuss the mechanical stability of a TSW with a generalized cosmic Chaplygin gas in Schwarzschild—de Sitter spacetime. In Section 2, the dynamics of TSW with a generalized cosmic Chaplygin gas in Schwarzschild—de Sitter spacetime is discussed. An outline of a general linearized stability analysis procedure is given in Section 3. A general conclusion is provided in Section 4.

2. DYNAMICS OF A THIN-SHELL WORMHOLE

The metric of a Schwarzschild-(anti-)de Sitter TSW is given by

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (1)$$

$$F(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \qquad (2)$$

where m is the gravitational mass, Λ is the cosmological constant. Let the equation of the shell be $r = R(\tau)$, the function $R(\tau)$ describes a time evolution of the shell. The intrinsic metric on Σ is

$$ds^2 = -d\tau^2 + R^2(\tau)(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (3)$$

where τ is proper time on the shell. Applying the Darmois-Israel formalism [15] to matter on the hypersurface Σ, the extrinsic curvature associated with two sides of the shell is defined as

$$K^{\pm}_{ij} = -n^{\pm}_{\gamma}\left(\frac{\partial^2 x^\gamma}{\partial\xi^i\partial\xi^j} + \Gamma^{\gamma}_{\alpha\beta}\frac{\partial x^\alpha}{\partial\xi^i}\frac{\partial x^\beta}{\partial\xi^j}\right), \qquad (4)$$

*E-mail: amaid@imamu.edu.sa

where $n_{\gamma}^{\pm}$ are the unit normal 4-vectors. The Einstein equations determine relations between the 3D intrinsic energy-momentum tensor and the extrinsic curvature through the Lanczos equations

$$t_{ij} = \frac{-1}{8\pi}([K_{ij}] + [K]g_{ij}), \quad (5)$$

where $[K]$ is the trace of $[K_{ij}] = K_{ij}^{+} - K_{ij}^{-}$, and t_{ij} is the surface stress-energy tensor on the hypersurface Σ, $t_j^i = \mathrm{diag}(-\sigma, p_\varphi, p_z)$, where σ and p are the surface energy density and pressure [9]. For a spherically symmetric thin shell, the Lanczos equations are then reduced to

$$\sigma = \frac{-1}{4\pi}[K_\theta^\theta], \quad (6)$$

$$p = p_\varphi = p_z = \frac{1}{8\pi}([K_\tau^\tau] + [K_\theta^\theta]). \quad (7)$$

These equations become

$$\sigma = \frac{-1}{2\pi R}\sqrt{\dot{R}^2 + F(R)}, \quad (8)$$

$$p = \frac{1}{8\pi R}\frac{2R\ddot{R} + 2F(R) + RF'(R) + 2\dot{R}^2}{\sqrt{\dot{R}^2 + F(R)}}, \quad (9)$$

where the dot and the prime denote derivatives with respect to τ and R, respectively. The equation of state (EoS) of the generalized cosmic Chaplygin gas is given by

$$p = -\frac{1}{\sigma^\gamma}\left[E + (\sigma^{1+\gamma} - E)^{-\omega}\right], \quad E = \frac{B}{1+\omega} - 1,$$
$$B \in (-\infty, \infty), \quad 0 < \gamma \leq 1, \quad -A < \omega < 0, \quad (10)$$

where $A > 1$ is a constant. This equation in the limit $\omega \to 0$ reduces to that of generalized Chaplygin gas. Inserting Eqs. (8) and (9) into Eq. (10), the dynamical equation becomes

$$(2R)^\gamma \left(R^2(2\ddot{R} + F'(R)) + 2R(F(R) + \dot{R}^2)\right) - 2(4\pi R^2)^{1+\gamma}\left[F(R) + \dot{R}^2\right]^{\frac{1}{2}(1-\gamma)} \times \left(E - E^{-\omega} + (2\pi R)^{\omega(1+\gamma)}(F(R) + \dot{R}^2)^{-\frac{1}{2}\omega(1+\gamma)}\right) = 0. \quad (11)$$

This differential equation should be satisfied by the throat radius of a TSW threaded by exotic matter with the EoS of a generalized cosmic Chaplygin gas. It is convenient to define the parameter space of the problem using m, γ, E, ω, and Λ as free parameters.

3. LINEARIZED STABILITY ANALYSIS

The dynamical equation (11) for the static solution (where $\dot{R} = \ddot{R} = 0$), becomes

$$(2R_0)^\gamma \left(R_0^2 F'(R_0) + 2R_0 F(R_0)\right) - 2(4\pi R_0^2)^{1+\gamma}\left[F(R_0)\right]^{\frac{1}{2}(1-\gamma)} \Big(E - E^{-\omega} + (2\pi R_0)^{\omega(1+\gamma)}(F(R_0))^{-\frac{1}{2}\omega(1+\gamma)}\Big) = 0. \quad (12)$$

The surface energy density and pressure are in the static

$$\sigma_0 = \frac{-1}{2\pi R_0}\sqrt{F(R_0)},$$
$$p = \frac{1}{8\pi R_0}\frac{2F(R_0) + R_0 F'(R_0)}{\sqrt{F(R_0)}}. \quad (13)$$

The conservation equation with (5) and (13) can be written as

$$\frac{d}{d\tau}\sigma A + p\frac{dA}{d\tau} = 0, \quad (14)$$

where $A = 4\pi R^2$ is the area of the wormhole throat. This equation represents the continuity equation for the energy-momentum tensor (i.e., the change in internal energy of the throat plus the work done by the throat's internal forces), and can be written in the form

$$\frac{d}{d\tau}\sigma = \frac{-2}{R}(\sigma + p)\frac{dR}{d\tau}, \quad (15)$$

or

$$R\sigma' = -2(\sigma + p). \quad (16)$$

From Eq. (8), the dynamic equation of motion of the TSW becomes

$$\dot{R}^2 + V(R) = 0 \quad (17)$$

where $V(R)$ is known as the potential function:

$$V(R) = F(R) - 4\pi^2 R^2 \sigma^2(R). \quad (18)$$

Its derivative is

$$V'(R) = F'(R) + 8\pi^2 R\sigma(R)(\sigma + 2p). \quad (19)$$

Taking the first derivative with respect to R in Eq. (10) and using (16), we obtain

$$\sigma' + 2p' = \sigma'\Big[2\omega(1+\gamma)(\sigma^{1+\gamma} - E)^{-1-\omega} + 1 - \frac{2\gamma p}{\sigma}\Big]. \quad (20)$$

The Taylor series expansion of $V(R)$ up to second order around R_0 is given by

$$V(a) = \sum_{n=0}^{2} b_n (a - a_0)^n, \quad b_n = \frac{V^{(n)}(a_0)}{n!}. \quad (21)$$

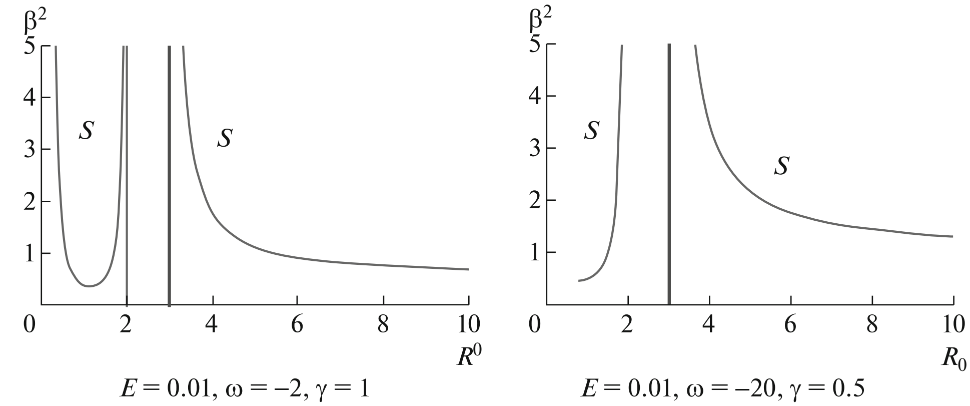

Fig. 1. Stability regions for TSW corresponding to $\Lambda = 0.0$ and the fixed value $m = 1$; the symbol S denotes stability regions.

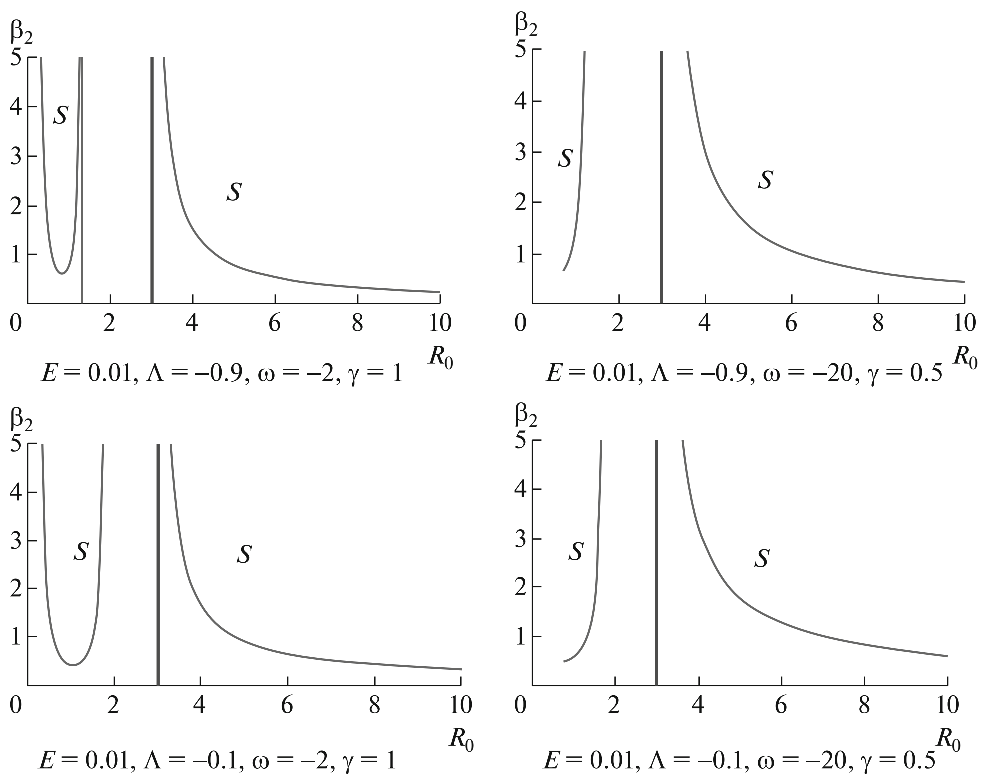

Fig. 2. Stability regions for TSWs corresponding to $\Lambda = -0.9, -0.1$ and the fixed value $m = 1$; S denoted stability regions.

The second derivative of $V(R)$ is

$$V''(R) = F''(R) - 8\pi^2\Big[(\sigma+2p)^2 + 2\sigma(\sigma+p) \times \left[1 - 2\gamma\beta^2 + 2\omega(1+\gamma)(\sigma^{1+\gamma} - E)^{-1-\omega}\right]\Big], \quad (22)$$

where $\beta^2 = p/\sigma$ is the squared sound velocity. The stability of static solutions at $R = R_0$ requires $V(R_0) = 0$ and $V'(R_0) = 0$, while $V''(R_0)$ becomes

$$V''(R_0) = F''(R_0) + \frac{(\gamma-1)F'^2(R_0)}{2F_0} + \left[1 - 2\omega(1+\gamma)Q\right]\frac{F'(R_0)}{R_0} - \frac{2F_0}{R_0^2}(1+\gamma)\left[1 - 2\omega Q\right],$$

$$Q = \left[\left(\frac{\sqrt{F_0}}{2\pi R_0}\right)^{1+\gamma} + E\right]^{-1-\omega},$$

$$F_0 = F(R_0) = 1 - 2m/R_0 - \Lambda R_0^2/3. \qquad (23)$$

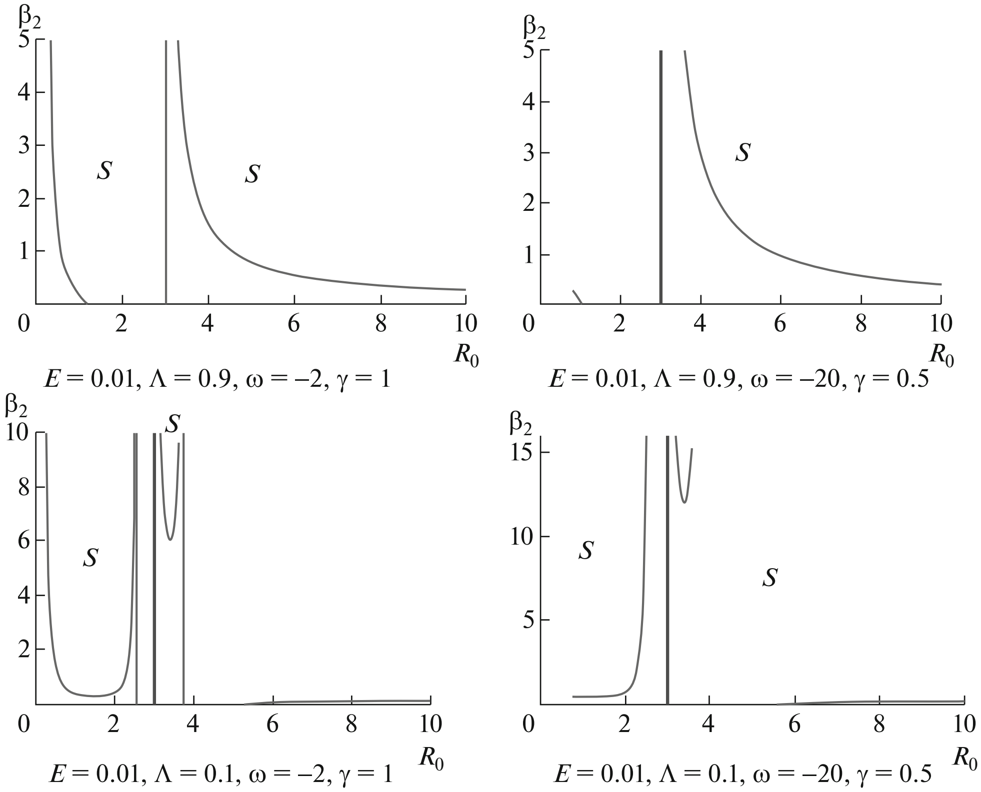

Fig. 3. Stability regions TSWs corresponding to $\Lambda = 0.9, 0.1$ and the fixed value $m = 1$; S denoted stability regions.

The surface energy density and pressure (13) can be written in the form

$$\sigma_0 = \frac{-1}{2\pi R_0\sqrt{3R_0}}\sqrt{3R_0 - 6m - \Lambda R_0^3}, \qquad (24)$$

$$p_0 = \frac{1}{4\pi R_0}\frac{3R_0 - 3m - 2\Lambda R_0^3}{\sqrt{3R_0(3R_0 - 6m - \Lambda R_0^3)}}. \qquad (25)$$

Using these equations in (12) and (23), the dynamic equation and the second derivative of the potential become

$$R_0 - m - \frac{2}{3}\Lambda R_0^3 - 2(2\pi)^{1+\gamma}R_0^{2+\gamma}F_0^{\frac{1}{2}(1-\gamma)} \times \left[E - E^{-\omega} + \left((2\pi)^{\omega}R_0^{\frac{1}{2}(1+\omega)}F_0^{-\frac{1}{2}\omega}\right)^{1+\gamma}\right] = 0, \qquad (26)$$

and

$$V''(R_0) = \frac{1}{R_0^4F_0}\Bigg[-2\left(2m + \frac{1}{3}\Lambda R_0^3\right)R_0F_0 + 2(\gamma - 1)\left(m - \frac{1}{3}\Lambda R_0^3\right)^2 + 2\left(m - \frac{1}{3}\Lambda R_0^3\right)R_0F_0[1 - 2\omega(1+\gamma)H^{-1-\omega}] - 2(1+\gamma)(R_0F_0)^2(1 - 2\omega H^{-1-\omega})\Bigg], \qquad (27)$$

where

$$H = E + (2\pi R_0)^{-1-\gamma}F_0^{\frac{1}{2}(1+\gamma)}.$$

Thus $\dot{R}^2 = -\frac{1}{2}V''(R_0)(R - R_0)^2 + O[(R - R_0)^3]$. The TSW is stable under radial perturbations if and only if $V''(R_0) > 0$, so that the motion of the throat is oscillatory with an angular frequency equal to $\sqrt{\frac{1}{2}V''(R_0)}$. But the TSW is unstable if $V''(R_0) < 0$, so that the motion will be exponential toward the initial perturbation. From Eq. (22), by letting $V''(R_0) = 0$, the squared sound velocity β^2 is given by

$$\beta^2 = \frac{1}{2\gamma} + \frac{\omega}{\gamma}(1+\gamma)\left[-E + \left[\frac{-\sqrt{F_0}}{2\pi R_0}\right]^{1+\gamma}\right]^{-1-\omega} + \frac{(2m/R_0)(1 - 3m/(2R_0) - \Lambda R_0^2) + \Lambda R_0^2/3}{2\gamma(1 - 3m/R_0)F_0}. \qquad (28)$$

The plot of this expression has a vertical asymptote. The right and left of the asymptote depends on the inequality, $\gamma(R_0 - 3m)R_0F_0 > 0$ and $\gamma(R_0 - 3m)R_0F_0 < 0$. Returning to Eq. (22), letting

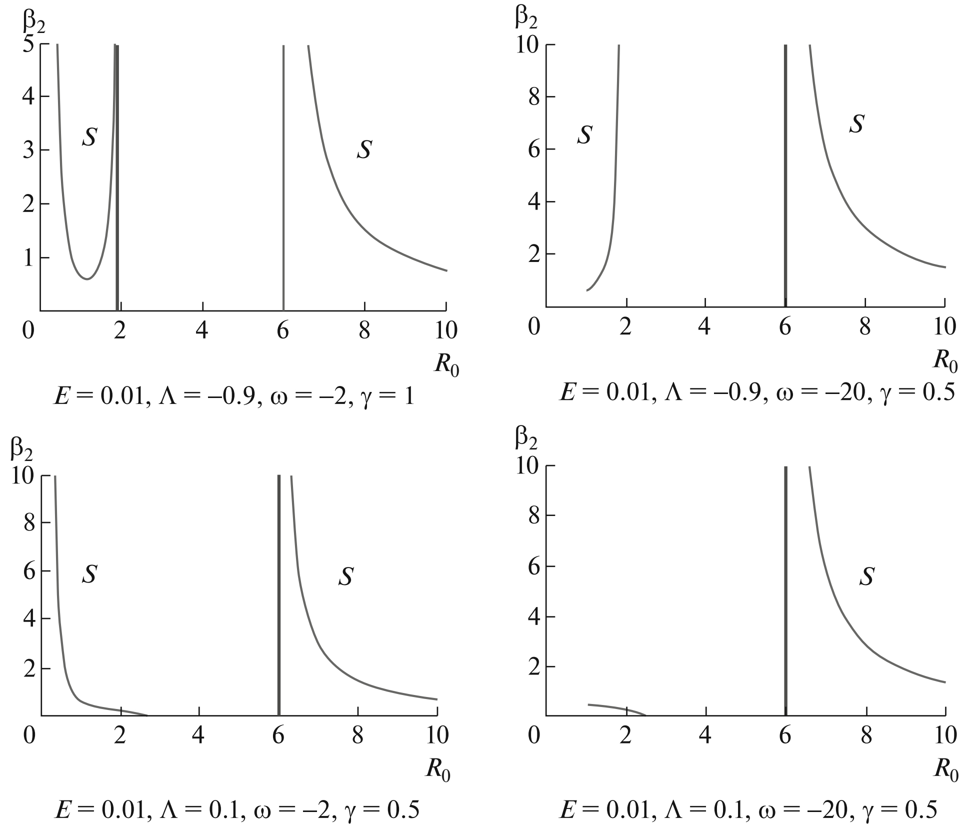

Fig. 4. Stability regions of TSWs corresponding to $\Lambda = -0.9$, 0.1 and the fixed value $m = 2$; S denotes stability regions.

$V''(R_0) > 0$, for the squared speed of sound β^2 we have

$$\beta^2 > \frac{1}{2\gamma(R_0 - 3m)R_0 F_0}\left[R_0(R_0 - 3m) + 3m\left(m - \frac{1}{3}\Lambda R_0^3\right)\right] + \frac{2\omega(\gamma+1)}{2\gamma}\left[-E + \left(\frac{-\sqrt{F_0}}{2\pi R_0}\right)^{1+\gamma}\right]^{-1-\omega}, \quad (29)$$

and the opposite inequality.

Variation of β^2 versus R_0 is plotted in Figs. 1–4 with different values of m, γ, E, ω, and Λ as free parameters. The regions of stability corresponding to $\Lambda = 0.0$ for the fixed value of $m = 1$ are shown in Fig. 1. These represent the stability regions of a Schwarzschild wormhole when $\Lambda = 0.0$, with $R_0 > 2m$.

The typical regions of stability corresponding to $\Lambda = -0.9$, -0.1 for the fixed value $m = 1$ are shown in Fig. 2. The stability regions can be extended by reducing the value of the cosmological constant. The stability regions corresponding to $\Lambda = 0.9$, 0.1 for the fixed value $m = 1$ are shown in Fig. 3. The stability regions for $\Lambda > 0$ are not of the typical form like in the case of a negative cosmological constant. The stability regions corresponding to $\Lambda = -0.9$, 0.1 for the fixed value $m = 2$ are shown in Fig. 4.

4. CONCLUSION

Spherically symmetric thin-shell wormholes supported by a generalized cosmic Chaplygin gas are constructed by using the usual cut and paste procedure (the Darmois-Israel formalism). This kind of exotic matter has been recently considered to be of particular interest in cosmology as it provides a possible explanation for the observed accelerated expansion of the Universe. The stability analysis of thin-shell wormholes with the equation of state of a generalized cosmic Chaplygin gas, with linearized spherically symmetric perturbation about a static equilibrium solution, has been carried out. The energy density and pressure at the throat were found as functions of the throat radius.

The EoS of a generalized cosmic Chaplygin gas reduces to that of a generalized Chaplygin gas if $\omega \to 0$ [5] and Chaplygin gas for $\gamma = 1$ [16]. The cosmological constant increases the linear stability region of spherically symmetric TSWs. This shows that the choice of the EoS plays a vital role in the existence of stable wormhole solutions.

The TSW is stable under radial perturbations if and only if $V''(R_0) > 0$, while for $V''(R_0) < 0$ the static solution is unstable. The TSWs can be stable or unstable, depending on the mass m, the parameters γ, E, ω, Λ, and the initial position R_0 of the dynamic shell. The stability regions have been plotted in the form of parameter β^2.

Some authors reported the stability analysis to $0 \leq \beta^2 \leq 1$. Eiroa [12] found that stable configurations did not lie in a realistic range, i.e., $0 < \beta^2 \leq 1$, for dilatonic TSWs. However, Rahaman et al. [13] and Usmani et al. [14] found stable configurations in this range for regular charged black holes as well as charged black holes in generalized dilaton-axion gravity, respectively. By this paper, the stability regions can be made fall in this range for the case of negative and positive cosmological constants by varying the range of a fixed value of mass. This shows the physical relevance of negative and positive cosmological constants to wormhole stability; these stable configurations are similar to those of Eiroa [12], Rahaman et al. [13], and Usmani et al. [14] for different values of the parameters.

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