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On periodicity of systems of rational difference equations

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Abstract

In this paper, we investigate the periodicity of two systems of rational sequences of second and third order, respectively. The systems include a permutation that gives the ability of changing the appearance of components of solutions in the equations of the systems. We find periods of systems in terms of the order of the permutation. The periodicity of two more systems of maximum type are studied. Finally, many illustrative examples are given.

Keywords: Systems of difference equations, permutation groups, periodicity.

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1. Introduction

The theory of permutation groups is important to diverse area of mathematics such as Galois theory, invariant theory, the representation theory of Lie groups, and combinatorics. See for instance [1, 8]. In general, the theory of abstract groups plays an important part in present day mathematics and science. Groups arise in a bewildering number of apparently unconnected subjects. Thus they appear in algebra and analysis, in geometry and topology, in crystallography and quantum mechanics, in physics and chemistry, and even in biology, see [2, 16]. In this paper, as another application of the theory of permutation groups, we investigate the periodicity of systems of difference equations. For the basics of the theory of difference equations, we refer the reader to the monographs [3, 10]. Recently, there has been a great interest in difference equations, because they describe naturally many real-life problems in biology, ecology, genetics, psychology, sociology, and so forth. Iricanian and Stevic [9], inspired by the work of Lyness [11–13] who proved that the difference equation

$$x_{n+1} = \frac{1+x_n}{x_{n-1}}, \quad n \in \mathbb{N}_0,$$

is periodic with period five, investigated the periodicity of the systems

$$x^{(1)}_{n+1} = \frac{x^{(2)}_n + 1}{x^{(3)}_{n-1}}, \quad x^{(2)}_{n+1} = \frac{x^{(3)}_n + 1}{x^{(4)}_{n-1}}, \quad \dots \quad x^{(k)}_{n+1} = \frac{x^{(1)}_n + 1}{x^{(2)}_{n-1}},$$

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and

$$x_{n+1}^{(1)} = \frac{x_n^{(2)} + x_{n-1}^{(3)} + 1}{x_{n-2}^{(4)}}, \quad x_{n+1}^{(2)} = \frac{x_n^{(3)} + x_{n-1}^{(4)} + 1}{x_{n-2}^{(5)}}, \quad \dots \quad x_{n+1}^{(k)} = \frac{x_n^{(1)} + x_{n-1}^{(2)} + 1}{x_{n-2}^{(3)}},$$

For some pertinent results about periodicity, we refer the reader to the interesting papers [4–6, 15, 17].

Motivated by the above results, we investigate the periodicity of the following systems

$$x_{n+1}^{(1)} = \frac{x_n^{\pi(1)} + 1}{x_{n-1}^{\pi^2(1)}}, \quad x_{n+1}^{(2)} = \frac{x_n^{\pi(2)} + 1}{x_{n-1}^{\pi^2(2)}}, \quad \dots \quad x_{n+1}^{(k)} = \frac{x_n^{\pi(k)} + 1}{x_{n-1}^{\pi^2(k)}}, \tag{1.1}$$

and

$$x_{n+1}^{(1)} = \frac{x_n^{\pi(1)} + x_{n-1}^{\pi^2(1)} + 1}{x_{n-2}^{\pi^3(1)}}, \quad x_{n+1}^{(2)} = \frac{x_n^{\pi(2)} + x_{n-1}^{\pi^2(2)} + 1}{x_{n-2}^{\pi^3(2)}}, \quad \dots \quad x_{n+1}^{(k)} = \frac{x_n^{\pi(k)} + x_{n-1}^{\pi^2(k)} + 1}{x_{n-2}^{\pi^3(k)}}, \tag{1.2}$$

where $\pi \in \mathcal{S}_k$. Every choice of a permutation π gives a system of difference equations. Thus in fact System (1.1) or System (1.2) represents $k!$ systems. It is well-known that for any permutation $\pi \in S_k$, there is a natural number l such that the property $\pi^l = I$ holds, where I is the identity permutation and π^l is the composition of π with itself l-times. The smallest l for which this property holds is called the order of π.

Definition 1.1. The system

$$\begin{aligned} x_{n+1}^{(1)} &= f_1(x_n^{\pi(1)}, x_{n-1}^{\pi^2(1)}, \dots, x_{n-s}^{\pi^{s+1}(1)}), \\ &\vdots \\ x_{n+1}^{(k)} &= f_k(x_n^{\pi(k)}, x_{n-1}^{\pi^2(k)}, \dots, x_{n-s}^{\pi^{s+1}(k)}) \end{aligned}$$

is called periodic with period d if every positive solution $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ satisfies

$$x_{n+d}^{(i)} = x_n^{(i)}, n \in \mathbb{Z}^{\geqslant 0}, i \in \mathbb{Z}_k.$$

A solution $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ is said to be positive if each component x_n^i is positive for every n.

2. Main results

In this section we study the periodicity of systems (1.1) and (1.2).

Theorem 2.1. *System (1.1) is periodic with period l if $l = 0 \mod 5$ and is periodic with period $5l$ if $l \neq 0 \mod 5$.*

Proof. Let $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ be a positive solution of system (1.1). We can see that

$$x_{n+d}^{(1)} = x_{n+d-5r}^{\pi^{5r}(1)}, n \in \mathbb{Z}^{\geqslant 0}, r \in \mathbb{N}, d \geqslant 5r. \tag{2.1}$$

Indeed, we have

$$x_{n+d}^{(1)} = \frac{x_{n+d-1}^{\pi(1)} + 1}{x_{n+d-2}^{\pi^2(1)}} = \frac{\left(\frac{x_{n+d-2}^{\pi^2(1)}+1}{x_{n+d-3}^{\pi^3(1)}}\right) + 1}{x_{n+d-2}^{\pi^2(1)}} = \frac{x_{n+d-2}^{\pi^2(1)} + 1 + x_{n+d-3}^{\pi^3(1)}}{x_{n+d-3}^{\pi^3(1)} x_{n+d-2}^{\pi^2(1)}} = \frac{\left(\frac{x_{n+d-3}^{\pi^3(1)}+1}{x_{n+d-4}^{\pi^4(1)}}\right) + 1 + x_{n+d-3}^{\pi^3(1)}}{x_{n+d-3}^{\pi^3(1)} \left(\frac{x_{n+d-3}^{\pi^3(1)}+1}{x_{n+d-4}^{\pi^4(1)}}\right)}.$$

It follows that

$$x_{n+d}^{(1)} = \frac{x_{n+d-3}^{\pi^3(1)} + 1 + x_{n+d-4}^{\pi^4(1)} + x_{n+d-3}^{\pi^3(1)} x_{n+d-4}^{\pi^4(1)}}{x_{n+d-3}^{\pi^3(1)} \left(x_{n+d-3}^{\pi^3(1)} + 1\right)}$$

$$
\begin{aligned}
&= \frac{x_{n+d-3}^{\pi^3(1)} + 1 + x_{n+d-4}^{\pi^4(1)}\left(1 + x_{n+d-3}^{\pi^3(1)}\right)}{x_{n+d-3}^{\pi^3(1)}\left(x_{n+d-3}^{\pi^3(1)} + 1\right)} \\
&= \frac{1 + x_{n+d-4}^{\pi^4(1)}}{x_{n+d-3}^{\pi^3(1)}} = x_{n+d-5}^{\pi^5(1)} \\
&\vdots \\
&= x_{n+d-5r}^{\pi^{5r}(1)}, n \in \mathbb{Z}^{\geqslant 0}, r \in \mathbb{N}.
\end{aligned}
$$

Assume that $l = 0 \mod 5$. Then $l = 5r$ for some $r \in \mathbb{N}$. Setting $d = 5r$ in (2.1), we get $x_{n+l}^{(1)} = x_n^{(1)}$. Similarly, $x_{n+l}^{(i)} = x_n^{(i)}, i = 1, \dots, k$. Therefore, system (1.1) is periodic with period l. For the case $l \neq 0 \mod 5$, putting $d = 5l$ and $r = l$ in (2.1), we obtain $x_{n+5l}^{(1)} = x_n^{(1)}$. Similarly, $x_{n+5l}^{(i)} = x_n^{(i)}, i = 1, \dots, k$. Therefore system (1.1) is periodic with period $5l$. □

The following Theorem establishes the periodicity of systems of second-order of maximum type.

Theorem 2.2. *The system*

$$
x_{n+1}^{(1)} = \frac{\max\left\{x_n^{\pi(1)}, 1\right\}}{x_{n-1}^{\pi^2(1)}}, \ x_{n+1}^{(2)} = \frac{\max\left\{x_n^{\pi(2)}, 1\right\}}{x_{n-1}^{\pi^2(2)}}, \dots, \ x_{n+1}^{(k)} = \frac{\max\left\{x_n^{\pi(k)}, 1\right\}}{x_{n-1}^{\pi^2(k)}}, \tag{2.2}
$$

is periodic with period l *if* $l = 0 \mod 5$ *and is periodic with period* $5l$ *if* $l \neq 0 \mod 5$.

Proof. Let $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ be a positive solution of system (2.2). Then, it satisfies (2.1). Indeed,

$$
\begin{aligned}
x_{n+d}^{(1)} = \frac{\max\left\{x_{n+d-1}^{\pi(1)}, 1\right\}}{x_{n+d-2}^{\pi^2(1)}} &= \frac{\max\left\{\max\left\{x_{n+d-2}^{\pi^2(1)}, 1\right\}, x_{n+d-3}^{\pi^3(1)}\right\}}{x_{n+d-2}^{\pi^2(1)} x_{n+d-3}^{\pi^3(1)}} \\
&= \frac{\max\left\{x_{n+d-2}^{\pi^2(1)}, 1, x_{n+d-3}^{\pi^3(1)}\right\}}{x_{n+d-2}^{\pi^2(1)} x_{n+d-3}^{\pi^3(1)}} \\
&= \frac{\max\left\{\frac{\max\left\{x_{n+d-3}^{\pi^3(1)}, 1\right\}}{x_{n+d-4}^{\pi^4(1)}}, 1, x_{n+d-3}^{\pi^3(1)}\right\}}{x_{n+d-3}^{\pi^3(1)} \frac{\max\left\{x_{n+d-3}^{\pi^3(1)}, 1\right\}}{x_{n+d-4}^{\pi^4(1)}}} \\
&= \frac{\max\left\{\max\left\{x_{n+d-3}^{\pi^3(1)}, 1\right\}, x_{n+d-4}^{\pi^4(1)}, x_{n+d-3}^{\pi^3(1)} x_{n+d-4}^{\pi^4(1)}\right\}}{x_{n+d-3}^{\pi^3(1)} \max\left\{x_{n+d-3}^{\pi^3(1)}, 1\right\}}. \\
&= \frac{\max\left\{\max\left\{x_{n+d-3}^{\pi^3(1)}, 1\right\}, x_{n+d-4}^{\pi^4(1)} \max\left\{x_{n+d-3}^{\pi^3(1)}, 1\right\}\right\}}{x_{n+d-3}^{\pi^3(1)} \max\left\{x_{n+d-3}^{\pi^3(1)}, 1\right\}}.
\end{aligned}
$$

It follows that

$$
x_{n+d}^{(1)} = \frac{\max\left\{1, x_{n+d-4}^{\pi^4(1)}\right\}}{x_{n+d-3}^{\pi^3(1)}} = x_{n+d-5}^{\pi^5(1)} = \cdots = x_{n+d-5r}^{\pi^{5r}(1)}, n \in \mathbb{Z}^{\geqslant 0}, r \in \mathbb{N}, d \geqslant 5r.
$$

For the case $l = 0 \mod 5$, there exists $r \in \mathbb{N}$ such that $l = 5r$. Set $d = 5r$ in (2.1) to obtain $x^{(1)}_{n+l} = x^{(1)}_n$. Similarly, we obtain $x^{(i)}_{n+l} = x^{(i)}_n, i = 1, \dots k$. So system (2.2) is periodic with period l.

Now, assume that $l \neq 0 \mod 5$. Putting $d = 5l$ and $r = l$ in (2.1), we obtain $x^{(1)}_{n+5l} = x^{(1)}_n$. Similarly, $x^{(i)}_{n+5l} = x^{(i)}_n, i = 1, \dots, k$. Therefore system (2.2) is periodic with period $5l$. □

In the following result, we prove the periodicity of system (1.2), with period $l2^{3-i}$, under the condition $\mathrm{GCD}(l, 8) = 2^i, i = 0, 1, 2, 3$. Here, $\mathrm{GCD}(a, b)$ is the greatest common divisor of a and b.

Theorem 2.3. *System (1.2) is periodic with period $l2^{3-i}$ if $\mathrm{GCD}(l, 8) = 2^i,\ i = 0, 1, 2, 3$.*

Proof. Let $(x^{(1)}_n, x^{(2)}_n, \dots, x^{(k)}_n)$ be a positive solution of system (1.2). We prove

$$x^{(1)}_{n+d} = x^{\pi^8(1)}_{n+d-8}, n \in \mathbb{Z}^{\geqslant 0}, d \geqslant 8. \tag{2.3}$$

We have

$$\begin{aligned}
x^{(1)}_{n+d} &= \frac{x^{\pi(1)}_{n+d-1} + x^{\pi^2(1)}_{n+d-2} + 1}{x^{\pi^3(1)}_{n+d-3}} \\
&= \frac{\left(\frac{x^{\pi^2(1)}_{n+d-2} + x^{\pi^3(1)}_{n+d-3} + 1}{x^{\pi^4(1)}_{n+d-4}}\right) + x^{\pi^2(1)}_{n+d-2} + 1}{x^{\pi^3(1)}_{n+d-3}} \\
&= \frac{x^{\pi^2(1)}_{n+d-2} + x^{\pi^3(1)}_{n+d-3} + 1 + x^{\pi^4(1)}_{n+d-4} x^{\pi^2(1)}_{n+d-2} + x^{\pi^4(1)}_{n+d-4}}{x^{\pi^3(1)}_{n+d-4} x^{\pi^3(1)}_{n+d-3}} \\
&= \frac{\left(x^{\pi^2(1)}_{n+d-2} + 1\right)\left(x^{\pi^4(1)}_{n+d-4} + 1\right) + x^{\pi^3(1)}_{n+d-3}}{x^{\pi^3(1)}_{n+d-3} x^{\pi^4(1)}_{n+d-4}} \\
&= \frac{\left(\frac{x^{\pi^3(1)}_{n+d-3} + x^{\pi^4(1)}_{n+d-4} + 1}{x^{\pi^5(1)}_{n+d-5}} + 1\right)\left(x^{\pi^4(1)}_{n+d-4} + 1\right) + x^{\pi^3(1)}_{n+d-3}}{x^{\pi^3(1)}_{n+d-3} x^{\pi^4(1)}_{n+d-4}} \\
&= \frac{\left(x^{\pi^3(1)}_{n+d-3} + x^{\pi^4(1)}_{n+d-4} + 1 + x^{\pi^5(1)}_{n+d-5}\right)\left(x^{\pi^4(1)}_{n+d-4} + 1\right) + x^{\pi^5(1)}_{n+d-5} x^{\pi^3(1)}_{n+d-3}}{x^{\pi^3(1)}_{n+d-3} x^{\pi^4(1)}_{n+d-4} x^{\pi^5(1)}_{n+d-5}} \\
&= \frac{\left(\frac{x^{\pi^4(1)}_{n+d-4} + x^{\pi^5(1)}_{n+d-5} + 1}{x^{\pi^6(1)}_{n+d-6}} + x^{\pi^4(1)}_{n+d-4} + 1 + x^{\pi^5(1)}_{n+d-5}\right)\left(x^{\pi^4(1)}_{n+d-4} + 1\right) + \frac{x^{\pi^4(1)}_{n+d-4} + x^{\pi^5(1)}_{n+d-5} + 1}{x^{\pi^6(1)}_{n+d-6}} x^{\pi^5(1)}_{n+d-5}}{\frac{x^{\pi^4(1)}_{n+d-4} + x^{\pi^5(1)}_{n+d-5} + 1}{x^{\pi^6(1)}_{n+d-6}} x^{\pi^4(1)}_{n+d-4} x^{\pi^5(1)}_{n+d-5}} \\
&= \frac{\left(x^{\pi^6(1)}_{n+d-6} + 1\right)\left(x^{\pi^4(1)}_{n+d-4} + 1\right) + x^{\pi^5(1)}_{n+d-5}}{x^{\pi^4(1)}_{n+d-4} x^{\pi^5(1)}_{n+d-5}}. \\
&= \frac{\left(x^{\pi^6(1)}_{n+d-6} + 1\right)\left(\frac{x^{\pi^5(1)}_{n+d-5} + x^{\pi^6(1)}_{n+d-6} + 1}{x^{\pi^7(1)}_{n+d-7}} + 1\right) + x^{\pi^5(1)}_{n+d-5}}{\left(\frac{x^{\pi^5(1)}_{n+d-5} + x^{\pi^6(1)}_{n+d-6} + 1}{x^{\pi^7(1)}_{n+d-7}}\right) x^{\pi^5(1)}_{n+d-5}}.
\end{aligned}$$

This implies that

$$
\begin{aligned}
x_{n+d}^{(1)} &= \frac{\left(x_{n+d-5}^{\pi^5(1)} + x_{n+d-6}^{\pi^6(1)} + 1 + x_{n+d-7}^{\pi^7(1)}\right)\left(x_{n+d-6}^{\pi^6(1)} + 1\right) + x_{n+d-5}^{\pi^5(1)} x_{n+d-7}^{\pi^7(1)}}{\left(x_{n+d-5}^{\pi^5(1)} + x_{n+d-6}^{\pi^6(1)} + 1\right) x_{n+d-5}^{\pi^5(1)}} \\
&= \frac{\left(x_{n+d-6}^{\pi^6(1)} + 1\right)\left(\frac{x_{n+d-6}^{\pi^6(1)} + x_{n+d-7}^{\pi^7(1)} + 1}{x_{n+d-8}^{\pi^8(1)}} + x_{n+d-6}^{\pi^6(1)} + 1 + x_{n+d-7}^{\pi^7(1)}\right) + \frac{x_{n+d-6}^{\pi^6(1)} + x_{n+d-7}^{\pi^7(1)} + 1}{x_{n+d-8}^{\pi^8(1)}} x_{n+d-7}^{\pi^7(1)}}{\left(\frac{x_{n+d-6}^{\pi^6(1)} + x_{n+d-7}^{\pi^7(1)} + 1}{x_{n+d-8}^{\pi^8(1)}} + x_{n+d-6}^{\pi^6(1)} + 1\right) x_{n+d-5}^{\pi^5(1)}} \\
&= \frac{x_{n+d-6}^{\pi^6(1)} + x_{n+d-7}^{\pi^7(1)} + 1}{x_{n+d-8}^{\pi^8(1)}} \frac{\left[\left(x_{n+d-6}^{\pi^6(1)} + 1\right)\left(x_{n+d-8}^{\pi^8(1)} + 1\right) + x_{n+d-7}^{\pi^7(1)}\right]}{\left(\frac{x_{n+d-6}^{\pi^6(1)} + x_{n+d-7}^{\pi^7(1)} + 1}{x_{n+d-8}^{\pi^8(1)}} + x_{n+d-6}^{\pi^6(1)} + 1\right) x_{n+d-5}^{\pi^5(1)}} \\
&= \frac{x_{n+d-6}^{\pi^6(1)} + x_{n+d-7}^{\pi^7(1)} + 1}{x_{n+d-5}^{\pi^5(1)}} = x_{n+d-8}^{\pi^8(1)}.
\end{aligned}
$$

We begin by the first possibility $i = 0$, that is $\mathrm{GCD}(l, 8) = 1$. Then l has the form $l = 2r - 1$ for some $r \in \mathbb{N}$. Setting $d = 8l$ in (2.3), we obtain

$$x_{n+8l}^{(1)} = x_{n+8l-8}^{\pi^8(1)} = x_{n+8l-8\times 3}^{\pi^{8\times 3}(1)} = \cdots = x_{n+8l-8l}^{\pi^{8l}(1)} = x_n^{(1)}, n \in \mathbb{Z}^{\geqslant 0}.$$

So, $x_{n+8l}^{(1)} = x_n^{(1)}$. Similarly, $x_{n+8l}^{(j)} = x_n^{(j)}$, $j = 1, 2, \ldots, k$. Therefore system (1.2) is periodic with period $8l$.

The second possibility $i = 1$ means that $\mathrm{GCD}(l, 8) = 2$. Then in this case l has the form $l = 4r - 2$ for some $r \in \mathbb{N}$. Use (2.3) with $d = 4l$ to conclude that

$$x_{n+4l}^{(1)} = x_{n+4l-8}^{\pi^8(1)} = x_{n+4l-4\times 2}^{\pi^{4\times 2}(1)} = x_{n+4l-4\times 6}^{\pi^{4\times 6}(1)} = \cdots = x_{n+4l-4(4r-2)}^{\pi^{4\times(4r-2)}(1)} = x_{n+4l-4l}^{\pi^{4l}(1)} = x_n^{(1)}.$$

Similarly, we can see that $x_{n+4l}^{(j)} = x_n^{(j)}$, $j = 1, 2, \ldots, k$. Therefore system (1.2) is periodic with period $4l$. Consider $i = 2$, i.e., $\mathrm{GCD}(l, 8) = 4$. Thus l has the form $l = 8r - 4$ for some $r \in \mathbb{N}$. Use (2.3) with $d = 2l$ to obtain

$$x_{n+2l}^{(1)} = x_{n+2l-8}^{\pi^8(1)} = x_{n+2l-2\times 4}^{\pi^{2\times 4}(1)} = x_{n+2l-2\times 12}^{\pi^{2\times 12}(1)} = \cdots = x_{n+2l-2\times(8r-4)}^{\pi^{2\times(8r-4)}(1)} = x_{n+2l-2l}^{\pi^{2l}(1)} = x_n^{(1)}.$$

Similarly we can see that $x_{n+2l}^{(j)} = x_n^{(j)}$, $j = 1, 2, \ldots, k$. Therefore the system (1.2) is periodic with period $2l$.

Finally, for $i = 3$, $\mathrm{GCD}(l, 8) = 8$ and consequently $l = 8r$ for some $r \in \mathbb{N}$. Use (2.3) with $d = l$ to conclude that

$$x_{n+l}^{(1)} = x_{n+l-8}^{\pi^8(1)} = x_{n+l-8\times 2}^{\pi^{8\times 2}(1)} = \cdots = x_{n+l-8r}^{\pi^{8r}(1)} = x_{n+l-l}^{\pi^{l}(1)} = x_n^{(1)},$$

so, $x_{n+l}^{(1)} = x_n^{(1)}$. Same calculations show that $x_{n+l}^{(j)} = x_n^{(j)}$, $j = 1, 2, \ldots, k$. Therefore system (1.2) is periodic with period l. □

Theorem 2.4. *The system*

$$x_{n+1}^{(1)} = \frac{\max\left\{x_n^{\pi(1)}, x_{n-1}^{\pi^2(1)}, 1\right\}}{x_{n-2}^{\pi^3(1)}}, \; x_{n+1}^{(2)} = \frac{\max\left\{x_n^{\pi(2)}, x_{n-1}^{\pi^2(2)}, 1\right\}}{x_{n-2}^{\pi^3(2)}}, \; \ldots \; x_{n+1}^{(k)} = \frac{\max\left\{x_n^{\pi(k)}, x_{n-1}^{\pi^2(k)}, 1\right\}}{x_{n-2}^{\pi^3(k)}}, \quad (2.4)$$

is periodic with period $l2^{3-i}$ *if* $\mathrm{GCD}(l, 8) = 2^i$, $i = 0, 1, 2, 3$.

Proof. Let $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ be a positive solution of system (2.4). We can show (2.3). Indeed,

$$
\begin{aligned}
x_{n+d}^{(1)} &= \frac{\max\left\{\max\left\{x_{n+d-3}^{\pi^3(1)}, x_{n+d-4}^{\pi^4(1)}, 1\right\}, x_{n+d-4}^{\pi^4(1)}\max\left\{x_{n+d-3}^{\pi^3(1)}, x_{n+d-4}^{\pi^4(1)}, 1\right\}, x_{n+d-5}^{\pi^5(1)}\max\left\{x_{n+d-3}^{\pi^3(1)}, x_{n+d-4}^{\pi^4(1)}, 1\right\}\right\}}{x_{n+d-5}^{\pi^5(1)} x_{n+d-4}^{\pi^4(1)} x_{n+d-3}^{\pi^3(1)}} \\
&= \frac{\max\left\{x_{n+d-3}^{\pi^3(1)}, x_{n+d-4}^{\pi^4(1)}, 1\right\}\max\left\{x_{n+d-4}^{\pi^4(1)}, x_{n+d-5}^{\pi^5(1)}, 1\right\}}{x_{n+d-5}^{\pi^5(1)} x_{n+d-4}^{\pi^4(1)} x_{n+d-3}^{\pi^3(1)}} \\
&= \frac{\max\left\{\frac{\max\left\{x_{n+d-4}^{\pi^4(1)}, x_{n+d-5}^{\pi^5(1)}, 1\right\}}{x_{n+d-6}^{\pi^6(1)}}, x_{n+d-4}^{\pi^4(1)}, 1\right\}\max\left\{x_{n+d-4}^{\pi^4(1)}, x_{n+d-5}^{\pi^5(1)}, 1\right\}}{x_{n+d-5}^{\pi^5(1)} x_{n+d-4}^{\pi^4(1)} \frac{\max\left\{x_{n+d-4}^{\pi^4(1)}, x_{n+d-5}^{\pi^5(1)}, 1\right\}}{x_{n+d-6}^{\pi^6(1)}}} \\
&= \frac{\max\left\{\max\left\{x_{n+d-4}^{\pi^4(1)}, x_{n+d-5}^{\pi^5(1)}, 1\right\}, x_{n+d-6}^{\pi^6(1)} x_{n+d-4}^{\pi^4(1)}, x_{n+d-6}^{\pi^6(1)}\right\}\max\left\{x_{n+d-4}^{\pi^4(1)}, x_{n+d-5}^{\pi^5(1)}, 1\right\}}{x_{n+d-5}^{\pi^5(1)} x_{n+d-4}^{\pi^4(1)}\max\left\{x_{n+d-4}^{\pi^4(1)}, x_{n+d-5}^{\pi^5(1)}, 1\right\}}.
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
x_{n+d}^{(1)} &= \frac{\max\left\{x_{n+d-4}^{\pi^4(1)}, x_{n+d-5}^{\pi^5(1)}, 1, x_{n+d-6}^{\pi^6(1)} x_{n+d-4}^{\pi^4(1)}, x_{n+d-6}^{\pi^6(1)}\right\}}{x_{n+d-5}^{\pi^5(1)} x_{n+d-4}^{\pi^4(1)}} \\
&= \frac{\max\left\{\frac{\max\left\{x_{n+d-5}^{\pi^5(1)}, x_{n+d-6}^{\pi^6(1)}, 1\right\}}{x_{n+d-7}^{\pi^7(1)}}, x_{n+d-5}^{\pi^5(1)}, 1, x_{n+d-6}^{\pi^6(1)}\frac{\max\left\{x_{n+d-5}^{\pi^5(1)}, x_{n+d-6}^{\pi^6(1)}, 1\right\}}{x_{n+d-7}^{\pi^7(1)}}, x_{n+d-6}^{\pi^6(1)}\right\}}{x_{n+d-5}^{\pi^5(1)}\frac{\max\left\{x_{n+d-5}^{\pi^5(1)}, x_{n+d-6}^{\pi^6(1)}, 1\right\}}{x_{n+d-7}^{\pi^7(1)}}} \\
&= \frac{\max\left\{\max\left\{x_{n+d-5}^{\pi^5(1)}, x_{n+d-6}^{\pi^6(1)}, 1\right\}, x_{n+d-7}^{\pi^7(1)} x_{n+d-5}^{\pi^5(1)}, x_{n+d-7}^{\pi^7(1)}, x_{n+d-6}^{\pi^6(1)}\max\left\{x_{n+d-5}^{\pi^5(1)}, x_{n+d-6}^{\pi^6(1)}, 1\right\}, x_{n+d-7}^{\pi^7(1)} x_{n+d-6}^{\pi^6(1)}\right\}}{x_{n+d-5}^{\pi^5(1)}\max\left\{x_{n+d-5}^{\pi^5(1)}, x_{n+d-6}^{\pi^6(1)}, 1\right\}} \\
&= \frac{\max\left\{\max\left\{x_{n+d-5}^{\pi^5(1)}, x_{n+d-6}^{\pi^6(1)}, 1\right\}, x_{n+d-7}^{\pi^7(1)}\max\left\{x_{n+d-5}^{\pi^5(1)}, x_{n+d-6}^{\pi^6(1)}, 1\right\}, x_{n+d-6}^{\pi^6(1)}\max\left\{x_{n+d-5}^{\pi^5(1)}, x_{n+d-6}^{\pi^6(1)}, 1\right\}\right\}}{x_{n+d-5}^{\pi^5(1)}\max\left\{x_{n+d-5}^{\pi^5(1)}, x_{n+d-6}^{\pi^6(1)}, 1\right\}} \\
&= \frac{\max\left\{x_{n+d-6}^{\pi^6(1)}, x_{n+d-7}^{\pi^7(1)}, 1\right\}}{x_{n+d-5}^{\pi^5(1)}} = x_{n+d-8}^{\pi^8(1)}.
\end{aligned}
$$

The rest of the proof follows similarly as in proof of Theorem 2.3. ☐

3. Illustrative examples

(i) The systems

$$
\begin{aligned}
&x_{n+1}^{(1)} = \frac{x_n^{(5)}+1}{x_{n-1}^{(3)}},\ x_{n+1}^{(2)} = \frac{x_n^{(7)}+1}{x_{n-1}^{(4)}},\ x_{n+1}^{(3)} = \frac{x_n^{(1)}+1}{x_{n-1}^{(5)}},\ x_{n+1}^{(4)} = \frac{x_n^{(8)}+1}{x_{n-1}^{(2)}},\ x_{n+1}^{(5)} = \frac{x_n^{(3)}+1}{x_{n-1}^{(1)}}, \\
&x_{n+1}^{(6)} = \frac{x_n^{(10)}+1}{x_{n-1}^{(9)}},\ x_{n+1}^{(7)} = \frac{x_n^{(4)}+1}{x_{n-1}^{(8)}},\ x_{n+1}^{(8)} = \frac{x_n^{(2)}+1}{x_{n-1}^{(7)}},\ x_{n+1}^{(9)} = \frac{x_n^{(6)}+1}{x_{n-1}^{(10)}},\ x_{n+1}^{(10)} = \frac{x_n^{(9)}+1}{x_{n-1}^{(6)}}
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
&x_{n+1}^{(1)}=\frac{\max\left\{x_n^{(5)},1\right\}}{x_{n-1}^{(3)}},\ x_{n+1}^{(2)}=\frac{\max\left\{x_n^{(7)},1\right\}}{x_{n-1}^{(4)}},\ x_{n+1}^{(3)}=\frac{\max\left\{x_n^{(1)},1\right\}}{x_{n-1}^{(5)}},\\
&x_{n+1}^{(4)}=\frac{\max\left\{x_n^{(8)},1\right\}}{x_{n-1}^{(2)}},\ x_{n+1}^{(5)}=\frac{\max\left\{x_n^{(3)},1\right\}}{x_{n-1}^{(1)}},\ x_{n+1}^{(6)}=\frac{\max\left\{x_n^{(10)},1\right\}}{x_{n-1}^{(9)}},\\
&x_{n+1}^{(7)}=\frac{\max\left\{x_n^{(4)},1\right\}}{x_{n-1}^{(8)}},x_{n+1}^{(8)}=\frac{\max\left\{x_n^{(2)}+1\right\}}{x_{n-1}^{(7)}},x_{n+1}^{(9)}=\frac{\max\left\{x_n^{(6)}+1\right\}}{x_{n-1}^{(10)}},x_{n+1}^{(10)}=\frac{\max\left\{x_n^{(9)}+1\right\}}{x_{n-1}^{(6)}}.
\end{aligned}
\tag{3.2}
$$

are periodic with period 60. The permutation which corresponds to each of these systems is $\pi = (1\ \ 5\ \ 3)(2\ \ 7\ \ 4\ \ 8)(6\ \ 10\ \ 9)$. Its order is $12 \neq 0 \mod 5$. By Theorem 2.1, systems (3.1) and (3.2) are periodic with period 60.

(ii) The systems

$$
\begin{aligned}
&x_{n+1}^{(1)}=\frac{x_n^{(5)}+x_{n-1}^{(9)}+1}{x_{n-2}^{(3)}},\ x_{n+1}^{(2)}=\frac{x_n^{(12)}+x_{n-1}^{(8)}+1}{x_{n-2}^{(10)}},\ x_{n+1}^{(3)}=\frac{x_n^{(4)}+x_{n-1}^{(7)}+1}{x_{n-2}^{(11)}},\\
&x_{n+1}^{(4)}=\frac{x_n^{(7)}+x_{n-1}^{(11)}+1}{x_{n-2}^{(2)}},\ x_{n+1}^{(5)}=\frac{x_n^{(9)}+x_{n-1}^{(3)}+1}{x_{n-2}^{(4)}},\ x_{n+1}^{(6)}=\frac{x_n^{(1)}+x_{n-1}^{(5)}+1}{x_{n-2}^{(9)}},\\
&x_{n+1}^{(7)}=\frac{x_n^{(11)}+x_{n-1}^{(2)}+1}{x_{n-2}^{(12)}},\ x_{n+1}^{(8)}=\frac{x_n^{(10)}+x_{n-1}^{(6)}+1}{x_{n-2}^{(1)}},\ x_{n+1}^{(9)}=\frac{x_n^{(3)}+x_{n-1}^{(4)}+1}{x_{n-2}^{(7)}},\\
&x_{n+1}^{(10)}=\frac{x_n^{(6)}+x_{n-1}^{(1)}+1}{x_{n-2}^{(5)}},\ x_{n+1}^{(11)}=\frac{x_n^{(2)}+x_{n-1}^{(12)}+1}{x_{n-2}^{(8)}},\ x_{n+1}^{(12)}=\frac{x_n^{(8)}+x_{n-1}^{(10)}+1}{x_{n-2}^{(6)}},
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
&x_{n+1}^{(1)}=\frac{\max\left\{x_n^{(5)}+x_{n-1}^{(9)},1\right\}}{x_{n-2}^{(3)}},\ x_{n+1}^{(2)}=\frac{\max\left\{x_n^{(12)}+x_{n-1}^{(8)},1\right\}}{x_{n-2}^{(10)}},\ x_{n+1}^{(3)}=\frac{x_n^{(4)}+x_{n-1}^{(7)}+1}{x_{n-2}^{(11)}},\\
&x_{n+1}^{(4)}=\frac{\max\left\{x_n^{(7)}+x_{n-1}^{(11)},1\right\}}{x_{n-2}^{(2)}},\ x_{n+1}^{(5)}=\frac{\max\left\{x_n^{(9)}+x_{n-1}^{(3)},1\right\}}{x_{n-2}^{(4)}},\ x_{n+1}^{(6)}=\frac{\max\left\{x_n^{(1)}+x_{n-1}^{(5)},1\right\}}{x_{n-2}^{(9)}},\\
&x_{n+1}^{(7)}=\frac{\max\left\{x_n^{(11)}+x_{n-1}^{(2)},1\right\}}{x_{n-2}^{(12)}},\ x_{n+1}^{(8)}=\frac{\max\left\{x_n^{(10)}+x_{n-1}^{(6)},1\right\}}{x_{n-2}^{(1)}},\ x_{n+1}^{(9)}=\frac{\max\left\{x_n^{(3)}+x_{n-1}^{(4)},1\right\}}{x_{n-2}^{(7)}},\\
&x_{n+1}^{(10)}=\frac{\max\left\{x_n^{(6)}+x_{n-1}^{(1)},1\right\}}{x_{n-2}^{(5)}},\ x_{n+1}^{(11)}=\frac{\max\left\{x_n^{(2)}+x_{n-1}^{(12)},1\right\}}{x_{n-2}^{(8)}},\ x_{n+1}^{(12)}=\frac{\max\left\{x_n^{(8)}+x_{n-1}^{(10)},1\right\}}{x_{n-2}^{(6)}},
\end{aligned}
\tag{3.4}
$$

are periodic with period 24. The permutation which corresponds to each of these systems is $\pi = (1\ \ 5\ \ 9\ \ 3\ \ 4\ \ 7\ \ 11\ \ 2\ \ 12\ \ 8\ \ 10\ \ 6)$. Its order is 12. This implies that systems (3.3) and (3.4) are periodic with period 24.

(iii) The systems

$$
x_{n+1}^{(1)}=\frac{x_n^{(2)}+1}{x_{n-1}^{(3)}},\ x_{n+1}^{(2)}=\frac{x_n^{(3)}+1}{x_{n-1}^{(5)}},\ x_{n+1}^{(3)}=\frac{x_n^{(5)}+1}{x_{n-1}^{(4)}},\ x_{n+1}^{(4)}=\frac{x_n^{(1)}+1}{x_{n-1}^{(2)}},\ x_{n+1}^{(5)}=\frac{x_n^{(4)}+1}{x_{n-1}^{(1)}},
\tag{3.5}
$$

and

$$x_{n+1}^{(1)}=\frac{\max\left\{x_n^{(2)},1\right\}}{x_{n-1}^{(3)}},\ x_{n+1}^{(2)}=\frac{\max\left\{x_n^{(3)},1\right\}}{x_{n-1}^{(5)}},$$
$$x_{n+1}^{(3)}=\frac{\max\left\{x_n^{(5)},1\right\}}{x_{n-1}^{(4)}},\ x_{n+1}^{(4)}=\frac{\max\left\{x_n^{(1)},1\right\}}{x_{n-1}^{(2)}},\ x_{n+1}^{(5)}=\frac{\max\left\{x_n^{(4)},1\right\}}{x_{n-1}^{(1)}}, \tag{3.6}$$

are periodic with period 5. Indeed, the permutation which corresponds to each of these systems is $\pi=(1\ \ 2\ \ 3\ \ 5\ \ 4)$. So, its order is 5. This implies that systems (3.5) and (3.6) are periodic with period 5.

(iv) The systems

$$x_{n+1}^{(1)}=\frac{x_n^{(3)}+1}{x_{n-1}^{(2)}},\ x_{n+1}^{(2)}=\frac{x_n^{(1)}+1}{x_{n-1}^{(3)}},\ x_{n+1}^{(3)}=\frac{x_n^{(2)}+1}{x_{n-1}^{(1)}},\ x_{n+1}^{(4)}=\frac{x_n^{(4)}+1}{x_{n-1}^{(4)}},\ x_{n+1}^{(5)}=\frac{x_n^{(5)}+1}{x_{n-1}^{(5)}}, \tag{3.7}$$

and

$$x_{n+1}^{(1)}=\frac{\max\left\{x_n^{(3)},1\right\}}{x_{n-1}^{(2)}},\ x_{n+1}^{(2)}=\frac{\max\left\{x_n^{(1)},1\right\}}{x_{n-1}^{(3)}},$$
$$x_{n+1}^{(3)}=\frac{\max\left\{x_n^{(2)},1\right\}}{x_{n-1}^{(1)}},\ x_{n+1}^{(4)}=\frac{\max\left\{x_n^{(4)},1\right\}}{x_{n-1}^{(4)}},\ x_{n+1}^{(5)}=\frac{\max\left\{x_n^{(5)},1\right\}}{x_{n-1}^{(5)}}, \tag{3.8}$$

are periodic with period 15. The permutation which corresponds to each of these systems is $\pi=(1\ \ 3\ \ 2)(4)(5)$. Its order is $3\neq 0 \mod 5$. By Theorem 2.1, systems (3.7) and (3.8) are periodic with period 15.

(v) The systems

$$x_{n+1}^{(1)}=\frac{x_n^{(3)}+1}{x_{n-1}^{(2)}},\ x_{n+1}^{(2)}=\frac{x_n^{(4)}+1}{x_{n-1}^{(5)}},\ x_{n+1}^{(3)}=\frac{x_n^{(2)}+1}{x_{n-1}^{(4)}},$$
$$x_{n+1}^{(4)}=\frac{x_n^{(5)}+1}{x_{n-1}^{(1)}},\ x_{n+1}^{(5)}=\frac{x_n^{(1)}+1}{x_{n-1}^{(3)}},\ x_{n+1}^{(6)}=\frac{x_n^{(7)}+1}{x_{n-1}^{(6)}},\ x_{n+1}^{(7)}=\frac{x_n^{(6)}+1}{x_{n-1}^{(7)}}, \tag{3.9}$$

and

$$x_{n+1}^{(1)}=\frac{\max\left\{x_n^{(3)},1\right\}}{x_{n-1}^{(2)}},\ x_{n+1}^{(2)}=\frac{\max\left\{x_n^{(4)},1\right\}}{x_{n-1}^{(5)}},\ x_{n+1}^{(3)}=\frac{\max\left\{x_n^{(2)},1\right\}}{x_{n-1}^{(4)}},$$
$$x_{n+1}^{(4)}=\frac{\max\left\{x_n^{(5)},1\right\}}{x_{n-1}^{(1)}},\ x_{n+1}^{(5)}=\frac{\max\left\{x_n^{(1)},1\right\}}{x_{n-1}^{(3)}},\ x_{n+1}^{(6)}=\frac{\max\left\{x_n^{(7)},1\right\}}{x_{n-1}^{(6)}},\ x_{n+1}^{(7)}=\frac{\max\left\{x_n^{(6)},1\right\}}{x_{n-1}^{(7)}}, \tag{3.10}$$

are periodic with period 10. The permutation which corresponds to each of these systems is $\pi=(1\ \ 3\ \ 2\ \ 4\ \ 5)(6\ \ 7)$. Its order is $10=0 \mod 5$. By Theorem 2.1, systems (3.9) and (3.10) are periodic with period 10.

(vi) The systems

$$x_{n+1}^{(1)}=\frac{x_n^{(3)}+x_{n-1}^{(2)}+1}{x_{n-2}^{(1)}},\ x_{n+1}^{(2)}=\frac{x_n^{(1)}+x_{n-1}^{(3)}+1}{x_{n-2}^{(2)}},\ x_{n+1}^{(3)}=\frac{x_n^{(2)}+x_{n-1}^{(1)}+1}{x_{n-2}^{(3)}},\ x_{n+1}^{(4)}=\frac{x_n^{(4)}+x_{n-1}^{(4)}+1}{x_{n-2}^{(4)}} \tag{3.11}$$

and

$$x_{n+1}^{(1)}=\frac{\max\left\{x_n^{(3)},x_{n-1}^{(2)},1\right\}}{x_{n-2}^{(1)}},\ x_{n+1}^{(2)}=\frac{\max\left\{x_n^{(1)},x_{n-1}^{(3)},1\right\}}{x_{n-2}^{(2)}},$$
$$x_{n+1}^{(3)}=\frac{\max\left\{x_n^{(2)},x_{n-1}^{(1)},1\right\}}{x_{n-2}^{(3)}},\ x_{n+1}^{(4)}=\frac{\max\left\{x_n^{(4)},x_{n-1}^{(4)},1\right\}}{x_{n-2}^{(4)}}, \tag{3.12}$$

are periodic with period 24. The permutation which corresponds to each of these systems is $\pi=(1\ 3\ 2)(4)$. Its order is 3. This implies that systems (3.11) and (3.12) are periodic with period 24.

(vii) The systems

$$x_{n+1}^{(1)}=\frac{x_n^{(2)}+x_{n-1}^{(1)}+1}{x_{n-2}^{(2)}},\ x_{n+1}^{(2)}=\frac{x_n^{(1)}+x_{n-1}^{(2)}+1}{x_{n-2}^{(1)}},$$
$$x_{n+1}^{(3)}=\frac{x_n^{(4)}+x_{n-1}^{(3)}+1}{x_{n-2}^{(4)}},\ x_{n+1}^{(4)}=\frac{x_n^{(3)}+x_{n-1}^{(4)}+1}{x_{n-2}^{(3)}},\ x_{n+1}^{(5)}=\frac{x_n^{(5)}+x_{n-1}^{(5)}+1}{x_{n-2}^{(5)}}, \tag{3.13}$$

and

$$x_{n+1}^{(1)}=\frac{\max\left\{x_n^{(2)},x_{n-1}^{(1)},1\right\}}{x_{n-2}^{(2)}},\ x_{n+1}^{(2)}=\frac{\max\left\{x_n^{(1)},x_{n-1}^{(2)},1\right\}}{x_{n-2}^{(1)}},$$
$$x_{n+1}^{(3)}=\frac{\max\left\{x_n^{(4)},x_{n-1}^{(3)},1\right\}}{x_{n-2}^{(4)}},\ x_{n+1}^{(4)}=\frac{\max\left\{x_n^{(3)},x_{n-1}^{(4)},1\right\}}{x_{n-2}^{(3)}},\ x_{n+1}^{(5)}=\frac{\max\left\{x_n^{(5)},x_{n-1}^{(5)},1\right\}}{x_{n-2}^{(5)}}, \tag{3.14}$$

are periodic with period 8. The permutation which corresponds to each of these systems is $\pi=(1\ 2)(3\ 4)(5)$. Its order is 2. This implies that systems (3.13) and (3.14) are periodic with period 8.

(viii) The systems

$$x_{n+1}^{(1)}=\frac{x_n^{(2)}+x_{n-1}^{(3)}+1}{x_{n-2}^{(4)}},\ x_{n+1}^{(2)}=\frac{x_n^{(3)}+x_{n-1}^{(4)}+1}{x_{n-2}^{(1)}},$$
$$x_{n+1}^{(3)}=\frac{x_n^{(4)}+x_{n-1}^{(1)}+1}{x_{n-2}^{(2)}},\ x_{n+1}^{(4)}=\frac{x_n^{(1)}+x_{n-1}^{(2)}+1}{x_{n-2}^{(3)}},\ x_{n+1}^{(5)}=\frac{x_n^{(5)}+x_{n-1}^{(5)}+1}{x_{n-2}^{(5)}}, \tag{3.15}$$

and

$$x_{n+1}^{(1)}=\frac{\max\left\{x_n^{(2)},x_{n-1}^{(3)},1\right\}}{x_{n-2}^{(4)}},\ x_{n+1}^{(2)}=\frac{\max\left\{x_n^{(3)},x_{n-1}^{(4)},1\right\}}{x_{n-2}^{(1)}},$$
$$x_{n+1}^{(3)}=\frac{\max\left\{x_n^{(4)},x_{n-1}^{(1)},1\right\}}{x_{n-2}^{(2)}},\ x_{n+1}^{(4)}=\frac{\max\left\{x_n^{(1)},x_{n-1}^{(2)},1\right\}}{x_{n-2}^{(3)}},\ x_{n+1}^{(5)}=\frac{\max\left\{x_n^{(5)},x_{n-1}^{(5)},1\right\}}{x_{n-2}^{(5)}}, \tag{3.16}$$

are periodic with period 8. The permutation which corresponds to each of these systems is $\pi=(1\ 2\ 3\ 4)(5)$. Its order is 4. This implies that systems (3.15) and (3.16) are periodic with period 8.

4. Conclusion

We extended the periodicity results of Iricanian and Stevic [9] to the systems

$$x_{n+1}^{(1)} = \frac{x_n^{\pi(1)}+1}{x_{n-1}^{\pi^2(1)}}, \quad x_{n+1}^{(2)} = \frac{x_n^{\pi(2)}+1}{x_{n-1}^{\pi^2(2)}}, \quad \dots \quad x_{n+1}^{(k)} = \frac{x_n^{\pi(k)}+1}{x_{n-1}^{\pi^2(k)}},$$

and

$$x_{n+1}^{(1)} = \frac{x_n^{\pi(1)}+x_{n-1}^{\pi^2(1)}+1}{x_{n-2}^{\pi^3(1)}}, \quad x_{n+1}^{(2)} = \frac{x_n^{\pi(2)}+x_{n-1}^{\pi^2(2)}+1}{x_{n-2}^{\pi^3(2)}}, \quad \dots \quad x_{n+1}^{(k)} = \frac{x_n^{\pi(k)}+x_{n-1}^{\pi^2(k)}+1}{x_{n-2}^{\pi^3(k)}},$$

where $\pi \in \mathcal{S}_k$. Every choice of a permutation π gives a system of difference equations.

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