J. Math. Computer Sci., 35 (2024), 336–347

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Hyers-Ulam and Hyers-Ulam-Rassias stability of first-order linear quantum difference equations

Alaa E. Hamza*, Maryam A. ALghamdi, Suha A. ALasmi

Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah, 21589, Saudi Arabia.

Abstract

In this paper, we investigate Hyers-Ulam and Hyers-Ulam-Rassias stability of first-order linear quantum difference equations associated with a general quantum difference operator. This operator includes as special cases Jackson q-difference and Hahn difference operators. At the end of the paper, an illustrative example is given to show the applicability of the theoretical results.

Keywords: Hyers-Ulam stability, Hyers-Ulam-Rassias stability.

2020 MSC: 34N05, 34D20, 39A30.

1. Introduction

Quantum difference operators allows us to deal with non-differentiable functions in the usual sense. They have an essential role due to their applications in several mathematical areas such as orthogonal polynomials, basic hypergeometric function, combinatorics, the calculus of variations and the theory of relativity (see [1, 4, 9]). New results in quantum calculus can be found in [8] and the references cited therein.

In a talk for Ulam at 1940, he discussed a number of important unsolved mathematical problems. After that, these problems were collected in [15]. The notion of Ulam stability arose from a question of these problems concerning the stability of group homomorphisms. In [10], Hyers gave a first affirmative partial answer to this question for Banach spaces. After many years, Rassias [12] extended the result of Hyers by allowing an unbounded Cauchy difference. Since then, stability problems of many functional equations have been extensively investigated in various abstract spaces [2, 11, 12]. Henceafter, many fine mathematicians are interesting in studying this type of stability, which is called Hyers-Ulam-Rassias stability (HURs), for many differential and functional equations, see also [5, 13].

In this paper, we consider the quantum difference operator D_Γ, which is defined by

$$D_\Gamma g(t) = \frac{g(\Gamma(t)) - g(t)}{\Gamma(t) - t}$$

*Corresponding author

Email addresses: AHSABDULLAH@uj.edu.sa (Alaa E. Hamza), maalghamdi@uj.edu.sa (Maryam A. ALghamdi), suhaalharbi@outlook.com (Suha A. ALasmi)

doi: 10.22436/jmcs.035.03.06

Received: 2023-11-03 *Revised:* 2023-12-11 *Accepted:* 2024-03-18

for every t with $\Gamma(t) \neq t$ and $D_\Gamma g(t) = g'(t)$ when $\Gamma(t) = t$ provided that $g'(t)$ exists in the usual sense. Here, Γ is a continuous function on an interval $\mathcal{I}$ for which $\Gamma(t) \in \mathcal{I}$ for any $t \in \mathcal{I}$, and g is an arbitrary function from $\mathcal{I}$ to a Banach space $\mathbb{B}$. A function is said to be Γ-differentiable on $\mathcal{I}$ if it is differentiable in the usual sense at every point $t \in \mathcal{I}$ at which $\Gamma(t) = t$. If $\Gamma(t) = qt$, $q \in (0, 1)$, then $D_\Gamma = D_q$, the Jackson q-difference operator and if $\Gamma(t) = qt + \omega$, $q \in (0, 1)$, $\omega > 0$, then $D_\Gamma = D_{q,w}$, the Hahn difference operator. Theory of quantum difference equations helps us to avoid proving results twice, once for Jackson q- difference equations and once for Hahn difference equations (see [8]).

Our objective is to investigate HURs of first-order linear quantum difference equations, that include D_Γ. We denote

$$\Gamma^k(t) := \underbrace{\Gamma \circ \Gamma \circ \Gamma \circ \cdots \circ \Gamma}_{k \text{ times}}(t),$$

$k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of natural numbers. For convenience $\Gamma^0(t) = t$ for all $t \in \mathcal{I}$. It is well known that a continuous function $\Gamma : [a, b] \to [a, b]$ has at least a fixed point.

Throughout the paper, we assume Γ is a continuous increasing function on $\mathcal{I}$ that has a unique fixed point $s_0 \in \mathcal{I}$ and satisfies the following inequality:

$$(t - s_0)(\Gamma(t) - t) \leqslant 0 \quad \text{for all } t \in \mathcal{I}.$$

Moreover, $\mathbb{B}$ denotes a Banach space endowed with a norm $\|\|$. Also, the Γ-interval is defined to be

$$[a, b]_\Gamma = \{\Gamma^k(a); k \in \mathbb{N}_0\} \cup \{\Gamma^k(b); k \in \mathbb{N}_0\} \cup \{s_0\}, \text{ where } a, b \in \mathcal{I}.$$

Finally, for any set $B \subset \mathbb{R}$, the set B^* is defined by

$$B^* = B \backslash \{s_0\}.$$

For $d \in [a, b]$, the following facts are commonly known to be true:

(1) for $d > s_0$, we have $\Gamma^k(d)$ is decreasing to s_0 as $k \to \infty$;
(2) for $d < s_0$, we have $\Gamma^k(d)$ is increasing to s_0 as $k \to \infty$.

Accordingly, it is convenient to set $\Gamma^\infty(t) = s_0, t \in [a, b]$. For more details about quantum difference calculus, we refer the reader to [7–9]. We only mention some fundamental definitions and theorems that will be useful in our investigations.

Definition 1.1. Let $g : \mathcal{I} \longrightarrow \mathbb{B}$ and $c, d \in \mathcal{I}$. The Γ-integral of g from c to d is defined by

$$\int_c^d g(t) d_\Gamma t = \int_{s_0}^d g(t) d_\Gamma t - \int_{s_0}^c g(t) d_\Gamma t,$$

where

$$\int_{s_0}^h g(t) d_\Gamma t = \sum_{k=0}^{\infty} (\Gamma^k(h) - \Gamma^{k+1}(h)) g(\Gamma^k(h)), \quad h \in \mathcal{I},$$

provided that the series converges at $h = c$ and $h = d$. The function g is called Γ-integrable on $\mathcal{I}$ if the series converges at c, d for all $c, d \in \mathcal{I}$. Clearly, if g is continuous at $s_0 \in \mathcal{I}$, then g is Γ-integrable on $\mathcal{I}$, see [8].

Theorem 1.2 ([8])**.** *Let* $g : \mathcal{I} \longrightarrow \mathbb{B}$ *be continuous at* s_0*. Define the function*

$$G(t) = \int_{s_0}^t g(s) d_\Gamma s, \quad t \in I.$$

Then G *is continuous at* $s_0, D_\Gamma G(t)$ *exists for all* $t \in \mathcal{I}$*, and* $D_\Gamma G(t) = g(t)$*.*

Theorem 1.3 ([8])**.** *Let $g : \mathcal{I} \to \mathbb{B}$ be Γ-differentiable on $\mathcal{I}$. Then*

$$\int_c^d D_\Gamma g(\eta)\, d_\Gamma \eta = g(d) - g(c), \quad c, d \in \mathcal{I}.$$

Theorem 1.4 ([8])**.** *If $\mathcal{Q} : \mathcal{I} \to \mathbb{B}$ is continuous at s_0, then the series $\sum_{k=0}^{\infty} \|(\Gamma^k(t) - \Gamma^{k+1}(t))\mathcal{Q}(\Gamma^k(t))\|$ is uniformly convergent on every compact interval $J \subseteq \mathcal{I}$ containing s_0.*

Finally, the following Hölder inequality in the quantum setting was proved in [6] (see also [9]).

Theorem 1.5 (Γ-Hölder inequality)**.** *If $f \in L^p([a, b]_\Gamma, \mathbb{R})$, and $g \in L^q([a, b]_\Gamma, \mathbb{B})$, where $p > 1, q = \frac{p}{p-1}$, then $fg \in (L^1[a, b]_\Gamma, \mathbb{B})$ and*

$$\|fg\|_1 \leqslant \|f\|_p \, \|g\|_q,$$

that is

$$\int_a^b \|f(t)g(t)\| d_\Gamma t \leqslant \Big(\int_a^b |f(t)|^p d_\Gamma t \Big)^{\frac{1}{p}} \Big(\int_a^b \|g(t)\|^q d_\Gamma t \Big)^{\frac{1}{q}}.$$

If we put $p = q = 2$ in the Γ-Hölder inequality we get the Γ-Cauchy-Schwarz inequality.

Our paper is organized as follows. Section 2 is devoted to quantum exponential functions and existence and uniqueness of solutions of first-order linear quantum difference equations. In Sections 3 (resp. 4), we establish sufficient conditions for Hyers-Ulam (resp. Hyers-Ulam-Rassias) stability of these equations. In Section 5, we give an illustrative example to show the applicability of the theoretical results.

2. Quantum exponential functions and first-order linear equations

Let $b \in \mathcal{I}$, $b > s_0$ and $(\mathbb{B}, \|\|)$ be a Banach space. As usual we denote by

$$C([s_0, b], \mathbb{B}) = \{\phi : [s_0, b] \to \mathbb{B} |\ \phi \text{ is continuous}\}$$

endowed with the supremum norm

$$\|\phi\|_\infty = \sup_{t \in [s_0, b]} \|\phi(t)\|.$$

Assume that $\mathcal{Q} : \mathcal{I} \to \mathbb{R}$ is continuous function at s_0. We recall the Γ-exponential function

$$e_{\Gamma,\mathcal{Q}}(t, s) = \frac{e_{\Gamma,\mathcal{Q}}(t)}{e_{\Gamma,\mathcal{Q}}(s)}, \tag{2.1}$$

where

$$e_{\Gamma,\mathcal{Q}}(t) = \frac{1}{\prod_{k=0}^{\infty} \Big[1 - \mathcal{Q}(\Gamma^k(t))(\Gamma^k(t) - \Gamma^{k+1}(t))\Big]}. \tag{2.2}$$

Henceforth, we assume for the convergence of this product that

$$1 - (x - \Gamma(x))\mathcal{Q}(x) \neq 0, \ x \in [s_0, b].$$

It is worth mentioning that, under this condition, $e_{\Gamma,\mathcal{Q}}(t)$ is the unique solution of the initial value problem

$$D_\Gamma y(t) = \mathcal{Q}(t)y(t), \quad t \in [s_0, b], \quad y(s_0) = y_0 \in \mathbb{B}.$$

The following result is of utmost importance in the theory of quantum difference equations.

Theorem 2.1 ([7])**.** *If $\mathcal{Q} : \mathcal{I} \to \mathbb{R}$ is continuous at s_0, then*

(i) $D_{\Gamma,\eta} e_{\Gamma,\mathcal{Q}}(t,\eta) = -\mathcal{Q}(\eta) e_{\Gamma,\mathcal{Q}}(t,\Gamma(\eta))$, *where* $D_{\Gamma,\eta}$ *is the* Γ*-derivative with respect to* η;

(ii) $\int_{s_0}^{t} e_{\Gamma,\mathcal{Q}}(t,\Gamma(\eta))\mathcal{Q}(\eta)\, d_\Gamma \eta = e_{\Gamma,\mathcal{Q}}(t) - 1$.

The following quantum Gronwall inequality was established in [9]. Likewise in the theory of differential equations, it is an efficient tool for establishing the existence and uniqueness of solutions.

Theorem 2.2 ([9])**.** *Let* $y, f, \mathcal{Q}$ *be real valued functions on* $\mathcal{I}$ *that are continuous at* s_0 *such that* $\mathcal{Q} \geqslant 0$*. Then*

$$y(t) \leqslant f(t) + \int_{s_0}^{t} y(\eta)\mathcal{Q}(\eta) d_\Gamma \eta, \quad t \in \mathcal{I},$$

implies

$$y(t) \leqslant f(t) + \int_{s_0}^{t} e_{\Gamma,\mathcal{Q}}(t,\Gamma(\eta)) f(\eta)\mathcal{Q}(\eta) d_\Gamma \eta, \ \ t \in \mathcal{I}.$$

Corollary 2.3 ([9])**.** *Let* $\mathcal{Q}(t) \geqslant 0$ *and* $\mu \in \mathbb{R}$*. Then*

$$y(t) \leqslant \mu + \int_{s_0}^{t} y(\eta)\mathcal{Q}(\eta) d_\Gamma \eta, \quad t \in \mathcal{I},$$

implies $y(t) \leqslant \mu e_{\Gamma,\mathcal{Q}}(t)$.

Now, we prove some useful properties of the Γ-exponential function which we use in our investigations.

Lemma 2.4. *If* $\mathcal{Q}$ *is continuous on* $[s_0, b]$*, the* Γ*-exponential* $e_{\Gamma,\mathcal{Q}}(t)$ *is positive and continuous on* $[s_0, b]$.

Proof. By Theorem 1.4, the series

$$\sum_{k=0}^{\infty} (\Gamma^k(t) - \Gamma^{k+1}(t))|\mathcal{Q}(\Gamma^k(t))|$$

is uniformly convergent on $[s_0, b]$. So, the product in (2.2) is uniformly and absolutely convergent on $[s_0, b]$, see [14]. Hence, $e_{\Gamma,\mathcal{Q}}(t)$ is continuous. Also it is positive because if it has different signs at two points, it will be zero at some intermediate point, which is impossible. □

Theorem 2.5. *If* $\mathcal{Q} \geqslant 0$ *is a continuous function on* $[s_0, b]$*, then* $e_{\Gamma,\mathcal{Q}}(t,s)$ *is increasing in* $t \in [s_0, b]_\Gamma$ *and decreasing in* $s \in [s_0, b]_\Gamma$.

Proof. Since $D_\Gamma\, e_{\Gamma,\mathcal{Q}}(t) = \mathcal{Q}(t)\, e_{\Gamma,\mathcal{Q}}(t) \geqslant 0$, $t \in [s_0, b]_\Gamma$, it follows that $e_{\Gamma,\mathcal{Q}}(t)$ is increasing in $t \in [s_0, b]_\Gamma$ (see [8]). This implies $e_{\Gamma,\mathcal{Q}}(t,s)$ is increasing in t and decreasing in s, from relation (2.1). □

Corollary 2.6. *If* $\mathcal{Q} \geqslant 0$ *is a continuous function on* $[s_0, b]$*, then* $0 < e_{\Gamma,\mathcal{Q}}(t,s) \leqslant e_{\Gamma,\mathcal{Q}}(b)$, $t, s \in [s_0, b]_\Gamma$.

Remark 2.7. Let $\mathcal{Q}$ be continuous at s_0. Under the condition $1 - (t - \Gamma(t))|\mathcal{Q}(t)| > 0$, $t \in [s_0, b]_\Gamma$, we can see

$$e_{\Gamma,\mathcal{Q}}(t) \leqslant e_{\Gamma,|\mathcal{Q}|}(t), \ \ t \in [s_0, b]_\Gamma.$$

Proof. In view of $\mathcal{Q}(\Gamma^k(t)) \leqslant |\mathcal{Q}(\Gamma^k(t))|$, $k \in \mathbb{N}_0$, we conclude that

$$\prod_{k=0}^{\infty}[1 - (\Gamma^k(t) - \Gamma^{k+1}(t))\mathcal{Q}(\Gamma^k(t))] \geqslant \prod_{k=0}^{\infty}[1 - (\Gamma^k(t) - \Gamma^{k+1}(t))|\mathcal{Q}(\Gamma^k(t))|],$$

from which we reassure the correctness of the required inequality. □

A sufficient condition for the existence and uniqueness of a solution of the initial value problem

$$D_\Gamma y(t) = \mathcal{Q}(t)y(t) + f(t), \quad t \in [s_0, b], \quad y(s_0) = y_0 \in \mathbb{B}, \tag{2.3}$$

is given in the following theorem.

Theorem 2.8 ([7]). *Assume that the following condition holds.*

(P1) *Let $f \in C([s_0, b], \mathbb{B})$ and $\mathcal{Q} \in C([s_0, b], \mathbb{R})$ such that $1 - (t - \Gamma(t))\mathcal{Q}(t) \neq 0$ for all $t \in [s_0, b]$.*

Then (2.3) has a unique solution y which is given by

$$y(t) = e_{\Gamma,\mathcal{Q}}(t)y_0 + \int_{s_0}^{t} e_{\Gamma,\mathcal{Q}}(t, \Gamma(\eta))f(\eta)\, d_\Gamma \eta.$$

Another sufficient condition for the existence of a unique solution of the initial value problem (2.3) is established in this section in Theorem 2.11.

Lemma 2.9. *Let $\mathcal{Q} \in C([s_0, b], \mathbb{R})$ and $f \in C([s_0, b], \mathbb{B})$. y is a solution of (2.3) if and only if y satisfies the integral equation*

$$y(t) = y_0 + \int_{s_0}^{t} (\mathcal{Q}(\eta)y(\eta) + f(\eta))d_\Gamma \eta, \quad t \in [s_0, b], \tag{2.4}$$

for some constant $y_0 \in \mathbb{B}$.

Proof. If y satisfies (2.3), then integrating both sides of the equation from s_0 to t we get that (2.4) holds with $y_0 = y(s_0)$. Conversely, if y satisfies (2.4), then we can differentiate y to get (2.3). □

Corollary 2.10. *Let $\mathcal{Q} \in C([s_0, b], \mathbb{R})$ and $f \in C([s_0, b], \mathbb{B})$. For $y_0 \in \mathbb{B}$, (2.3) has at most one solution y satisfying $y(s_0) = y_0$.*

Proof. Let $y_0 \in \mathbb{B}$. Assume that y_1 and y_2 are solutions of (2.3) with $y_1(s_0) = y_2(s_0) = y_0$. Then, by Lemma 2.9, both y_1 and y_2 satisfy (2.4). This implies

$$\|y_1(t) - y_2(t)\| \leqslant \int_{s_0}^{t} |\mathcal{Q}(\eta)|\, \|y_1(\eta) - y_2(\eta)\|\, d_\Gamma \eta, \quad t \in [s_0, b].$$

Now we let in Gronwall's inequality, Theorem 2.2, $y(t) = \|y_1(t) - y_2(t)\|$, $f = 0$. Hence, the assumptions in Theorem 2.2 are satisfied, and

$$y(t) \leqslant f(t) + \int_{s_0}^{t} y(\eta)|\mathcal{Q}(\eta)|d_\Gamma \eta, \quad t \in [s_0, b],$$

holds. Thus, by Theorem 2.2,

$$y(t) \leqslant f(t) + \int_{s_0}^{t} e_{\Gamma,|\mathcal{Q}|}(t, \Gamma(\eta))f(\eta)|\mathcal{Q}(\eta)|d_\Gamma(\eta) = 0, \quad t \in [s_0, b],$$

i.e., $\|y_1(t) - y_2(t)\| \leqslant 0$ for all $t \in [s_0, b]$, so $y_1 = y_2$. □

Theorem 2.11. *Assume that the following condition holds.*

(P2) $f \in C([s_0, b], \mathbb{B})$, $\mathcal{Q} \in C([s_0, b], \mathbb{R})$ *and there exists $l \in (0, 1)$ such that*

$$\int_{s_0}^{t} |\mathcal{Q}(s)|\, d_\Gamma s \leqslant l, \quad t \in [s_0, b].$$

For any $y_0 \in \mathbb{B}$*, equation* (2.3) *has a unique solution* y *with* $y(s_0) = y_0$*.*

Proof. Fix $y_0 \in \mathbb{B}$. The operator $\mathcal{T} : C([s_0, b], \mathbb{B}) \to C([s_0, b], \mathbb{B})$ defined by

$$\mathcal{T}y(t) := y_0 + \int_{s_0}^{t} (\mathcal{Q}(\eta)y(\eta) + f(\eta))d_\Gamma\eta, \quad t \in [s_0, b],$$

is a contraction. Indeed, for $y_1, y_2 \in C([s_0, b], \mathbb{B})$, we have

$$\|\mathcal{T}y_1(t) - \mathcal{T}y_2(t)\| \leqslant \int_{s_0}^{t} |\mathcal{Q}(\eta)| \, \|y_1(\eta) - y_2(\eta)\| \; d_\Gamma\eta \leqslant l \, \|y_1 - y_2\|_\infty , \quad t \in [s_0, b].$$

Hence

$$\|\mathcal{T}y_1 - \mathcal{T}y_2\|_\infty \leqslant l \, \|y_1 - y_2\|_\infty .$$

Therefore, $\mathcal{T}$ has a unique fixed point y, which is the unique solution of (2.4) satisfying $y(s_0) = y_0$. Thus, by Lemma 2.9, y is the unique solution of (2.3) with the condition $y(s_0) = y_0$. □

Remark 2.12. If **(P2)** holds, then condition **(P1)** holds but the converse is not true. Assume **(P1)** does not hold, there exists $x_0 \in [s_0, b]$ such that $\mathcal{Q}(x_0)(x_0 - \Gamma(x_0)) = 1$, then

$$\int_{s_0}^{x_0} |\mathcal{Q}(s)| \, d_\Gamma s = \sum_{k=0}^{\infty} (\Gamma^k(x_0) - \Gamma^{k+1}(x_0))|\mathcal{Q}(\Gamma^k(x_0))| \geqslant (x_0 - \Gamma(x_0)|\mathcal{Q}(x_0)| = 1.$$

The converse is not true. Assume $\mathcal{Q}(t) = 1, t \in [0, 1]$ and $\Gamma : [0, 1] \to [0, 1]$ such that $\Gamma(0) = 0$ and $\Gamma(t) < t$, $t \in (0, 1]$. Then $t - \Gamma(t) < 1$, $t \in [0, 1]$. We have $\int_0^t |\mathcal{Q}(s)| d_\Gamma s = t$, so $\sup \int_0^t |\mathcal{Q}(s)| d_\Gamma s = 1$. It follows that condition **(P1)** holds but **(P2)** dose not hold.

3. Hyers-Ulam stability of first-order quantum difference equations

We need the following space

$$C^1([s_0, b], \mathbb{B}) = \{\phi \in C([s_0, b], \mathbb{B}) : D_\Gamma\phi \text{ exists and } D_\Gamma\phi \in C([s_0, b], \mathbb{B})\}.$$

In this section, we establish Hyers-Ulam stability of first order linear quantum difference equation (2.3). From now on assume $f \in C([s_0, b], \mathbb{B})$ and $\mathcal{Q} \in C([s_0, b], \mathbb{R})$ (see [3]).

First, we introduce the concept of Hyers-Ulam stability.

Definition 3.1. Equation (2.3) is said to have Hyers-Ulam stability, if there exists a constant $K > 0$, a so-called HUs constant, with the following property. For any $\epsilon > 0$, if $y \in C^1([s_0, b], \mathbb{B})$ is such that

$$\|D_\Gamma y(t) - \mathcal{Q}(t)y(t) - f(t)\| \leqslant \epsilon, \;\; t \in [s_0, b]_\Gamma,$$

there exists a solution u of equation (2.3) such that

$$\|y(t) - u(t)\| \leqslant K\epsilon, \;\; t \in [s_0, b]_\Gamma.$$

Theorem 3.2. *Assume that the following condition holds.*

(P3) *For every* $y_0 \in \mathbb{B}$*, equation* (2.3) *has a solution* y *that satisfies* $y(s_0) = y_0$*.*

Then (2.3) *has Hyers-Ulam stability with HUs constant given by*

$$K = (b - s_0)e_{\Gamma,|\mathcal{Q}|}(b). \tag{3.1}$$

Proof. Let $\epsilon > 0$ and $y \in C^1([s_0, b], \mathbb{B})$ satisfies

$$\|D_\Gamma y(t) - \mathcal{Q}(t)y(t) - f(t)\| \leqslant \epsilon, \qquad t \in [s_0, b]_\Gamma.$$

Set

$$\sigma(t) = D_\Gamma y(t) - \mathcal{Q}(t)y(t) - f(t), \quad y(s_0) = y_0.$$

By Lemma 2.9, y solves

$$y(t) = y_0 + \int_{s_0}^{t} (\mathcal{Q}(\eta)y(\eta) + f(\eta) + \sigma(\eta))\, d_\Gamma\eta. \tag{3.2}$$

In view of condition **(P3)**, (2.3) has a solution u that satisfies $u(s_0) = y_0$. Again, by Lemma 2.9, u satisfies

$$u(t) = y_0 + \int_{s_0}^{t} (\mathcal{Q}(\eta)u(\eta) + f(\eta))\, d_\Gamma\eta,\ t \in [s_0, b]. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} \|y(t) - u(t)\| &\leqslant \int_{s_0}^{t} \|\sigma(\eta)\| d_\Gamma\eta + \int_{s_0}^{t} |\mathcal{Q}(\eta)| \|y(\eta) - u(\eta)\|\, d_\Gamma\eta \\ &\leqslant \epsilon(b - s_0) + \int_{s_0}^{t} |\mathcal{Q}(\eta)| \|y(\eta) - u(\eta)\|\, d_\Gamma\eta,\ t \in [s_0, b]_\Gamma. \end{aligned} \tag{3.4}$$

By Theorem 2.2 and (3.4), we obtain

$$\|y(t) - u(t)\| \leqslant \epsilon(b - s_0)e_{\Gamma,|\mathcal{Q}|}(t) \leqslant \epsilon(b - s_0)e_{\Gamma,|\mathcal{Q}|}(b),\ \ t \in [s_0, b]_\Gamma.$$

□

Regarding to Theorem 2.11, we can establish the following result.

Theorem 3.3. *Assume condition* **(P2)** *holds. Then* (2.3) *has Hyers-Ulam stability with HUs constant* K *given by* (3.1).

4. Hyers-Ulam-Rassias stability

The following describes Hyers-Ulam-Rassias stability of (2.3) (see [10]).

Definition 4.1. Let $\mathcal{F} \subseteq C([s_0, b], (0, \infty))$. Equation (2.3) is said to have Hyers-Ulam-Rassias stability of type $\mathcal{F}$ if there exists a constant $K > 0$, a so-called HURs$_\mathcal{F}$ constant, with the following property. For any $\phi \in \mathcal{F}$, if $y \in C^1([s_0, b], \mathbb{B})$ is such that

$$\|D_\Gamma y(t) - \mathcal{Q}(t)y(t) - f(t)\| \leqslant \phi(t),\ \ t \in [s_0, b]_\Gamma, \tag{4.1}$$

then there exists a solution u of (2.3) such that

$$\|y(t) - u(t)\| \leqslant K\phi(t), \ \ \text{for all}\ \ t \in [s_0, b]_\Gamma.$$

Theorem 4.2. *Let* $\mathcal{F}^* := \{\phi \in C([s_0, b], (0, \infty)) :\ \phi$ *is nondecreasing on* $[s_0, b]_\Gamma\}$. *If* **(P3)** *holds, then* (2.3) *has Hyers-Ulam-Rassias stability of type* $\mathcal{F}^*$ *with HURs*$_{\mathcal{F}^*}$ *constant*

$$K := (b - s_0)e_{\Gamma,|\mathcal{Q}|}(b). \tag{4.2}$$

Proof. Suppose $\phi \in \mathcal{F}^*$ and $y \in C^1([s_0, b], \mathbb{B})$ satisfies (4.1). Set

$$\sigma(t) := D_\Gamma y(t) - \mathcal{Q}(t)y(t) - f(t). \tag{4.3}$$

Clearly, $\sigma \in C([s_0, b], \mathbb{B})$. Let $y_0 = y(s_0)$. By Lemma 2.9, y solves

$$y(t) = y_0 + \int_{s_0}^{t} (\mathcal{Q}(\eta)y(\eta) + f(\eta) + \sigma(\eta))\, d_\Gamma\eta. \tag{4.4}$$

In view of condition **(P3)**, (2.3) has a solution u that satisfies $u(s_0) = y_0$. Again, by Lemma 2.9, u satisfies

$$u(t) = y_0 + \int_{s_0}^{t} (\mathcal{Q}(\eta)u(\eta) + f(\eta))\, d_\Gamma\eta, \ \ t \in [s_0, b]. \tag{4.5}$$

Subtracting (4.5) from (4.4), we find, for all $t \in [s_0, b]$,

$$\|y(t) - u(t)\| \leqslant \int_{s_0}^{t} \|\sigma(\eta)\|\, d_\Gamma\eta + \int_{s_0}^{t} |\mathcal{Q}(\eta)|\|y(\eta) - u(\eta)\|\, d_\Gamma\eta.$$

This implies

$$\|y(t) - u(t)\| \leqslant \int_{s_0}^{t} \phi(\eta) d_\Gamma\eta + \int_{s_0}^{t} |\mathcal{Q}(\eta)|\|y(\eta) - u(\eta)\| d_\Gamma\eta, \ \ t \in [s_0, b]_\Gamma, \tag{4.6}$$

from which, it follows by the nondecreasing property of ϕ, that

$$\begin{aligned} \|y(t) - u(t)\| &\leqslant \int_{s_0}^{t} \phi(t)\, d_\Gamma\eta + \int_{s_0}^{t} |\mathcal{Q}(\eta)|\|y(\eta) - u(\eta)\| d_\Gamma\eta \\ &= (b - s_0)\phi(t) + \int_{s_0}^{t} |\mathcal{Q}(\eta)|\|y(\eta) - u(\eta)\|\, d_\Gamma\eta, \ t \in [s_0, b]_\Gamma. \end{aligned}$$

Thus, by Gronwall's inequality, Theorem 2.2, we obtain, for $t \in [s_0, b]_\Gamma$,

$$\begin{aligned} \|y(t) - u(t)\| &\leqslant (b - s_0)\phi(t) + \int_{s_0}^{t} e_{\Gamma,|\mathcal{Q}|}(t, \Gamma(\eta))|\mathcal{Q}(\eta)|(b - s_0)\phi(\eta) d_\Gamma\eta \\ &\leqslant (b - s_0)\phi(t) + \int_{s_0}^{t} e_{\Gamma,|\mathcal{Q}|}(t, \Gamma(\eta))|\mathcal{Q}(\eta)|(b - s_0)\phi(t) d_\Gamma\eta \\ &= (b - s_0)\phi(t)\Big(1 + \int_{s_0}^{t} e_{\Gamma,|\mathcal{Q}|}(t, \Gamma(\eta))|\mathcal{Q}(\eta)| d_\Gamma\eta\Big) \\ &= (b - s_0)\phi(t)\Big(1 + e_{\Gamma,|\mathcal{Q}|}(t) - 1\Big) = (b - s_0)e_{|\mathcal{Q}|}(t)\phi(t) \leqslant (b - s_0)e_{\Gamma,|\mathcal{Q}|}(b)\phi(t) = K\phi(t), \end{aligned}$$

where we have used Theorem 2.1 (ii). Therefore, (2.3) has Hyers-Ulam-Rassias stability of type $\mathcal{F}^*$ with HURs$_{\mathcal{F}^*}$ constant given by (4.2). □

Throughout the rest of the paper, we denote, for $p \geqslant 1$,

$$\mathcal{F}_p := \left\{ \phi \in C([s_0, b], (0, \infty)) : \int_{s_0}^{t} \phi^p(\eta) d_\Gamma\eta \leqslant \phi^p(t) \text{ for all } t \in [s_0, b]_\Gamma \right\}.$$

If we consider $\phi \in \mathcal{F}^* \cap \mathcal{F}_1$, then we can improve the HURs constant (if $b > s_0 + 1$) as follows.

Theorem 4.3. *If* **(P3)** *holds, then* (2.3) *has Hyers-Ulam-Rassias stability of type* $\mathcal{F}^* \cap \mathcal{F}_1$ *with HURs*$_{\mathcal{F}^* \cap \mathcal{F}_1}$ *constant*

$$K := e_{\Gamma,|\mathcal{Q}|}(b). \tag{4.7}$$

Proof. Let $\varphi \in \mathcal{F}^* \cap \mathcal{F}_1$. Suppose $y \in C^1([s_0, b], \mathbb{B})$ satisfies (4.1). Defining σ as in (4.3), we see that $\sigma \in C([s_0, b], \mathbb{B})$. Let $y_0 = y(s_0)$. By Lemma 2.9, (4.4) holds. By **(P3)**, there exists a solution u of (2.3) satisfying $u(s_0) = y_0$. Equivalently, by Lemma 2.9, (4.5) holds. Subtracting (4.4) from (4.5), we get (4.6). This implies

$$\|y(t) - u(t)\| \leqslant \int_{s_0}^{t} \varphi(\eta) d_\Gamma \eta + \int_{s_0}^{t} |\Omega(\eta)| \|y(\eta) - u(\eta)\| d_\Gamma \eta \leqslant \varphi(t) + \int_{s_0}^{t} |\Omega(\eta)| \|y(\eta) - u(\eta)\| d_\Gamma \eta,\ t \in [s_0, b]_\Gamma.$$

By Gronwall's inequality, Theorem 2.2, we obtain

$$\begin{aligned} \|y(t) - u(t)\| &\leqslant \varphi(t) + \int_{s_0}^{t} e_{\Gamma,|\Omega|}(t, \Gamma(\eta)) |\Omega(\eta)| \varphi(\eta)\, d_\Gamma \eta. \\ &\leqslant \varphi(t) + \varphi(t) \int_{s_0}^{t} e_{\Gamma,|\Omega|}(t, \Gamma(\eta)) |\Omega(\eta)|\, d_\Gamma \eta \\ &= \varphi(t)(1 + e_{\Gamma,|\Omega|}(t) - 1) \leqslant \varphi(t) e_{\Gamma,|\Omega|}(t) \leqslant \varphi(t) e_{\Gamma,|\Omega|}(b),\ t \in [s_0, b]_\Gamma. \end{aligned}$$

Therefore, (2.3) has Hyers-Ulam-Rassias stability of type $\mathcal{F}^* \cap \mathcal{F}_1$ with HURs constant given by (4.7). □

Theorem 4.4. *If* **(P3)** *holds, then* (2.3) *has Hyers-Ulam-Rassias stability of type $\mathcal{F}_1$ with HURs $_{\mathcal{F}_1}$ constant*

$$K := 1 + e_{\Gamma,|\Omega|}(b) \|\Omega\|_\infty. \tag{4.8}$$

Proof. Let $\varphi \in \mathcal{F}_1$. Suppose $y \in C^1([s_0, b], \mathbb{B})$ satisfies (4.1). Defining σ as in (4.3), we see that $\sigma \in C([s_0, b], \mathbb{B})$. Let $y_0 = y(s_0)$. By Lemma 2.9, (4.4) holds. By **(P3)**, there exists a solution u of (2.3) satisfying $u(s_0) = y_0$. Equivalently, by Lemma 2.9, (4.5) holds. Subtracting (4.4) from (4.5), we get (4.6). This implies

$$\|y(t) - u(t)\| \leqslant \int_{s_0}^{t} \varphi(\eta) d_\Gamma \eta + \int_{s_0}^{t} |\Omega(\eta)| \|y(\eta) - u(\eta)\|\, d_\Gamma \eta \leqslant \varphi(t) + \int_{s_0}^{t} |\Omega(\eta)| \|y(\eta) - u(\eta)\| d_\Gamma \eta, t \in [s_0, b]_\Gamma.$$

Thus, by Gronwall's inequality, Theorem 2.2, we obtain, for all $t \in [s_0, b]_\Gamma$,

$$\|y(t) - u(t)\| \leqslant \varphi(t) + \int_{s_0}^{t} e_{\Gamma,|\Omega|}(t, \Gamma(\eta)) |\Omega(\eta)| \varphi(\eta) d_\Gamma \eta \leqslant \varphi(t) + e_{\Gamma,|\Omega|}(b) \|\Omega\|_\infty \int_{s_0}^{t} \varphi(\eta) d_\Gamma \eta \leqslant K\varphi(t).$$

Therefore, (2.3) indeed has Hyers-Ulam-Rassias stability of type $\mathcal{F}_1$ with HURs constant given by (4.8). □

Theorem 4.5. *If* **(P3)** *holds, then* (2.3) *has Hyers-Ulam-Rassias stability of type $\mathcal{F}_2$ with HURs $_{\mathcal{F}_2}$ constant*

$$K := \sqrt{b - s_0}\left(1 + \sqrt{b - s_0}\, e_{\Gamma,|\Omega|}(b) \|\Omega\|_\infty\right). \tag{4.9}$$

Proof. Let $\varphi \in \mathcal{F}_2$ and $y \in C^1([s_0, b], \mathbb{B})$ satisfies (4.1). Defining σ as in (4.3), and following the same steps as in the proof of Theorem 4.4, we find, for all $t \in [s_0, b]_\Gamma$,

$$\begin{aligned} \|y(t) - u(t)\| &\leqslant \int_{s_0}^{t} \varphi(\eta) d_\Gamma \eta + \int_{s_0}^{t} |\Omega(\eta)| \|y(\eta) - u(\eta)\|\, d_\Gamma \eta \\ &\leqslant \sqrt{t - s_0} \sqrt{\int_{s_0}^{t} \varphi^2(\eta)\, d_\Gamma \eta} + \int_{s_0}^{t} |\Omega(\eta)| \|y(\eta) - u(\eta)\|\, d_\Gamma \eta \\ &\leqslant \sqrt{b - s_0} \sqrt{\varphi^2(t)} + \int_{s_0}^{t} |\Omega(\eta)| \|y(\eta) - u(\eta)\| d_\Gamma \eta \\ &= \sqrt{b - s_0}\, \varphi(t) + \int_{s_0}^{t} |\Omega(\eta)| \|y(\eta) - u(\eta)\| d_\Gamma \eta, \end{aligned}$$

where we have used of Cauchy-Schwarz inequality [9] . Thus, by Gronwall's inequality, Theorem 2.2, we obtain, for all $t \in [s_0, b]_\Gamma$,

$$\begin{aligned}\|y(t)-u(t)\| &\leqslant \sqrt{b-s_0}\phi(t) + \int_{s_0}^{t} e_{\Gamma,|\Omega|}(t,\Gamma(\eta))|\Omega(\eta)|\sqrt{b-s_0}\phi(\eta)d_\Gamma\eta \\ &\leqslant \sqrt{b-s_0}\phi(t) + e_{\Gamma,|\Omega|}(b)\|\Omega\|_\infty\sqrt{b-s_0}\int_{s_0}^{t}\phi(\eta)d_\Gamma\eta \leqslant K\phi(t),\end{aligned}$$

where we have used of Cauchy-Schwarz inequality once more. Therefore, (2.3) indeed has Hyers-Ulam-Rassias stability of type $\mathcal{F}_2$ with HURs constant given by (4.9). □

Theorem 4.6. *Let* $p > 1$ *and* $a := p/(p-1)$*. If* **(P3)** *holds, then* (2.3) *has Hyers-Ulam-Rassias stability of type* $\mathcal{F}_p$ *with HURs* $_{\mathcal{F}_p}$ *constant*

$$K := \sqrt[a]{b-s_0}\Big(1 + \sqrt[a]{b-s_0}e_{\Gamma,|\Omega|}(b)\|\Omega\|_\infty\Big). \tag{4.10}$$

Proof. Let $\phi \in \mathcal{F}_p$ and $y \in C^1([s_0, b], \mathbb{B})$ satisfies (4.1). Define σ as in (4.3). Again, we get (4.6). From which we obtain for all $t \in [s_0, b]_\Gamma$,

$$\begin{aligned}\|y(t)-u(t)\| &\leqslant \int_{s_0}^{t}\phi(\eta)\, d_\Gamma\eta + \int_{s_0}^{t}|\Omega(\eta)|\|y(\eta)-u(\eta)\|\, d_\Gamma\eta \\ &\leqslant \sqrt[a]{t-s_0}\sqrt[p]{\int_{s_0}^{t}\phi^p(\eta)d_\Gamma\eta} + \int_{s_0}^{t}|\Omega(\eta)|\|y(\eta)-u(\eta)\|\, d_\Gamma\eta \\ &\leqslant \sqrt[a]{b-s_0}\sqrt[p]{\phi^p(t)} + \int_{s_0}^{t}|\Omega(\eta)|\|y(\eta)-u(\eta)\|\, d_\Gamma\eta \\ &= \sqrt[a]{b-s_0}\,\phi(t) + \int_{s_0}^{t}|\Omega(\eta)|\|y(\eta)-u(\eta)\|\, d_\Gamma\eta,\ \ t \in [s_0, b]_\Gamma,\end{aligned}$$

where we have used of Hölder inequality [9]. Thus, by Gronwall's inequality, Theorem 2.2, we obtain, for all $t \in [s_0, b]_\Gamma$,

$$\begin{aligned}\|y(t)-u(t)\| &\leqslant \sqrt[a]{b-s_0}\,\phi(t) + \int_{s_0}^{t} e_{\Gamma,|\Omega|}(t,\Gamma(\eta))|\Omega(\eta)|\sqrt[a]{b-s_0}\,\phi(\eta)\, d_\Gamma\eta \\ &\leqslant \sqrt[a]{b-s_0}\,\phi(t) + e_{\Gamma,|\Omega|}(b)\|\Omega\|_\infty\sqrt[a]{b-s_0}\int_{s_0}^{t}\phi(\eta)\, d_\Gamma\eta \leqslant K\phi(t),\ \ t \in [s_0, b]_\Gamma,\end{aligned}$$

where we have used of Hölder inequality once more. Therefore, (2.3) has Hyers-Ulam-Rassias stability of type $\mathcal{F}_2$ with HURs constant, given by (4.10). □

Since both of conditions **(P1)** and **(P2)** imply **(P3)**, then we conclude following. Theorems 4.2, 4.3, 4.4, 4.5, and 4.6 are true if we replace **(P3)** by either **(P1)** or **(P2)**.

For the last result, we denote, for $p > 1$ and $r \geqslant 0$, by $\mathcal{F}_p^r$ the family

$$\mathcal{F}_p^r := \left\{\phi \in C([s_0, b], (0, \infty)) : \int_{s_0}^{t}\phi^p(\eta)d_\Gamma\eta \leqslant r\phi^p(t) \text{ for all } t \in [s_0, b]_\Gamma\right\}.$$

Theorem 4.7. *Let* $a := p/(p-1)$*, where* $p > 1$*. If either* **(P1)**, **(P2)** *or* **(P3)** *hold, then* (2.3) *has Hyers-Ulam-Rassias stability of type* $\mathcal{F}_p^r$ *with HURs* $_{\mathcal{F}_p^r}$ *constant*

$$K := \sqrt[a]{b-s_0}\,\sqrt[p]{r}\left(1 + \sqrt[a]{b-s_0}\,\sqrt[p]{r}\, e_{\Gamma,|\Omega|}(b)\|\Omega\|_\infty\right). \tag{4.11}$$

Proof. Let $\phi \in \mathcal{F}_p^r$ and $y \in C^1([s_0, b], \mathbb{B})$ satisfies (4.1). Defining σ as in (4.3), we see that $\sigma \in C^1([s_0, b], \mathbb{B})$. Let $y_0 = y(s_0)$. By Lemma 2.9, (4.4) holds. By **(P1)**, **(P2)** or **(P3)**, there exists a solution u of (2.3) satisfying $u(s_0) = y_0$. Equivalently, by Lemma 2.9, (4.5) holds. Subtracting (4.4) from (4.5), we get (4.6). We find, for all $t \in [s_0, b]_\Gamma$,

$$\begin{aligned}\|y(t) - u(t)\| &\leqslant \int_{s_0}^{t} \phi(\eta)\, d_\Gamma\eta + \int_{s_0}^{t} |\Omega(\eta)|\|y(\eta) - u(\eta)\|\, d_\Gamma\eta \\ &\leqslant \sqrt[a]{t - s_0}\sqrt[p]{\int_{s_0}^{t} \phi^p(\eta) d_\Gamma\eta} + \int_{s_0}^{t} |\Omega(\eta)|\|y(\eta) - u(\eta)\|\, d_\Gamma\eta \\ &\leqslant \sqrt[a]{b - s_0}\sqrt[p]{r\phi^p(t)} + \int_{s_0}^{t} |\Omega(\eta)|\|y(\eta) - u(\eta)\|\, d_\Gamma\eta \\ &= \sqrt[a]{b - s_0}\sqrt[p]{r}\phi(t) + \int_{s_0}^{t} |\Omega(\eta)|\|y(\eta) - u(\eta)\|\, d_\Gamma\eta,\end{aligned}$$

where we have used Hölder's inequality, Theorem [9]. Thus, by applying Gronwall's inequality, Theorem 2.2, we get, for all $t \in [s_0, b]_\Gamma$,

$$\begin{aligned}\|y(t) - u(t)\| &\leqslant \sqrt[a]{b - s_0}\sqrt[p]{r}\phi(t) + \int_{s_0}^{t} e_{|\Omega|}(t, \Gamma(\eta))|\Omega(\eta)| \sqrt[a]{b - s_0}\sqrt[p]{r}\phi(\eta)\, d_\Gamma\eta \\ &\leqslant \sqrt[a]{b - s_0}\sqrt[p]{r}\phi(t) + e_{|\Omega|}(b)\|\Omega\|_\infty \sqrt[a]{b - s_0}\sqrt[p]{r}\int_{s_0}^{t} \phi(\eta)\, d_\Gamma\eta \\ &\leqslant \sqrt[a]{b - s_0}\sqrt[p]{r}\phi(t) + e_{\Gamma,|\Omega|}(b)\|\Omega\|_\infty \sqrt[a]{b - s_0}\sqrt[p]{r}(\sqrt[a]{b - s_0}\sqrt[p]{r}\phi(t)) \\ &\leqslant \sqrt[a]{b - s_0}\sqrt[p]{r}\phi(t)(1 + \sqrt[a]{b - s_0}\sqrt[p]{r}e_{\Gamma,|\Omega|}(b)\|\Omega\|_\infty) = K\phi(t).\end{aligned}$$

Therefore, (2.3) has Hyers-Ulam-Rassias stability of type $\mathcal{F}_p^r$ with constant K given in (4.11). □

5. Example

The following example shows the applicability of the theoretical results.

Example 5.1. Consider the equation

$$D_\Gamma y(t) = \theta e_{\Gamma,\lambda}(t)y(t) + f(t), t \in [s_0, b],$$

where $\lambda > 0$ and $0 < \theta < \frac{\lambda}{e_{\Gamma,\lambda}(b)-1}$. Set $l = \frac{\theta}{\lambda}(e_{\Gamma,\lambda}(b) - 1)$. One can see that condition **(P2)** holds. Indeed,

$$\int_{s_0}^{t} \Omega(s) d_\Gamma = \frac{\theta}{\lambda}(e_{\Gamma,\lambda}(t) - 1) = l < 1.$$

Therefore, we can apply Theorems 3.2, 4.2, 4.3, 4.4, 4.5, 4.6, and 4.7.

Acknowlegements

This work was funded by the University of Jeddah, Saudi Arabia, under grant No. (UJ-23-FR-68). Therefore, the authors thank the University of Jeddah for its technical and finantial support.

References

[1] K. A. Aldwoah, A. B. Malinowska, D. F. M. Torres, *The power quantum calculus and variational problems*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, **19** (2012), 93–116. 1

[2] C. Alsina, R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl., **2** (1998), 373–380. 1

[3] D. R. Anderson, M. Onitsuka, *Hyers-Ulam stability of first-order homogeneous linear dynamic equations on time scales*, Demonstr. Math., **51** (2018), 198–210. 3

[4] M. H. Annaby, A. E. Hamza, K. A. Aldwoah, *Hahn difference operator and associated Jackson-Nörlund integrals*, J. Optim. Theory Appl., **154** (2012), 133–153. 1

[5] N. Faried, E. M. Shehata, R. M. El Zafarani, *Quantum exponential functions in a Banach algebra*, J. Fixed Point Theory Appl., **22** (2020), 13 pages. 1

[6] A. E. Hamza, M. A. Alghamdi, S. A. Alasmi, *Quantum* L^p*-spaces and inequalities of hardy's type*, J. Math. Comput. Sci., **31** (2023), 274–286. 1

[7] A. E. Hamza, A.-S. M. Sarhan, E. M. Shehata, *Exponential, trigonometric and hyperbolic functions associated with a general quantum difference operator*, Adv. Dyn. Syst. Appl., **12** (2017), 25–38. 1, 2.1, 2.8

[8] A. E. Hamza, A.-S. M. Sarhan, E. M. Shehata, K. A. Aldwoah, *A general quantum difference calculus*, Adv. Difference Equ., **2015** (2015), 1–19. 1, 1.1, 1.2, 1.3, 1.4, 2

[9] A. E. Hamza, E. M. Shehata, *Some inequalities based on a general quantum difference operator*, J. Inequal. Appl., **2015** (2015), 12 pages. 1, 1, 1, 2, 2.2, 2.3, 4, 4, 4

[10] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224. 1, 4

[11] S.-M. Jung, *Hyers–Ulam–Rassias stability of functional equations in nonlinear analysis*, Springer, New York, (2011). 1

[12] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300. 1

[13] E. M. Shehata, N. Faried, R. M. El Zafarani, *Stability of first order linear general quantum difference equations in a banach algebra*, Appl. Math. Inf. Sci., **16** (2022), 101–108. 1

[14] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, Oxford, (1939). 2

[15] S. M. Ulam, *A collection of mathematical problems*, Interscience Publishers, New York-London, (1960). 1