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On Hyers–Ulam and Hyers–Ulam–Rassias Stability of a Nonlinear Second-Order Dynamic Equation on Time Scales

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Abstract: In this paper, we obtain sufficient conditions for Hyers–Ulam and Hyers–Ulam–Rassias stability of an abstract second–order nonlinear dynamic equation on bounded time scales. An illustrative example is given to show the applicability of the theoretical results.

Keywords: time scales; second order nonlinear dynamic equations on time scales; Hyers–Ulam stability; Hyers–Ulam–Rassias stability

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1. Introduction

In 1940, the audience of the Mathematics Club of the University of Wisconsin had the pleasure to listen to the talk of S. M. Ulam presenting a list of unsolved problems. See [1]. Such problems have been taken up by Hyers [2], Rassias [3] and other fine mathematicians. Since then, the stability problems of many function equations have been extensively investigated in various abstract spaces [4–6]. Obloza [7] appears to be the first author who investigated the Hyers–Ulam stability of a differential equation, followed by Alsina and Ger [8]. Then, a generalized result was given by S. E. Takahasi, T. Miura and S. Miyajima [9], in which they investigated the stability of the Banach space valued linear differential equation of first order (see also [10,11]).

Many interesting results concerning the Ulam stability of different types have been established. For example, see [12–23]. Some studies dealing with difference equations were published in [24,25]. Recently, many articles studied the Hyers–Ulam stability of Dynamic equations on time scales [26–30]. Hamza and Yaseen [31] generalized and extended the work of Douglas R. Anderson, Ben Gates and Dylan Heuer [26] for unbounded time scales. In [32], Hamza et al. obtained new sufficient conditions for Hyers–Ulam–Rassias stability of an abstract second-order linear dynamic equation on time scales.

In this paper, we investigate sufficient conditions for Hyers–Ulam and Hyers–Ulam–Rassias stability of second-order nonlinear dynamic equations on time scales of the form

$$\psi^{\Delta^2}(t) = \mathcal{Q}(t)\psi(t) + \mathcal{G}(t, \psi(t), h(\psi(t))) + f(t), \quad t \in \mathcal{J}^\kappa, \quad \psi^{\Delta^i}(a) = a_i \in \mathbb{X}, i = 0, 1, \quad (1)$$

where $\mathcal{J} := [a, b] \cap \mathbb{T}$ with a time scale $\mathbb{T} \subset \mathbb{R}$, $a, b \in \mathbb{T}, a < b$, and $\mathbb{X}$ is a Banach space endowed with a norm $\|.\|$. Additionally, $\mathcal{G}(t, x, y) : \mathcal{J} \times \mathbb{X}^2 \to \mathbb{X}$ is such that $\mathcal{G}(., x, y)$ is rd-continuous and $\mathcal{G}(t, ., y)$ and $\mathcal{G}(t, x, .)$ are continuous for all $t \in \mathcal{J}$ and $x, y \in \mathbb{X}$. Additionally, $\mathcal{Q} \in \mathcal{R}$, the family of all regressive and rd-continuous functions from $\mathcal{J}$ to $\mathbb{R}$, $f \in C_{rd}(\mathcal{J}, \mathbb{X})$ the space of all rd-continuous functions from $\mathcal{J}$ to $\mathbb{X}$, and $h : \mathbb{X} \to \mathbb{X}$ is continuous. As usual, for a bounded function $\Phi : X \to Y$ from a normed space X to a normed space Y, we denote

$$\|\Phi\|_\infty = \sup_{x \in X} \|\Phi(x)\|.$$

For the time scale terminology, we refer the reader to Bohner and Peterson [33,34]. We introduce the notion of the Lipschitz condition with some constants.

Definition 1. *A function $f : \mathbb{T} \times \mathbb{X}^k \to \mathbb{X}$ is said to satisfy the Lipschitz condition with constant $L > 0$ if*

$$\|f(t, x_1, \dots, x_k) - f(t, y_1, \dots, y_k)\| \leqslant L \sum_{i=1}^{k} \|x_i - y_i\| \tag{2}$$

for all $x_i, y_i \in \mathbb{X}$ and all $t \in \mathbb{T}$.

As usual, a function $h : \mathbb{X}^k \to \mathbb{X}$ is said to satisfy the Lipschitz condition with constant $\gamma > 0$ if

$$\|h(x_1, \dots, x_k) - h(y_1, \dots, y_k)\| \leqslant \gamma \sum_{i=1}^{k} \|x_i - y_i\|. \tag{3}$$

2. Sufficient Conditions for Existence and Uniqueness of Solutions

Theorem 1. *Let $\mathcal{K}(t, x) : \mathcal{J} \times \mathbb{X} \to \mathbb{X}$ be rd-continuous in $t \in \mathcal{J}$ for every $x \in \mathbb{X}$, and continuous in x for every $t \in \mathcal{J}$. Then, ψ is a solution of*

$$\psi^{\Delta^2}(t) = \mathcal{K}(t, \psi(t)), \quad t \in \mathcal{J}^\kappa, \quad \psi^{\Delta^i}(a) = a_i \in \mathbb{X}, i = 0, 1, \tag{4}$$

if, and only if ψ solves the integral equation

$$\psi(t) = a_0 + a_1(t - a) - \int_a^t (s - t + \mu(s))\mathcal{K}(s, \psi(s))\Delta s, \quad t \in \mathcal{J}, \tag{5}$$

for some constants $a_0, a_1 \in \mathbb{X}$.

Proof. Assume that ψ satisfies the integral Equation (5). We denote by

$$\mathcal{M}(t) = -\int_a^t (s - t + \mu(s))\mathcal{K}(s, \psi(s))\Delta s.$$

By Theorem 1.117(i) in [33], we conclude that

$$\mathcal{M}^\Delta(t) = \int_a^t \mathcal{K}(s, \psi(s))\Delta s,$$

and

$$\mathcal{M}^{\Delta^2}(t) = \mathcal{K}(t, \psi(t)).$$

This implies that $\psi^{\Delta^2}(t) = \mathcal{K}(t, \psi(t))$. To prove the other direction, assume ψ is a solution of Equation (4). We denote by

$$G(t) = \int_a^t \mathcal{K}(s, \psi(s))\, \Delta s,$$

and

$$\mathcal{L}(t) = \int_a^t G(s)\, \Delta s.$$

By integrating both sides of (4) twice, we have

$$\psi(t) = a_0 + a_1(t - a) + \mathcal{L}(t).$$

Here, $a_i = \psi^{\Delta^i}(a), i = 0, 1$. It is readily seen that $\mathcal{M}(t) = \mathcal{L}(t)$ for every t. Indeed, we have

$$\begin{aligned}\mathcal{L}^{\Delta}(t) &= G(t)\\ &= \int_a^t \mathcal{K}(s, \psi(s))\,\Delta\,s\\ &= \mathcal{M}^{\Delta}(t).\end{aligned}$$

Consequently, $\mathcal{M}(t) = \mathcal{L}(t) + C, t \in [a, b]_{\mathbb{T}}$. We have $C = \mathcal{M}(a) - \mathcal{L}(a) = 0$. Therefore, ψ satisfies Equation (5). □

As a direct consequence, setting $\mathcal{K}(t, \psi(t)) = \mathcal{Q}(t)\psi(t) + \mathcal{G}(t, \psi(t), h(\psi(t))) + f(t)$, we get the following:

Corollary 1. *ψ is a solution of Equation (1) if and only if ψ solves the integral equation*

$$\psi(t) = a_0 + a_1(t-a) - \int_a^t (s - t + \mu(s))(\mathcal{Q}(s)\psi(s) + \mathcal{G}(s, \psi(s), h(\psi(s))) + f(s))\Delta s, \quad t \in \mathcal{J}, \tag{6}$$

for some constants $a_0, a_1 \in \mathbb{X}$.

Throughout the rest of the paper, we use the following conditions.

(**A**) $\mathcal{Q} \in \mathcal{R}$ and $f \in C_{rd}(\mathcal{J}, \mathbb{X})$.
(**B**) $\mathcal{G}$ and h satisfy the Lipschitz conditions with constants β and γ, respectively.
(**C**) For any $a_0, a_1 \in \mathbb{X}$, (1) has a solution ϕ satisfying $\phi^{\Delta^i}(a) = a_i, i = 0, 1$.
(**D**) There is $\alpha \in (0, 1)$ such that

$$\sup_{t \in \mathcal{J}} \int_a^t |\mathcal{Q}(s)|\Delta s \le \frac{\alpha}{b-a} - \beta(1+\gamma)(b-a).$$

Theorem 2. *Assume* (**A**), (**B**), *and* (**D**). *If $a_0, a_1 \in \mathbb{X}$, then* (1) *has a unique solution ϕ satisfying $\phi^{\Delta^i}(a) = a_i, i = 0, 1$.*

Proof. Fix $a_0, a_1 \in \mathbb{X}$. Define the operator $T : C_{rd}(\mathcal{J}, \mathbb{X}) \to C_{rd}(\mathcal{J}, \mathbb{X})$ by

$$T\psi(t) = a_0 + a_1(t-a) + \int_a^t (s - t + \mu(s))(-\mathcal{Q}(s)\psi(s) - \mathcal{G}(s, \psi(s), h(\psi(s))) - f(s))\Delta s.$$

For $\psi_1, \psi_2 \in (\mathcal{J}, \mathbb{X})$, we have

$$\begin{aligned}\|T\psi_1(t) - T\psi_2(t)\| \le \int_a^t &|s - t + \mu(s)|(|\mathcal{Q}(s)|\|\psi_1(s) - \psi_2(s)\|\\ &+ \|\mathcal{G}(s, \psi_1(s), h(\psi_1(s))) - \mathcal{G}(s, \psi_2(s), h(\psi_2(s)))\|)\Delta s.\end{aligned} \tag{7}$$

It follows from (**B**) and (**D**) that

$$\begin{aligned}
\|T\psi_1(t) - T\psi_2(t)\| &\leq \int_a^t |s-t+\mu(s)|\;|\mathcal{Q}(s)|\|\psi_1(s)-\psi_2(s)\|\Delta s \\
&\quad + \beta\int_a^t (\|\psi_1(s)-\psi_2(s)\| + \|h(\psi_1(s)) - h(\psi_2(s))\|)\Delta\, s \\
&\leq \int_a^t |s-t+\mu(s)||\mathcal{Q}(s)|\|\psi_1(s)-\psi_2(s)\|\Delta\, s \\
&\quad + \beta\int_a^t (\|\psi_1(s)-\psi_2(s)\| + \gamma\|\psi_1(s)-\psi_2(s)\|)\Delta\, s \\
&\leq \int_a^t |s-t+\mu(s)|[|\mathcal{Q}(s)|\|\psi_1(s)-\psi_2(s)\| + \beta(1+\gamma)\|\psi_1(s)-\psi_2(s)\|]\Delta\, s \\
&\leq (b-a)\|\psi_1-\psi_2\|_\infty(\int_a^t |\mathcal{Q}(s)|\Delta s + \beta(1+\gamma)(b-a)) \\
&\leq \alpha\|\psi_1-\psi_2\|_\infty.
\end{aligned}$$

This implies that T is a contraction. Therefore, T has a unique fixed point ϕ, which is the unique solution of the integral Equation (6). By Corollary 1, ϕ is the unique solution of (1) satisfying the initial conditions. □

3. Hyers–Ulam Stability Results

In this section, we assume that $\mathcal{Q} \in C_{rd}(\mathcal{J}, \mathbb{R})$ and $f \in C_{rd}(\mathcal{J}, \mathbb{X})$. We investigate the Hyers–Ulam stability of (1). For a function $\psi \in C^2_{rd}(\mathcal{J}, \mathbb{X})$, the space of all rd-continuous functions whose first and second derivatives exist and are rd-continuous, we denote

$$\mathcal{H}_\psi(t) = \mathcal{Q}(t)\psi(t) + \mathcal{G}(t, \psi(t), h(\psi(t))) + f(t), \tag{8}$$

and

$$g_\psi(t) := \psi^{\Delta^2}(t) - \mathcal{H}_\psi(t). \tag{9}$$

First, we recall the concept of Hyers–Ulam stability. See [12].

Definition 2 (Hyers–Ulam Stability)**.** *We say that* (1) *has Hyers–Ulam stability if there exists a constant $L > 0$, a so-called HUS constant, with the following property. For any $\varepsilon > 0$, if $\psi \in C^2_{rd}(\mathcal{J}, \mathbb{X})$ is such that*

$$\|g_\psi(t)\| \leqslant \varepsilon \quad for\ all \quad t \in \mathcal{J}^\kappa, \tag{10}$$

then there exists a solution $\phi : \mathcal{J} \to \mathbb{X}$ of (1) *such that*

$$\|\psi(t) - \phi(t)\| \leqslant L\varepsilon \quad for\ all \quad t \in \mathcal{J}. \tag{11}$$

The next Theorem establishes sufficient conditions for the Hyers–Ulam stability of (1).

Theorem 3. *If* (**A**), (**B**), *and* (**C**) *hold, then* (1) *has Hyers–Ulam stability with HUS constant*

$$L := (b-a)^2 e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b, a). \tag{12}$$

Proof. Let $\epsilon > 0$ and $\psi \in C^2_{rd}(\mathcal{J}, \mathbb{X})$ such that (10) holds. Then ψ satisfies the equation

$$\psi^{\Delta^2}(t) = \mathcal{H}_\psi(t) + g_\psi(t), \;\; t \in \mathcal{J}^{\kappa^2}. \tag{13}$$

Let $a_i = \psi^{\Delta^i}(a), i = 0, 1$. By Theorem 1, ψ satisfies the integral equation

$$\psi(t) = a_0 + a_1(t-a) - \int_a^t (s-t+\mu(s))(\mathcal{H}_\psi(s) + g_\psi(s))\Delta s. \tag{14}$$

By $(\mathbf{C})$, there exists a solution ϕ of (1) with $\phi^{\Delta^i}(a) = a_i, i = 0, 1$, that is, by Corollary 1,

$$\phi(t) = a_0 + a_1(t-a) - \int_a^t (s - t + \mu(s))\mathcal{H}_\phi(s)\Delta s. \tag{15}$$

Subtracting (15) from (14), we find, for all $t \in \mathcal{J}$,

$$\begin{aligned} \|\psi(t) - \phi(t)\| \leqslant \| \int_a^t (s-t+\mu(s))g_\psi(s)\Delta s\| + \| \int_a^t (s-t+\mu(s))Q(s)(\psi(s)-\phi(s))\Delta s\| \\ + \| \int_a^t (s-t+\mu(s))[\mathcal{G}(s,\psi(s),h(\psi(s))) - \mathcal{G}(s,\phi(s),h(\phi(s)))]\Delta s\|. \end{aligned} \tag{16}$$

Taking into account $(\mathbf{B})$, we get

$$\begin{aligned} \|\psi(t) - \phi(t)\| &\leqslant (b-a)\int_a^t \|g_\psi(s)\|\Delta s + \int_a^t (b-a)\|\mathcal{Q}(s)(\psi(s)-\phi(s))\|\Delta s \\ &\quad + \int_a^t (b-a)\beta[\|\psi(s)-\phi(s)\| + \|h(\psi(s)) - h(\phi(s))\|]\Delta s \\ &\leqslant (b-a)\int_a^t \|g_\psi(s)\|\Delta s + \int_a^t (b-a)\|\mathcal{Q}(s)(\psi(s)-\phi(s))\|\Delta s \\ &\quad + \int_a^t (b-a)\beta[\|\psi(s)-\phi(s)\| + \gamma\|\psi(s)-\phi(s)\|]\Delta\, s \\ &\leqslant (b-a)\int_a^t \|g_\psi(s)\|\Delta s + \int_a^t (b-a)\|\mathcal{Q}(s)(\psi(s)-\phi(s))\|\Delta s \\ &\quad + \int_a^t (b-a)\beta(1+\gamma)\|\psi(s)-\phi(s)\|\Delta s. \end{aligned}$$

Hence,

$$\|\psi(t) - \phi(t)\| \leqslant (b-a)\int_a^t \|g_\psi(s)\|\Delta s + \int_a^t (b-a)[|\mathcal{Q}(s)| + \beta(1+\gamma)]\|\psi(s)-\phi(s)\|\Delta s. \tag{17}$$

Since $\|g_\psi(t)\| \leqslant \varepsilon$ holds for $t \in \mathcal{J}$, we have

$$\|\psi(t) - \phi(t)\| \leqslant \varepsilon(b-a)^2 + \int_a^t (b-a)[|\mathcal{Q}(s)| + \beta(1+\gamma)]\|\psi(s)-\phi(s)\|\Delta s.$$

Thus, by Gronwall's inequality, ([33] Corollary 6.7), we conclude that

$$\|\psi(t) - \phi(t)\| \leq \epsilon(b-a)^2 e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b,a). \tag{18}$$

Therefore, (1) has Hyers–Ulam stability with HUS constant L given in (12). □

Theorem 4. *If* $(\mathbf{A})$, $(\mathbf{B})$, *and* $(\mathbf{D})$ *hold, then* (1) *has Hyers–Ulam stability with HUS constant*

$$L := \frac{(b-a)^2}{1-\alpha}. \tag{19}$$

Proof. Let $\epsilon > 0$ and $\psi \in C^2_{rd}(\mathcal{J}, \mathbb{X})$ such that (10) holds. Set $g_\psi(t)$ as in (9). Then ψ satisfies (13). Let $a_i = \psi^{\Delta^i}(a), i = 0, 1$. By Theorem 1, (14) holds. By Theorem 2, there exists a unique solution ϕ of (1) with $\phi^{\Delta^i}(a) = a_i, i = 0, 1$. By Corollary 1, $\phi(t)$ is as in (15). By subtracting (15) from (14) and as in the proof of Theorem 3, we get

$$\begin{aligned}\|\psi(t) - \phi(t)\| &\leqslant \epsilon(b-a)^2 + \int_a^t (b-a)[|\mathcal{Q}(s)| + \beta(1+\gamma)]\|\psi(s) - \phi(s)\|\Delta s \\ &\leqslant \epsilon(b-a)^2 + (b-a)\|\psi - \phi\|_\infty \int_a^t [|\mathcal{Q}(s)| + \beta(1+\gamma)]\Delta s \\ &\leqslant \epsilon(b-a)^2 + \alpha\|\psi - \phi\|_\infty, t \in \mathcal{J}.\end{aligned}$$

This implies that

$$\|\psi - \phi\|_\infty \leq \frac{(b-a)^2}{1-\alpha}\epsilon. \tag{20}$$

Therefore, (1) has Hyers–Ulam stability with HUS constant L given in (19). □

4. Hyers–Ulam–Rassias Stability

In this section, we introduce the Hyers–Ulam–Rassias Stability of (1).

Definition 3 (Hyers–Ulam–Rassias stability)**.** *Let $\mathcal{N}$ be a family of positive rd-continuous functions on $\mathcal{J}$. We say that Equation (1) has Hyers-Ulam-Rassias stability of type $\mathcal{N}$ if there exist a constant $L > 0$, a so-called HURS$_\mathcal{N}$ constant, with the following property. For any $\omega \in \mathcal{N}$, if $\psi \in C^2_{rd}(\mathcal{J}, \mathbb{X})$ is such that*

$$\|g_\psi(t)\| \leqslant \omega(t) \quad for\ all \quad t \in \mathcal{J}^{\kappa^2}, \tag{21}$$

then there exists a solution $\phi : \mathcal{J} \to \mathbb{X}$ of (1) such that

$$\|\psi(t) - \phi(t)\| \leqslant L\omega(t) \quad for\ all \quad t \in \mathcal{J}. \tag{22}$$

We note that Hyers–Ulam–Rassias stability yields Hyers–Ulam stability, when

$$\mathcal{N} = \{l_\epsilon : \epsilon > 0\},$$

where $l_\epsilon(t) = \epsilon, t \in \mathcal{J}$. We use the notations (8) and (9),

$$\mathcal{N}^* := \{\omega \in C_{rd}(\mathcal{J}, (0, \infty)) : \ \omega \ \text{ is nondecreasing}\} \tag{23}$$

and for $\Lambda \geqslant 1, \delta > 0$

$$\mathcal{N}^\delta_\Lambda := \left\{\omega \in C_{rd}(\mathcal{J}, (0, \infty)) : \int_a^t \omega^\Lambda(s)\,\Delta s \leq \delta\,\omega^\Lambda(t) \text{ for all } t \in \mathcal{J}\right\}. \tag{24}$$

The following theorem is concerned with Hyers–Ulam–Rassias stability.

Theorem 5. *If* (**A**), (**B**), *and* (**C**) *hold, then* (1) *has Hyers–Ulam–Rassias stability of type $\mathcal{N}^*$ with HURS$_{\mathcal{N}^*}$ constant*

$$L := (b-a)^2 e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b, a). \tag{25}$$

Proof. Let $\omega \in \mathcal{N}^*$ and $\psi \in C^2_{rd}(\mathcal{J}, \mathbb{X})$ be such that (21) holds. Then ψ satisfies (13). Let $a_i = \psi^{\Delta^i}(a), i = 0, 1$. By Theorem 1, (14) holds. By (**C**), there exists a solution ϕ of (1) that satisfies $\phi^{\Delta^i}(a) = a_i, i = 0, 1$. Then, (15) holds. Subtracting (15) from (14), we obtain inequality (16), and by taking into account (**B**), we get inequality (17).

Since $\|g_\psi(t)\| \leq \omega(t)$ for $t \in \mathcal{J}$, we get

$$\|\psi(t) - \phi(t)\| \leq (b-a)^2\omega(t) + \int_a^t (b-a)[|\mathcal{Q}(s)| + \beta(1+\gamma)]\|\psi(s) - \phi(s)\|\Delta s.$$

Using Gronwall's inequality, ([33] Theorem 6.4), and by ([33] Theorem 2.39), we get, for all $t \in \mathcal{J}$,

$$\begin{aligned}\|\psi(t)-\phi(t)\| \leq{}& (b-a)^2\omega(t)\\ &+\int_a^t e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,\sigma(s))(b-a)^2\omega(s)(b-a)[|\mathcal{Q}(s)|+\beta(1+\gamma)]\Delta s\\ \leqslant{}& (b-a)^2\omega(t)\\ &+(b-a)^2\omega(t)\int_a^t e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,\sigma(s))(b-a)[|\mathcal{Q}(s)|+\beta(1+\gamma)]\Delta s\\ \leqslant{}& (b-a)^2\omega(t)\\ &+(b-a)^2\omega(t)\Big(e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,a)-e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,t)\Big)\\ \leqslant{}& (b-a)^2 e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b,a)\omega(t).\end{aligned}$$

Therefore, (1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}^*$ with a constant L defined by (25). □

Theorem 6. *If* (**A**), (**B**), *and* (**C**) *hold, then* (1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{N}^* \cap \mathcal{N}_1^\delta$ *with* $HURS_{\mathcal{N}^*\cap\mathcal{N}_1^\delta}$ *constant*

$$L := \delta(b-a)e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b,a). \tag{26}$$

Proof. Let $\omega \in \mathcal{N}^* \cap \mathcal{N}_1^\delta$ and $\psi \in C_{rd}^2(\mathcal{J},\mathbb{X})$ be such that (21) holds. Then ψ satisfies (13). Let $a_i = \psi^{\Delta^i}(a), i = 0,1$. By Theorem 1, (14) holds. By (**C**), there exists a solution ϕ of (1) that satisfies $\phi^{\Delta^i}(a) = a_i, i = 0,1$. By Corollary 1, (15) holds. Subtracting (15) from (14), we obtain inequality (16), and by taking into account (**B**), we get inequality (17).

Since $\|g_\psi(t)\| \leq \omega(t)$ for $t \in \mathcal{J}$, we get, for all $t \in \mathcal{J}$,

$$\begin{aligned}\|\psi(t)-\phi(t)\| \leq{}& (b-a)\int_a^t \omega(s)\Delta s+\int_a^t |s-t+\mu(s)|\|\mathcal{Q}(s)(\psi(s)-\phi(s))\|\Delta s\\ &+\int_a^t |s-t+\mu(s)|[\beta\|\psi(s)-\phi(s)\|+\|h(\psi(s))-h(\phi(s))\|]\Delta s.\\ \leq{}& (b-a)\int_a^t \omega(s)\Delta s+\int_a^t (b-a)\|\mathcal{Q}(s)(\psi(s)-\phi(s))\|\Delta s\\ &+\int_a^t (b-a)\beta(1+\gamma)\|\psi(s)-\phi(s)\|\Delta s\\ \leq{}& \delta(b-a)\omega(t)+\int_a^t (b-a)[|\mathcal{Q}(s)|+\beta(1+\gamma)]\|\psi(s)-\phi(s)\|\Delta s.\end{aligned}$$

Applying Gronwall's inequality, ([33] Theorem 6.4), and by ([33] Theorem 2.39), we get, for all $t \in \mathcal{J}$,

$$\begin{aligned}\|\psi(t)-\phi(t)\| \leq{}& \delta(b-a)\omega(t)\\ &+\int_a^t e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,\sigma(s))(b-a)\delta\omega(s)(b-a)[|\mathcal{Q}(s)|+\beta(1+\gamma)]\Delta s\\ \leqslant{}& \delta(b-a)\omega(t)\\ &+\delta(b-a)\omega(t)\int_a^t e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,\sigma(s))(b-a)[|\mathcal{Q}(s)|+\beta(1+\gamma)]\Delta s\\ \leqslant{}& \delta(b-a)\omega(t)\\ &+\delta(b-a)\omega(t)\Big(e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,a)-e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,t)\Big)\\ \leqslant{}& \delta(b-a)e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b,a)\omega(t).\end{aligned}$$

Therefore, (1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}^* \cap \mathcal{N}_1^\delta$ with a constant L given in (26). □

Theorem 7. *Let $\Lambda > 1$ and $\Gamma := \Lambda/(\Lambda - 1)$. If* (**A**)*,* (**B**)*, and* (**C**) *hold, then* (1) *has Hyers–Ulam–Rassias stability of type $\mathcal{N}^* \cap \mathcal{N}_\Lambda^\delta$ with HURS$_{\mathcal{N}^* \cap \mathcal{N}_\Lambda^\delta}$ constant*

$$L := \delta^{\frac{1}{\Lambda}} (b-a)^{\frac{\Gamma+1}{\Gamma}} e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b,a). \tag{27}$$

Proof. Let $\omega \in \mathcal{N}^* \cap \mathcal{N}_\Lambda^\delta$ and $\psi \in C_{rd}^2(\mathcal{J}, \mathbb{X})$ be such that (21) holds. Then ψ satisfies (13). Let $a_i = \psi^{\Delta^i}(a), i = 0, 1$. By Theorem 1, (14) holds. By (**C**), there exists a solution ϕ of (1) that satisfies $\phi^{\Delta^i}(a) = a_i, i = 0, 1$. By Corollary 1, (15) holds. Subtracting (15) from (14), we obtain inequality (16), and by taking into account (**B**), we get inequality (17).

Since $\|g_\psi(t)\| \le \omega(t)$ for $t \in \mathcal{J}$, we get, for all $t \in \mathcal{J}$,

$$\begin{aligned}
\|\psi(t) - \phi(t)\| &\leqslant (b-a)\int_a^t \omega(s)\Delta s + (b-a)\int_a^t \|\mathcal{Q}(s)(\psi(s) - \phi(s))\|\Delta s \\
&\quad + (b-a)\beta(1+\gamma)\int_a^t \|\psi(s) - \phi(s)\|\Delta s \\
&\leqslant (b-a)\sqrt[\Gamma]{t-a}\sqrt[\Lambda]{\int_a^t \omega^\Lambda(s)\Delta s} + (b-a)\int_a^t \|\mathcal{Q}(s)(\psi(s) - \phi(s))\|\Delta s \\
&\quad + (b-a)\beta(1+\gamma)\int_a^t \|\psi(s) - \phi(s)\|\Delta s \\
&\leqslant \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(t) + \int_a^t (b-a)[|\mathcal{Q}(s)| + \beta(1+\gamma)]\|\psi(s) - \phi(s)\|\Delta s,
\end{aligned}$$

where we have used the Hölder inequality, ([33] Theorem 6.13). Thus, by applying Gronwall's inequality, ([33] Theorem 6.4), and by applying ([33] Theorem 2.39), we get, for all $t \in \mathcal{J}$,

$$\begin{aligned}
\|\psi(t) - \phi(t)\| &\le \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(t) \\
&\quad + \int_a^t e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,\sigma(s))\delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(s)(b-a)[|\mathcal{Q}(s)| + \beta(1+\gamma)]\Delta s \\
&\le \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(t) \\
&\quad + \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(t)\int_a^t e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,\sigma(s))(b-a)[|\mathcal{Q}(s)| + \beta(1+\gamma)]\Delta s \\
&\le \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(t) \\
&\quad + \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(t)\left(e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,a) - e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,t)\right) \\
&\leqslant \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}} e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b,a)\omega(t).
\end{aligned}$$

Therefore, Equation (1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}^* \cap \mathcal{N}_\Lambda^\delta$ with constant L defined in (27) . □

Theorem 8. *Let $\Lambda > 1$ and $\Gamma := \Lambda/(\Lambda - 1)$. If* (**A**)*,* (**B**)*, and* (**C**) *hold, then* (1) *has Hyers–Ulam–Rassias stability of type $\mathcal{N}_\Lambda^\delta$ with HURS$_{\mathcal{N}_\Lambda^\delta}$ constant*

$$L := \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\left(1 + \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}[\|\mathcal{Q}\|_\infty + \beta(1+\gamma)]e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b,a)\right). \tag{28}$$

Proof. Let $\omega \in \mathcal{N}_\Lambda^\delta$ and $\psi \in C_{rd}^2(\mathcal{J}, \mathbb{X})$ be such that (21) holds. Then ψ satisfies (13). Let $a_i = \psi^{\Delta^i}(a), i = 0, 1$. By Theorem 1, (14) holds. By (**C**), there exists a unique solution ϕ of

(1) that satisfies $\phi^{\Delta^i}(a) = a_i, i = 0, 1$. By Theorem 1, (15) holds. Subtracting (15) from (14), we obtain inequality (16), and by taking into account $(\mathbf{B})$, we get inequality (17). Since $\|g_\psi(t)\| \leq \omega(t)$ for $t \in \mathcal{J}$, we obtain, for all $t \in \mathcal{J}$,

$$\begin{aligned}
\|\psi(t) - \phi(t)\| &\leqslant (b-a)\int_a^t \omega(s)\Delta s + \int_a^t |s - t + \mu(s)|\|\mathcal{Q}(s)(\psi(s) - \phi(s))\|\Delta\, s \\
&\quad + \int_a^t |s - t + \mu(s)|\beta[\|\psi(s) - \phi(s)\| + \|h(\psi(s)) - h(\phi(s))\|]\Delta\, s. \\
&\leqslant (b-a)\sqrt[\Gamma]{t-a}\sqrt[\Lambda]{\int_a^t \omega^\Lambda(s)\Delta s} + (b-a)\int_a^t \|\mathcal{Q}(s)(\psi(s) - \phi(s))\|\Delta s \\
&\quad + (b-a)\beta(1+\gamma)\int_a^t \|\psi(s) - \phi(s)\|\Delta s \\
&\leqslant \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(t) + \int_a^t (b-a)[|\mathcal{Q}(s)| + \beta(1+\gamma)]\|\psi(s) - \phi(s)\|\Delta s,
\end{aligned}$$

where we have applied the Hölder inequality, ([33] Theorem 6.13). Thus, by using Gronwall's inequality, ([33] Theorem 6.4), and by applying ([33] Theorem 2.39), we get, for all $t \in \mathcal{J}$,

$$\begin{aligned}
\|\psi(t) - \phi(t)\| &\leq \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(t) \\
&\quad + \int_a^t e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(t,\sigma(s))\delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(s)(b-a)[|\mathcal{Q}(s)| + \beta(1+\gamma)]\Delta\, s \\
&\leq \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(t) \\
&\quad + \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{2\Gamma+1}{\Gamma}}[\|\mathcal{Q}\|_\infty + \beta(1+\gamma)]e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b,a)\int_a^t \omega(s)\Delta\, s \\
&\leq \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(t) \\
&\quad + \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{2\Gamma+2}{\Gamma}}[\|\mathcal{Q}\|_\infty + \beta(1+\gamma)]e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b,a)\left(\int_a^t \omega^\Lambda(s)\Delta s\right)^{\frac{1}{\Lambda}} \\
&\leq \delta^{\frac{1}{\Lambda}}(b-a)^{\frac{\Gamma+1}{\Gamma}}\omega(t) + \delta^{\frac{2}{\Lambda}}(b-a)^{\frac{2\Gamma+2}{\Gamma}}[\|\mathcal{Q}\|_\infty \\
&\quad + \beta(1+\gamma)]e_{(b-a)[|\mathcal{Q}|+\beta(1+\gamma)]}(b,a)\omega(t) \\
&= L\omega(t).
\end{aligned}$$

Therefore, Equation (1) has Hyers–Ulam–Rassias stability of type $\mathcal{N}_\Lambda^\delta$ with constant L given in (28). □

Remark 1. *Since condition* $(\mathbf{D})$ *implies condition* $(\mathbf{C})$*, all results in Sections 3 and 4 are true, if we replace* $(\mathbf{C})$ *by* $(\mathbf{D})$.

Example 1. *Now, we give an example for which conditions* $(\mathbf{A})$, $(\mathbf{B})$ *and* $(\mathbf{D})$ *are satisfied. Let* $\mathbb{T} := \cup_{k=0}^\infty [2k, 2k+1]$. *Let* $m \in \mathbb{N}, a = 0$ *and* $b = 2m+1$. *Fix* $\alpha \in (0,1)$ *and* $\beta \in (0, \frac{\alpha}{2(2m+1)^2})$. *Assume* $f \in C_{rd}$, $\mathcal{G}(t,x,y) = \beta(\cos x + y)$, *and* $h(x) = \sin x$. *Choose a positive number* C *such that* $C \geq \frac{2^m e^{m+1}(2m+1)}{\alpha - 2\beta(2m+1)^2}$. *Take* $\mathcal{Q}(t) = \frac{e_1(t,0)}{C}$. *Equation (1) takes the form*

$$\psi^{\Delta^2}(t) = \frac{e_1(t,0)}{C}\psi(t) + \beta(\cos(\psi(t)) + \sin(\psi(t))) + f(t).$$

Clearly, condition (**A**) *holds. Additionally, condition* (**B**) *is true, since* $\mathcal{G}$ *and* h *satisfy Lipschitz conditions with constants* β *and* $\gamma = 1$*, respectively. Finally, we check that* (**D**) *holds. Indeed,*

$$\begin{aligned}
\int_0^b \mathcal{Q}(s)\,\Delta s &= \frac{1}{C}(e_1(b,0)-1)\\
&\le \frac{1}{C}e_1(b,0)\\
&= \frac{1}{C}e_1(2m+1,0)\\
&= \frac{1}{C}2^m e^{m+1}\\
&\le \frac{\alpha-2\beta(2m+1)^2}{2^m e^{m+1}(2m+1)}2^m e^{m+1}\\
&= \frac{\alpha-2\beta(2m+1)^2}{2m+1}
\end{aligned}$$

where according to ([33] Example 2.58), we have $e_1(2m+1,0) = 2^m e^{m+1}$.

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Abbreviations

The following abbreviations are used in this manuscript:

HUS	Hyers–Ulam stability
HUSR	Hyers–Ulam–Rassias stability

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