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HYERS–ULAM AND HYERS–ULAM–RASSIAS STABILITY OF FIRST-ORDER LINEAR DYNAMIC EQUATIONS

Maryam A. Alghamdi, Alaa Aljehani, Martin Bohner, and Alaa E. Hamza

Abstract. We present several new sufficient conditions for Hyers–Ulam and Hyers–Ulam–Rassias stability of first-order linear dynamic equations for functions defined on a time scale with values in a Banach space.

1. Preliminaries and Introduction

In 1940, Ulam [**28**] proposed to "give conditions in order for a linear mapping near an approximately linear mapping to exist". The case of approximately additive mappings was solved by Hyers [**12**], who proved that the Cauchy equation is stable in Banach spaces. Since then, this type of stability, founded by Ulam and Hyers, is famed for Hyers–Ulam stability. There appeared hundreds of papers concerning Hyers–Ulam stability, due to its applications in control theory, numerical analysis, and other areas of applied mathematics. In 1978, Rassias [**21**] extended the Hyers–Ulam stability concept and called it Hyers–Ulam–Rassias stability. For more details, we refer the reader to the monograph of Jung [**14**]. In 1998, Alsina and Ger [**1**] were the first authors to investigate Hyers–Ulam stability of the differential equation

$$\psi' - \psi = 0$$

and to obtain a Hyers–Ulam stability constant 3 on a real interval. Hyers–Ulam stability of linear differential equations of first order was also investigated in [**13**, **22**, **23**, **29**]. Generalizations of these results were offered by Miura and others in [**17–19**, **27**]. Popa proved Hyers–Ulam stability of linear recurrence equations with constant coefficients [**20**]. For more studies concerning difference equations, we also refer to [**4**, **5**, **7**, **10**, **11**]. Recently, several articles have studied Hyers–Ulam stability of dynamic equations on time scales [**2**, **3**, **6**, **15**, **16**, **24**, **26**, **30**].

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In this paper, we investigate new sufficient conditions for Hyers–Ulam and Hyers–Ulam–Rassias stability of first-order linear dynamic equations on time scales of the form

$$\psi^{\Delta}(t) + \wp(t)\psi(t) = f(t), \quad t \in \mathcal{I}^{\kappa}, \tag{1.1}$$

where $\mathcal{I} = [a,b] \cap \mathbb{T}$ with a time scale $\mathbb{T} \subset \mathbb{R}$, $a,b \in \mathbb{T}$, $a < b$, $\wp \in \mathrm{C_{rd}}(\mathcal{I},\mathbb{R})$, $f \in \mathrm{C_{rd}}(\mathcal{I},\mathbb{X})$, and $\mathbb{X}$ is a Banach space. Our results depend basically on finding an equivalent integral equation to (1.1). The main result of the paper is a sufficient condition for (1.1) to have Hyers–Ulam stability, namely, the existence of a unique solution ψ of (1.1) satisfying the initial condition $\psi(a) = x_0$ for any initial value $x_0 \in \mathbb{X}$.

For the terminology and notations used here, we refer the reader to Bohner and Peterson [**8**, **9**]. Here, we only recall the dynamic version of Gronwall's inequality, as it is an essential tool in our investigations.

Theorem 1.1 (See [**8**, Theorem 6.4]). *Let be given functions $y, f \in \mathrm{C_{rd}}(\mathcal{I},\mathbb{R})$ and $p \in \mathrm{C_{rd}}(\mathcal{I},[0,\infty))$. Then*

$$y(t) \leqslant f(t) + \int_a^t y(s)p(s)\Delta s \quad \textit{for all } t \in \mathcal{I}$$

implies

$$y(t) \leqslant f(t) + \int_a^t e_p(t,\sigma(s))f(s)p(s)\Delta s \quad \textit{for all } t \in \mathcal{I}.$$

In this paper, we denote the norm on the Banach space $\mathbb{X}$ by $\|\cdot\|$, and for a bounded function $f : \mathcal{I} \to \mathbb{X}$, we also use the notation $\|f\|_\infty = \sup_{t\in\mathcal{I}} \|f(t)\|$.

2. An Existence and Uniqueness Result

Let $\wp \in \mathrm{C_{rd}}(\mathcal{I},\mathbb{R})$. It is well known that if $\wp$ is regressive, i.e., $1 + \mu(t)\wp(t) \neq 0$ for all $t \in \mathcal{I}$, then (1.1) has a unique solution ψ satisfying the initial condition $\psi(a) = x_0$, for every $x_0 \in \mathbb{X}$ (see [**8**, §8.2]). Another sufficient condition for the existence of a unique solution of initial value problems involving (1.1) is established in this section. We begin by the following lemma.

Lemma 2.1. *ψ solves (1.1) if and only if ψ satisfies the integral equation*

$$\psi(t) = x_0 - \int_a^t (\wp(s)\psi(s) - f(s))\Delta s \quad \textit{for all } t \in \mathcal{I} \tag{2.1}$$

for some constant $x_0 \in \mathbb{X}$.

Proof. If ψ satisfies (1.1), then we can integrate ψ^{Δ} from a to t to see that (2.1) holds with $x_0 = \psi(a)$. Conversely, if ψ satisfies (2.1), then we can differentiate ψ to see that (1.1) holds. □

Corollary 2.1. *For $x_0 \in \mathbb{X}$, (1.1) has at most one solution ψ satisfying $\psi(a) = x_0$.*

PROOF. Let $x_0 \in \mathbb{X}$. Assume that ψ_1 and ψ_2 are solutions of (1.1) with $\psi_1(a) = \psi_2(a) = x_0$. Then, by Lemma 2.1, both ψ_1 and ψ_2 satisfy (2.1). This implies

$$\|\psi_1(t) - \psi_2(t)\| \leqslant \int_a^t |\wp(s)| \, \|\psi_1(s) - \psi_2(s)\| \, \Delta s \quad \text{for all } t \in \mathcal{I}.$$

Now we let in Gronwall's inequality, Theorem 1.1, $y = \|\psi_1 - \psi_2\|$, $f = 0$, and $p = |\wp|$. Hence, the assumptions in Theorem 1.1 are satisfied, and

$$y(t) \leqslant f(t) + \int_a^t y(s)p(s)\Delta s \quad \text{for all } t \in \mathcal{I}$$

holds. Thus, by Theorem 1.1,

$$y(t) \leqslant f(t) + \int_a^t e_p(t, \sigma(s))f(s)p(s)\Delta s = 0 \quad \text{for all } t \in \mathcal{I},$$

i.e., $\|\psi_1(t) - \psi_2(t)\| \leqslant 0$ for all $t \in \mathcal{I}$, so $\psi_1 = \psi_2$. □

THEOREM 2.1. *Assume that there exists $\alpha \in (0, 1)$ such that*

$$\int_a^t |\wp(s)| \, \Delta s \leqslant \alpha \quad \textit{for all } t \in \mathcal{I}. \tag{2.2}$$

If $x_0 \in \mathbb{X}$, then (1.1) has a unique solution ψ satisfying $\psi(a) = x_0$.

PROOF. Fix $x_0 \in \mathbb{X}$. Define the operator $T : \mathrm{C_{rd}}(\mathcal{I}, \mathbb{X}) \to \mathrm{C_{rd}}(\mathcal{I}, \mathbb{X})$ by

$$T\psi(t) := x_0 - \int_a^t (\wp(s)\psi(s) - f(s))\Delta s, \quad t \in \mathcal{I}.$$

For $\psi_1, \psi_2 \in \mathrm{C}(\mathcal{I}, \mathbb{X})$, we have

$$\|T\psi_1(t) - T\psi_2(t)\| \leqslant \|\psi_1 - \psi_2\|_\infty \int_a^t |\wp(s)| \, \Delta s \leqslant \alpha \, \|\psi_1 - \psi_2\|_\infty \, , \quad t \in \mathcal{I},$$

Hence, $\|T\psi_1 - T\psi_2\|_\infty \leqslant \alpha \, \|\psi_1 - \psi_2\|_\infty$, so T is a contraction. Therefore, T has a unique fixed point ψ, which is the unique solution of (2.1) satisfying $\psi(a) = x_0$. Thus, by Lemma 2.1, ψ is the unique solution of (1.1) satisfying $\psi(a) = x_0$. □

REMARK 2.1. Assume $\mathbb{T}$ is a discrete time scale. We can see that if there exists $\alpha \in (0, 1)$ such that (2.2) holds, then $\wp$ is regressive, and the converse is not true. Indeed, if $\wp$ is nonregressive, then there exists $t_0 \in \mathcal{I}^\kappa$ such that $\wp(t_0)\mu(t_0) = -1$. We have

$$\int_a^b |\wp(s)| \, \Delta s = \sum_{t \in \mathcal{I}^\kappa} |\wp(t)| \, \mu(t) \geqslant |\wp(t_0)| \, \mu(t_0) = 1.$$

This is a contradiction. To see that the converse is not true, the function $\wp(t) \equiv 1$ is regressive on $\mathbb{N}_0$, but (2.2) does not hold for any $\alpha \in (0, 1)$.

3. Hyers–Ulam Stability

In this section, we assume that $\wp \in \mathrm{C_{rd}}(\mathcal{I}, \mathbb{R})$ and $f \in \mathrm{C_{rd}}(\mathcal{I}, \mathbb{X})$. We investigate Hyers–Ulam stability of (1.1). First, we recall the concept of Hyers–Ulam stability, see [**3**, Definition 3.1].

Definition 3.1 (Hyers–Ulam Stability). We say that (1.1) has Hyers–Ulam stability if there exists a constant $L > 0$, a so-called HUS constant, with the following property. For any $\varepsilon > 0$, if $\psi \in \mathrm{C^1_{rd}}(\mathcal{I}, \mathbb{X})$ is such that

$$\left\| \psi^{\Delta}(t) + \wp(t)\psi(t) - f(t) \right\| \leqslant \varepsilon \quad \text{for all } t \in \mathcal{I}^{\kappa},$$

then there exists a solution $\phi : \mathcal{I} \to \mathbb{X}$ of (1.1) such that $\|\psi(t) - \phi(t)\| \leqslant L\varepsilon$ for all $t \in \mathcal{I}$.

The following result establishes a new sufficient condition for Hyers–Ulam stability of (1.1). We introduce the assumption

(H) For any $x_0 \in \mathbb{X}$, (1.1) has a solution ϕ satisfying $\phi(a) = x_0$.

Theorem 3.1. *If* (H) *holds, then* (1.1) *has Hyers–Ulam stability with HUS constant* $L := (b-a)e_{|\wp|}(b,a)$.

Proof. Note that $|\wp| \in \mathrm{C_{rd}}(\mathcal{I}, [0,\infty))$, and so L is well defined and $L > 0$. Let $\varepsilon > 0$. Suppose $\psi \in \mathrm{C^1_{rd}}(\mathcal{I}, \mathbb{X})$ is such that

$$\left\| \psi^{\Delta}(t) + \wp(t)\psi(t) - f(t) \right\| \leqslant \varepsilon \quad \text{for all } t \in \mathcal{I}^{\kappa}. \tag{3.1}$$

Defining $h(t) := \psi^{\Delta}(t) + \wp(t)\psi(t) - f(t)$, we see that $h \in \mathrm{C_{rd}}(\mathcal{I}, \mathbb{X})$. Moreover, ψ satisfies the equation $\psi^{\Delta}(t) + \wp(t)\psi(t) = f(t) + h(t)$ for all $t \in \mathcal{I}$. Let $x_0 = \psi(a)$. By Lemma 2.1,

$$\psi(t) = x_0 - \int_a^t (\wp(s)\psi(s) - (f(s) + h(s)))\Delta s \quad \text{for all } t \in \mathcal{I}. \tag{3.2}$$

By (H), there exists a solution ϕ of (1.1) satisfying $\phi(a) = x_0$. Equivalently, by Lemma 2.1,

$$\phi(t) = x_0 - \int_a^t (\wp(s)\phi(s) - f(s))\Delta s \quad \text{for all } t \in \mathcal{I}. \tag{3.3}$$

Subtracting (3.3) from (3.2), we find, for all $t \in \mathcal{I}$,

$$\begin{aligned}
\|\psi(t) - \phi(t)\| &= \left\| \int_a^t h(s)\Delta s + \int_a^t \wp(s)(\phi(s) - \psi(s))\Delta s \right\| \\
&\leqslant \int_a^t \|h(s)\| \,\Delta s + \int_a^t |\wp(s)| \,\|\psi(s) - \phi(s)\| \,\Delta s \\
&\overset{(3.1)}{\leqslant} \varepsilon(t-a) + \int_a^t |\wp(s)| \,\|\psi(s) - \phi(s)\| \,\Delta s \\
&\leqslant \varepsilon(b-a) + \int_a^t |\wp(s)| \,\|\psi(s) - \phi(s)\| \,\Delta s.
\end{aligned}$$

Thus, by Gronwall's inequality, Theorem 1.1, we obtain, for all $t \in \mathcal{I}$,

$$\begin{aligned}\|\psi(t)-\phi(t)\| &\leqslant \varepsilon(b-a)+\int_a^t e_{|\wp|}(t,\sigma(s))\,|\wp(s)|\,\varepsilon(b-a)\Delta s\\ &= \varepsilon(b-a)\left(1+\int_a^t e_{|\wp|}(t,\sigma(s))\,|\wp(s)|\,\Delta s\right)\\ &= \varepsilon(b-a)\left(1+e_{|\wp|}(t,a)-e_{|\wp|}(t,t)\right)\\ &= (b-a)e_{|\wp|}(t,a)\varepsilon \leqslant (b-a)e_{|\wp|}(b,a)\varepsilon = L\varepsilon,\end{aligned}$$

where we used [**8**, Theorem 2.39] to evaluate the integral, the fact that $e_{|\wp|}(t,t)=1$, and the nondecreasing nature of $e_{|\wp|}(\cdot,a)$, which follows from the fact that the derivative of $e_{|\wp|}(t,s)$ with respect to t (for fixed s) is $|\wp(t)|\,e_{|\wp|}(t,s)$. Therefore, (1.1) indeed has Hyers–Ulam stability. □

Remark 3.1. Note that Theorem 3.1 coincides with the result given in [**25**, Corollary 2.3] in case of $\mathbb{T}=\mathbb{R}$, where the constant L is found as

$$L=(b-a)\exp\left(\int_a^b |\wp(t)|\,\mathrm{d}t\right).$$

Since a regressive equation always has a unique solution satisfying any initial condition, we get the following result.

Theorem 3.2. *If $\wp$ is regressive, then* (1.1) *has Hyers–Ulam stability.*

Moreover, combining Theorem 2.1 and Theorem 3.1 yields the following new sufficient condition for Hyers–Ulam stability of (1.1).

Theorem 3.3. *If there exists $\alpha \in (0,1)$ such that* (2.2) *holds, then* (1.1) *has Hyers–Ulam stability.*

4. Hyers–Ulam–Rassias Stability

We introduce Hyers–Ulam–Rassias stability of (1.1) as follows.

Definition 4.1 (Hyers–Ulam–Rassias Stability). Let Ω be a family of positive rd-continuous functions defined on $\mathcal{I}$. We say that (1.1) has Hyers–Ulam–Rassias stability of type Ω if there exists a constant $L>0$, a so-called HURS_Ω constant, with the following property. For any $\omega \in \Omega$, if $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I},\mathbb{X})$ is such that

$$\left\|\psi^\Delta(t)+\wp(t)\psi(t)-f(t)\right\| \leqslant \omega(t) \quad \text{for all } t \in \mathcal{I}^\kappa,$$

then there exists a solution $\phi:\mathcal{I}\to\mathbb{X}$ of (1.1) such that $\|\psi(t)-\phi(t)\| \leqslant L\omega(t)$ for all $t \in \mathcal{I}$.

The following results are concerned with Hyers–Ulam–Rassias stability.

Theorem 4.1. *Let $\Omega^*{:=}\{\omega \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I},(0,\infty)) : \omega \ \text{ is nondecreasing}\}$. If* (H) *holds, then* (1.1) *has Hyers–Ulam–Rassias stability of type Ω^* with $HURS_{\Omega^*}$ constant $L:=(b-a)e_{|\wp|}(b,a)$.*

Proof. As before, L is well defined and $L > 0$. Let $\varepsilon > 0$. Suppose that $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$ satisfies

$$\left\|\psi^{\Delta}(t) + \wp(t)\psi(t) - f(t)\right\| \leqslant \omega(t) \quad \text{for all } t \in \mathcal{I}^{\kappa}. \tag{4.1}$$

Defining $h(t) := \psi^{\Delta}(t) + \wp(t)\psi(t) - f(t)$, we see that $h \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$. Let $x_0 = \psi(a)$. By Lemma 2.1, (3.2) holds. By (H), there exists a solution ϕ of (1.1) satisfying $\phi(a) = x_0$. Equivalently, by Lemma 2.1, (3.3) holds. Subtracting (3.3) from (3.2), we find, for all $t \in \mathcal{I}$,

$$\begin{aligned}
\|\psi(t) - \phi(t)\| &\leqslant \int_a^t \|h(s)\|\,\Delta s + \int_a^t |\wp(s)|\,\|\psi(s) - \phi(s)\|\,\Delta s \\
&\overset{(4.1)}{\leqslant} \int_a^t \omega(s)\Delta s + \int_a^t |\wp(s)|\,\|\psi(s) - \phi(s)\|\,\Delta s \\
&\leqslant \int_a^t \omega(t)\Delta s + \int_a^t |\wp(s)|\,\|\psi(s) - \phi(s)\|\,\Delta s \\
&\leqslant (b-a)\omega(t) + \int_a^t |\wp(s)|\,\|\psi(s) - \phi(s)\|\,\Delta s.
\end{aligned}$$

Thus, by Gronwall's inequality, Theorem 1.1, we obtain, for all $t \in \mathcal{I}$,

$$\begin{aligned}
\|\psi(t) - \phi(t)\| &\leqslant (b-a)\omega(t) + \int_a^t e_{|\wp|}(t, \sigma(s))\,|\wp(s)|\,(b-a)\omega(s)\Delta s \\
&\leqslant (b-a)\omega(t) + \int_a^t e_{|\wp|}(t, \sigma(s))\,|\wp(s)|\,(b-a)\omega(t)\Delta s \\
&= (b-a)\omega(t)\left(1 + \int_a^t e_{|\wp|}(t, \sigma(s))\,|\wp(s)|\,\Delta s\right) \\
&= (b-a)e_{|\wp|}(t,a)\omega(t) \leqslant (b-a)e_{|\wp|}(b,a)\omega(t) = L\omega(t).
\end{aligned}$$

Therefore, (1.1) indeed has Hyers–Ulam–Rassias stability of type Ω^*. □

Throughout the rest of the paper, we denote, for $p \geqslant 1$,

$$\Omega_p := \left\{\omega \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I}, (0,\infty)) : \int_a^t \omega^p(s)\Delta s \leqslant \omega^p(t) \text{ for all } t \in \mathcal{I}\right\}.$$

If we consider $\omega \in \Omega^* \cap \Omega_1$, then we can improve the HURS constant (if $b > a+1$) as follows.

Theorem 4.2. *If* (H) *holds, then* (1.1) *has Hyers–Ulam–Rassias stability of type* $\Omega^* \cap \Omega_1$ *with HURS*$_{\Omega^* \cap \Omega_1}$ *constant* $L := e_{|\wp|}(b,a)$.

Proof. As before, L is well defined and $L > 0$. Let $\varepsilon > 0$. Suppose that $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$ satisfies (4.1). Defining $h(t) := \psi^{\Delta}(t) + \wp(t)\psi(t) - f(t)$, we see that $h \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$. Let $x_0 = \psi(a)$. By Lemma 2.1, (3.2) holds. By (H), there exists a solution ϕ of (1.1) satisfying $\phi(a) = x_0$. Equivalently, by Lemma 2.1, (3.3) holds. Subtracting (3.3) from (3.2), we find, for all $t \in \mathcal{I}$,

$$\|\psi(t) - \phi(t)\| \leqslant \int_a^t \|h(s)\|\,\Delta s + \int_a^t |\wp(s)|\,\|\psi(s) - \phi(s)\|\,\Delta s$$

$$\overset{(4.1)}{\leqslant} \int_a^t \omega(s)\Delta s + \int_a^t |\wp(s)| \, \|\psi(s) - \phi(s)\| \, \Delta s$$
$$\leqslant \omega(t) + \int_a^t |\wp(s)| \, \|\psi(s) - \phi(s)\| \, \Delta s.$$

Thus, by Gronwall's inequality, Theorem 1.1, we obtain, for all $t \in \mathcal{I}$,

$$\begin{aligned}\|\psi(t) - \phi(t)\| &\leqslant \omega(t) + \int_a^t e_{|\wp|}(t, \sigma(s)) \, |\wp(s)| \, \omega(s)\Delta s \\ &\leqslant \omega(t) + \int_a^t e_{|\wp|}(t, \sigma(s)) \, |\wp(s)| \, \omega(t)\Delta s \\ &= \omega(t) \left(1 + \int_a^t e_{|\wp|}(t, \sigma(s)) \, |\wp(s)| \, \Delta s\right) \\ &= e_{|\wp|}(t, a)\omega(t) \leqslant e_{|\wp|}(b, a)\omega(t) = L\omega(t).\end{aligned}$$

Therefore, (1.1) indeed has Hyers–Ulam–Rassias stability of type $\Omega^* \cap \Omega_1$. ☐

REMARK 4.1. Note that for $\mathbb{T} = \mathbb{N}_0$, $\Omega^* \cap \Omega_1 = \Omega_1$, since any $\omega \in \Omega_1$ satisfies

$$\omega(t+1) \geqslant \sum_{s=a}^{t} \omega(s) \geqslant \omega(t) \quad \text{for } t \in \mathcal{I}^\kappa.$$

Therefore, by Theorem 4.2, if $\mathbb{T} = \mathbb{N}_0$ and (H) holds, then (1.1) has Hyers–Ulam–Rassias stability of type Ω_1 with HURS_{Ω_1} constant $L = \prod_{s=a}^{b-1} (1 + |\wp(s)|)$.

Now we consider only $\omega \in \Omega_1$.

THEOREM 4.3. *If* (H) *holds, then* (1.1) *has Hyers–Ulam–Rassias stability of type* Ω_1 *with HURS*$_{\Omega_1}$ *constant*

$$L := 1 + e_{|\wp|}(b, a) \, |\wp|_\infty \,, \quad \textit{where } |\wp|_\infty := \sup_{t \in \mathcal{I}} |\wp(t)| \,.$$

PROOF. Note that $|\wp| \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I}, [0, \infty))$, and so, also in view of [**8**, Theorem 1.65], L is well defined and $L \geqslant 1$. Let $\varepsilon > 0$. Suppose $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$ satisfies (4.1). Defining $h(t) := \psi^\Delta(t) + \wp(t)\psi(t) - f(t)$, we see that $h \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$. Let $x_0 = \psi(a)$. By Lemma 2.1, (3.2) holds. By (H), there exists a solution ϕ of (1.1) satisfying $\phi(a) = x_0$. Equivalently, by Lemma 2.1, (3.3) holds. Subtracting (3.3) from (3.2), we find, for all $t \in \mathcal{I}$,

$$\begin{aligned}\|\psi(t) - \phi(t)\| &\leqslant \int_a^t \|h(s)\| \, \Delta s + \int_a^t |\wp(s)| \, \|\psi(s) - \phi(s)\| \, \Delta s \\ &\overset{(4.1)}{\leqslant} \int_a^t \omega(s)\Delta s + \int_a^t |\wp(s)| \, \|\psi(s) - \phi(s)\| \, \Delta s \\ &\leqslant \omega(t) + \int_a^t |\wp(s)| \, \|\psi(s) - \phi(s)\| \, \Delta s.\end{aligned}$$

Thus, by Gronwall's inequality, Theorem 1.1, we obtain, for all $t \in \mathcal{I}$,

$$\|\psi(t) - \phi(t)\| \leqslant \omega(t) + \int_a^t e_{|\wp|}(t, \sigma(s)) \, |\wp(s)| \, \omega(s)\Delta s$$

$$\leqslant \omega(t) + e_{|\wp|}(b,a)\,|\wp|_\infty \int_a^t \omega(s)\Delta s \leqslant L\omega(t),$$

where, in addition to the nondecreasing nature of $e_{|\wp|}(\cdot,a)$, we have also used the nonincreasing nature of $e_{|\wp|}(b,\cdot)$, which follows from the fact that the derivative of $e_{|\wp|}(t,s)$ with respect to s (for fixed t) is $-\,|\wp(s)|\,e_{|\wp|}(t,\sigma(s))$. Therefore, (1.1) indeed has Hyers–Ulam–Rassias stability of type Ω_1. $\square$

THEOREM 4.4. *If* (H) *holds, then* (1.1) *has Hyers–Ulam–Rassias stability of type* Ω_2 *with HURS*$_{\Omega_2}$ *constant* $L := \sqrt{b-a} + (b-a)e_{|\wp|}(b,a)\,|\wp|_\infty$.

PROOF. As before, L is well defined and $L \geqslant \sqrt{b-a}$. Let $\varepsilon > 0$. Suppose that $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I},\mathbb{X})$ satisfies (4.1). Defining $h(t) := \psi^\Delta(t) + \wp(t)\psi(t) - f(t)$, we see that $h \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I},\mathbb{X})$. Let $x_0 = \psi(a)$. By Lemma 2.1, (3.2) holds. By (H), there exists a solution ϕ of (1.1) satisfying $\phi(a) = x_0$. Equivalently, by Lemma 2.1, (3.3) holds. Subtracting (3.3) from (3.2), we find, for all $t \in \mathcal{I}$,

$$\begin{aligned}
\|\psi(t)-\phi(t)\| &\leqslant \int_a^t \|h(s)\|\,\Delta s + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&\overset{(4.1)}{\leqslant} \int_a^t \omega(s)\Delta s + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&\leqslant \sqrt{t-a}\sqrt{\int_a^t \omega^2(s)\Delta s} + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&\leqslant \sqrt{b-a}\sqrt{\omega^2(t)} + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s \\
&= \sqrt{b-a}\,\omega(t) + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s,
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality [**8**, Theorem 6.15] on time scales. Thus, by Gronwall's inequality, Theorem 1.1, we obtain, for all $t \in \mathcal{I}$,

$$\begin{aligned}
\|\psi(t)-\phi(t)\| &\leqslant \sqrt{b-a}\,\omega(t) + \int_a^t e_{|\wp|}(t,\sigma(s))\,|\wp(s)|\,\sqrt{b-a}\,\omega(s)\Delta s \\
&\leqslant \sqrt{b-a}\,\omega(t) + e_{|\wp|}(b,a)\,|\wp|_\infty\,\sqrt{b-a}\int_a^t \omega(s)\Delta s \leqslant L\omega(t),
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality once more. Therefore, (1.1) indeed has Hyers–Ulam–Rassias stability of type Ω_2. $\square$

THEOREM 4.5. *Let* $p > 1$ *and* $q := p/(p-1)$. *If* (H) *holds, then* (1.1) *has Hyers–Ulam–Rassias stability of type* Ω_p *with HURS*$_{\Omega_p}$ *constant*

$$L := \sqrt[q]{b-a} + (b-a)^{2/q}e_{|\wp|}(b,a)\,|\wp|_\infty\,.$$

PROOF. As before, L is well defined and $L \geqslant \sqrt[q]{b-a}$. Let $\varepsilon > 0$. Suppose that $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I},\mathbb{X})$ satisfies (4.1). Defining $h(t) := \psi^\Delta(t) + \wp(t)\psi(t) - f(t)$, we see that $h \in \mathrm{C}_{\mathrm{rd}}(\mathcal{I},\mathbb{X})$. Let $x_0 = \psi(a)$. By Lemma 2.1, (3.2) holds. By (H), there exists a solution ϕ of (1.1) satisfying $\phi(a) = x_0$. Equivalently, by Lemma 2.1, (3.3) holds. Subtracting (3.3) from (3.2), we find, for all $t \in \mathcal{I}$,

$$\begin{aligned}\|\psi(t)-\phi(t)\| &\leqslant \int_a^t \|h(s)\|\,\Delta s + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s\\ &\overset{(4.1)}{\leqslant} \int_a^t \omega(s)\Delta s + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s\\ &\leqslant \sqrt[q]{t-a}\sqrt[p]{\int_a^t \omega^p(s)\Delta s} + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s\\ &\leqslant \sqrt[q]{b-a}\,\sqrt[p]{\omega^p(t)} + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s\\ &= \sqrt[q]{b-a}\,\omega(t) + \int_a^t |\wp(s)|\,\|\psi(s)-\phi(s)\|\,\Delta s,\end{aligned}$$

where we have used the Hölder inequality [**8**, Theorem 6.13] on time scales. Thus, by Gronwall's inequality, Theorem 1.1, we obtain, for all $t \in \mathcal{I}$,

$$\begin{aligned}\|\psi(t)-\phi(t)\| &\leqslant \sqrt[q]{b-a}\,\omega(t) + \int_a^t e_{|\wp|}(t,\sigma(s))\,|\wp(s)|\,\sqrt[q]{b-a}\,\omega(s)\Delta s\\ &\leqslant \sqrt[q]{b-a}\,\omega(t) + e_{|\wp|}(b,a)\,|\wp|_\infty\,\sqrt[q]{b-a}\int_a^t \omega(s)\Delta s \leqslant L\omega(t),\end{aligned}$$

where we have used the Hölder inequality once more. Therefore, (1.1) indeed has Hyers–Ulam–Rassias stability of type Ω_p. ☐

The results in this section imply the following.

Theorem 4.6. *If $\wp$ is regressive, then* (1.1) *has Hyers–Ulam–Rassias stability of types Ω^* and Ω_p for all $p \geqslant 1$.*

Combining Theorem 2.1 and the results in this section, we obtain a new sufficient condition for Hyers–Ulam–Rassias stability of (1.1).

Theorem 4.7. *If there exists $\alpha \in (0,1)$ such that* (2.2) *holds, then* (1.1) *has Hyers–Ulam–Rassias stability of types Ω^* and Ω_p for all $p \geqslant 1$.*

References

1. C. Alsina, R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2**(4) (1998), 373–380.
2. D. R. Anderson, B. Gates, D. Heuer, *Hyers–Ulam stability of second-order linear dynamic equations on time scales*, Commun. Appl. Anal. **16**(3) (2012), 281–291.
3. D. R. Anderson, M. Onitsuka, *Hyers–Ulam stability of first-order homogeneous linear dynamic equations on time scales*, Demonstr. Math. **51**(1) (2018), 198–210.
4. ______, *Best constant for Hyers–Ulam stability of second-order h-difference equations with constant coefficients*, Results Math. **74**(4) (2019), Paper No. 151, 16 pages.
5. ______, *Hyers–Ulam stability for a discrete time scale with two step sizes*, Appl. Math. Comput. **344/345** (2019), 128–140.

6. S. András, A. R. Mészáros, *Ulam–Hyers stability of dynamic equations on time scales via Picard operators*, Appl. Math. Comput. **219**(9) (2013), 4853–4864.
7. A. R. Baias, D. Popa, *On Ulam stability of a linear difference equation in Banach spaces*, Bull. Malays. Math. Sci. Soc. **43**(2) (2020), 1357–1371.
8. M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction With Applications*, Birkhäuser Boston, Boston, MA, 2001.
9. ———, *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, Boston, MA, 2003.
10. J. Brzdęk, P. Wójcik, *On approximate solutions of some difference equations*, Bull. Aust. Math. Soc. **95**(3) (2017), 476–481.
11. C. Chen, M. Bohner, B. Jia, *Ulam–Hyers stability of Caputo fractional difference equations*, Math. Methods Appl. Sci. **42**(18) (2019), 7461–7470.
12. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
13. S.-M. Jung, *Hyers–Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **17**(10) (2004), 1135–1140.
14. ———, *Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer Optim. Appl. **48**, Springer, New York, 2011.
15. Y. Li, Y. Shen, *Hyers–Ulam stability of nonhomogeneous linear differential equations of second order*, Int. J. Math. Math. Sci. (2009), Art. ID 576852, 7 pages.
16. ———, *Hyers–Ulam stability of linear differential equations of second order*, Appl. Math. Lett. **23**(3) (2010), 306–309.
17. T. Miura, *On the Hyers–Ulam stability of a differentiable map*, Sci. Math. Jpn. **55**(1) (2002), 17–24.
18. T. Miura, S. Miyajima, S.-E. Takahasi, *A characterization of Hyers–Ulam stability of first order linear differential operators*, J. Math. Anal. Appl. **286**(1) (2003), 136–146.
19. T. Miura, S.-E. Takahasi, H. Choda, *On the Hyers–Ulam stability of real continuous function valued differentiable map*, Tokyo J. Math. **24**(2) (2001), 467–476.
20. D. Popa, *Hyers–Ulam stability of the linear recurrence with constant coefficients*, Adv. Difference Equ. (2005), no. 2, 101–107.
21. T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72**(2) (1978), 297–300.
22. I. A. Rus, *Ulam stability of ordinary differential equations*, Stud. Univ. Babeş–Bolyai Math. **54**(4) (2009), 125–133.
23. ———, *Gronwall lemma approach to the Hyers–Ulam–Rassias stability of an integral equation*; in: P. M. Pardalos, (ed.) et al., *Nonlinear Analysis and Variational Problems. In Honor of George Isac*, Springer Optim. Appl. **35**, Springer, New York, 2010, pp. 147–152.
24. Y. Shen, *The Ulam stability of first order linear dynamic equations on time scales*, Results Math. **72**(4) (2017), 1881–1895.
25. Y. Shen, Y. Li, *A general method for the Ulam stability of linear differential equations*, Bull. Malays. Math. Sci. Soc. **42**(6) (2019), 3187–3211.
26. ———, *Hyers–Ulam stability of first order nonhomogeneous linear dynamic equations on time scales*, Commun. Math. Res. **35**(2) (2019), 139–148.
27. S.-E. Takahasi, T. Miura, S. Miyajima, *On the Hyers–Ulam stability of the Banach space-valued differential equation $y' = \lambda y$*, Bull. Korean Math. Soc. **39**(2) (2002), 309–315.
28. S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts Pure Appl. Math. **8**, Interscience, New York–London, 1960.
29. G. Wang, M. Zhou, L. Sun, *Hyers–Ulam stability of linear differential equations of first order*, Appl. Math. Lett. **21**(10) (2008), 1024–1028.

30. A. Zada, S. O. Shah, S. Ismail, T. Li, *Hyers–Ulam stability in terms of dichotomy of first order linear dynamic systems*, Punjab Univ. J. Math. (Lahore) **49**(3) (2017), 37–47.

University of Jeddah, College of Science
Department of Mathematics
Jeddah, Saudi Arabia
maaalghamdi4@gmail.com
a.a418@hotmail.com

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Missouri S&T, Department of Mathematics
and Statistics
Rolla, MO, USA
bohner@mst.edu

University of Jeddah, College of Science
Department of Mathematics
Jeddah, Saudi Arabia
and
Cairo University, Faculty of Science
Department of Mathematics
Giza, Egypt
hamzaaeg2003@yahoo.com