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Hyers–Ulam and Hyers–Ulam–Rassias Stability of First-Order Nonlinear Dynamic Equations

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Abstract
We investigate Hyers–Ulam and Hyers–Ulam–Rassias stability of first-order nonlinear dynamic equations for functions defined on a time scale with values in a Banach space.

Keywords Time scales · First-order nonlinear dynamic equations · Hyers–Ulam stability · Hyers–Ulam–Rassias stability

Mathematics Subject Classification 34N05 · 34D20 · 39A30

1 Preliminaries and Introduction

The study of stability problems for various functional equations was triggered by an intriguing and famous talk presented by Ulam at University of Wisconsin in 1940. In his talk, Ulam discussed a number of important unsolved mathematical problems. After that, these problems were collected in [23]. The notion of Ulam stability arose from a question of these problems concerning the stability of group homomorphisms. In [10], Hyers gave a first affirmative partial answer to this question for Banach

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spaces. After many years, Rassias [20] extended the result of Hyers by allowing an unbounded Cauchy difference. Since then, stability problems of many functional equations have been extensively investigated in various abstract spaces [8,11,21]. However, concerning functional equations, Obłoza seems to be the first mathematician who has investigated Hyers–Ulam stability of linear differential equations (see [17,18]). Thereafter, Alsina and Ger [2] published their paper, handling Hyers–Ulam stability of linear differential equations. Soon after, Miura and Takahasi et al. [14–16] deeply and systematically studied Ulam stability of differential equations in various abstract spaces. Many articles dealing with Ulam, Hyers–Ulam, and Hyers–Ulam–Rassias stability in various contexts were edited by Rassias [19]. In [12], Li and Shen introduced Hyers–Ulam stability of scalar second-order differential equations of the form

$$y'' + p(x)y' + q(x)y + r(x) = 0,$$

that is, if y is an approximate solution of this equation, then there exists an exact solution of the equation near y. Also, Li and Shen [13] proved Hyers–Ulam stability of homogeneous linear differential equations of second order. Paşu Gavruta [9] proved Hyers–Ulam stability of second-order linear differential equations with boundary conditions and initial conditions. That is, if y is an approximate solution of the differential equation

$$y'' + \beta(x)y = 0 \quad \text{with} \quad y(a) = y(b),$$

then there exists a solution of the differential equation near y. Hyers–Ulam stability of exact second-order linear differential equations was proved in [7]. Also in [7], Hyers–Ulam stability of second-order linear differential equations with constant coefficients, Euler, Hermite, Cheybyshev, and Legendre differential equations was proved. These results generalize the main results of Li and Shen [12,13]. In 2012, Anderson et al. [3] established Hyers–Ulam stability of scalar second-order linear nonhomogeneous dynamic equations on time scales of the form

$$x^{\Delta\Delta}(t) + p(t)x^{\Delta}(t) + r(t)x(t) = f(t), \quad t \in [a, b]_{\mathbb{T}},$$

where the given functions $p, r, f \in \mathrm{C}_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathbb{R})$, the space of all real-valued rd-continuous functions on $[a, b]_{\mathbb{T}}$. They extended the work of Li and Shen [12,13] to prove Hyers–Ulam stability of homogenous and nonhomogenous linear dynamic equations of second-order on time scales. Also, in 2013, András and Mészáros [4] studied Hyers–Ulam stability of some linear and nonlinear dynamic equations and integral equations on time scales, based on the theory of Picard operators. In 2017, Shen [22] established Hyers–Ulam stability of first-order linear dynamic equations and its adjoint equations on time scales by using the integrating factor method. Recently, there has been a great interest in studying stability of dynamic equations on time scales. In [1], Alghamdi et al. investigated Hyers–Ulam–Rassias stability of first-order linear dynamic equations on time scales.

In this paper, we investigate Hyers–Ulam stability and Hyers–Ulam–Rassias stability of first-order nonlinear dynamic equations on time scales of the form

$$\psi^{\Delta}(t) = \wp(t)\psi(t) + \mathcal{F}(t, \psi(t), h(\psi(t))) + f(t), \quad t \in \mathcal{I}^{\kappa}, \quad \psi(a) = a_0 \in \mathbb{X}, \tag{1.1}$$

where $\mathcal{I} := [a, b] \cap \mathbb{T}$ with a time scale $\mathbb{T} \subset \mathbb{R}$, $a, b \in \mathbb{T}$, $a < b$, and $\mathbb{X}$ is a Banach space. Also, $\mathcal{F} : \mathcal{I} \times \mathbb{X}^2 \to \mathbb{X}$ is such that $\mathcal{F}(\cdot, x, y)$ is rd-continuous and $\mathcal{F}(t, \cdot, y)$ and $\mathcal{F}(t, x, \cdot)$ are continuous for all $t \in \mathcal{I}$ and $x, y \in \mathbb{X}$, $\wp : \mathcal{I} \to \mathbb{R}$ is regressive and rd-continuous, $f : \mathcal{I} \to \mathbb{X}$ is rd-continuous, and $h : \mathbb{X} \to \mathbb{X}$ is continuous. Here, $\mathbb{X}$ is a Banach space with norm $\|\cdot\|$ and $\mathbb{X}^2$ is a Banach space endowed with $\|(x, y)\| = \|x\| + \|y\|$. As usual, for a bounded function $\Phi : X \to Y$ from a normed space X to a normed space Y, we denote

$$\|\Phi\|_{\infty} = \sup_{x \in X} \|\Phi(x)\|.$$

For the time scale terminology, we refer the reader to Bohner and Peterson [5,6]. Here, we only recall some definitions and results pertinent to the rest of the paper.

Definition 1.1 A function $f : \mathbb{T} \times \mathbb{X}^k \to \mathbb{X}$ is said to satisfy a Lipschitz condition with constant $L > 0$ if

$$\|f(t, x_1, \ldots, x_k) - f(t, y_1, \ldots, y_k)\| \le L \sum_{i=1}^{k} \|x_i - y_i\|$$

for all $x_i, y_i \in \mathbb{X}$ and all $t \in \mathbb{T}$.

Theorem 1.1 (See [5, Theorem 1.117(i)]) *If* $G(t) = \int_a^t f(t, s)\Delta s$, *then*

$$G^{\Delta}(t) = \int_a^t f^{\Delta}(t, s)\Delta s + f(\sigma(t), t).$$

Theorem 1.2 (Gronwall's inequality, see [5, Theorem 6.4]) *Let* $y, f \in \mathrm{C_{rd}}(\mathcal{I}, \mathbb{R})$ *and* $p \in \mathrm{C_{rd}}(\mathcal{I}, [0, \infty))$. *Then*

$$y(t) \le f(t) + \int_a^t y(s)p(s)\Delta s \quad \textit{for all} \quad t \in \mathcal{I}$$

implies

$$y(t) \le f(t) + \int_a^t e_p(t, \sigma(s)) f(s)p(s)\Delta s \quad \textit{for all} \quad t \in \mathcal{I}.$$

Theorem 1.3 (Hölder's inequality, see [5, Theorem 6.13]) *For* $f, g \in \mathrm{C_{rd}}(\mathcal{I}, \mathbb{R})$, *we have*

$$\int_a^b |f(t)g(t)|\Delta t \le \left\{\int_a^b |f(t)|^p \Delta t\right\}^{\frac{1}{p}} \left\{\int_a^b |g(t)|^q \Delta t\right\}^{\frac{1}{q}},$$

where $p > 1$ *and* $q = p/(p-1)$.

Theorem 1.4 *If* $\wp \in \mathcal{R}$ *and* $a, b, c \in \mathbb{T}$, *then*

$$\left[e_\wp(c,\cdot)\right]^\Delta = -\wp\left[e_\wp(c,\cdot)\right]^\sigma$$

and

$$\int_a^b \wp(t)e_\wp(c,\sigma(t))\Delta t = e_\wp(c,a) - e_\wp(c,b).$$

Remark 1.1 In view of the increasing nature (resp. decreasing nature) of the function $e_{|\wp|}(t,s)$ in the first argument (resp. in the second argument), we conclude that

$$|e_\wp(t,\sigma(s))| \le e_{|\wp|}(t,\sigma(s)) \le e_{|\wp|}(b,a) \quad \text{for all} \quad t, s \in \mathcal{I}.$$

2 Existence and Uniqueness Results

In this section, we investigate sufficient conditions for the existence and uniqueness of solutions of (1.1) by applying Banach's fixed point theorem. We need the following theorem to obain our results.

Theorem 2.1 *If* $\wp \in \mathcal{R}$, *then* ψ *solves* (1.1) *if and only if*

$$\psi(t) = e_\wp(t,a)a_0 + \int_a^t e_\wp(t,\sigma(s))[\mathcal{F}(s,\psi(s),h(\psi(s))) + f(s)]\Delta s, \quad t \in \mathcal{I}. \tag{2.1}$$

Proof We denote

$$H_\psi(s) := \mathcal{F}(s,\psi(s),h(\psi(s))) + f(s).$$

Assume ψ solves (2.1). Then

$$\begin{aligned}\psi^\Delta(t) &= \wp(t)e_\wp(t,a)a_0 + \int_a^t \wp(t)e_\wp(t,\sigma(s))H_\psi(s)\Delta s + H_\psi(t)\\ &= \wp(t)\left(e_\wp(t,a)a_0 + \int_a^t e_\wp(t,\sigma(s))H_\psi(s)\Delta s\right) + H_\psi(t)\\ &= \wp(t)\psi(t) + H_\psi(t)\end{aligned}$$

and $\psi(a) = a_0$. Hence ψ solves (1.1). To prove the other direction, assume that ψ solves (1.1), i.e.,

$$\psi^{\Delta}(t) = \wp(t)\psi(t) + H_{\psi}(t), \quad t \in \mathcal{I}, \quad \psi(a) = a_0.$$

By [5, Theorem 2.77],

$$\psi(t) = e_{\wp}(t, a)a_0 + \int_a^t e_{\wp}(t, \sigma(s))H_{\psi}(s)\Delta s.$$

Therefore, ψ satisfies (2.1). □

Throughout the rest of the paper, we use the following conditions.

(H$_1$) $\wp \in \mathcal{R}$ and $f \in \mathrm{C_{rd}}$.
(H$_2$) $\mathcal{F}$ and h satisfy Lipschitz conditions with constants β and γ, respectively.
(H$_3$) For any $a_0 \in \mathbb{X}$, (1.1) has a solution ϕ satisfying $\phi(a) = a_0$.
(H$_4$) $\theta := \sup_{t \in \mathcal{I}} \int_a^t |e_{\wp}(t, \sigma(s))|\Delta s < \dfrac{1}{\beta(1+\gamma)}$.
(H$_5$) $(b-a)e_{|\wp|}(b, a) < \dfrac{1}{\beta(1+\gamma)}$.

Theorem 2.2 *Assume* (H$_1$), (H$_2$), *and* (H$_4$). *If* $a_0 \in \mathbb{X}$, *then* (1.1) *has a unique solution* ψ *satisfying* $\psi(a) = a_0$.

Proof Fix $a_0 \in \mathbb{X}$. Define the operator $T : \mathrm{C_{rd}}(\mathcal{I}, \mathbb{X}) \to \mathrm{C_{rd}}(\mathcal{I}, \mathbb{X})$ by

$$T\psi(t) = e_{\wp}(t, a)a_0 + \int_a^t e_{\wp}(t, \sigma(s))[\mathcal{F}(s, \psi(s), h(\psi(s))) + f(s)]\Delta s.$$

For $\psi_1, \psi_2 \in \mathrm{C_{rd}}(\mathcal{I}, \mathbb{X})$, we have

$$\begin{aligned}
&\|T\psi_1(t) - T\psi_2(t)\| \\
&\quad \le \int_a^t |e_{\wp}(t, \sigma(s))| \, \|\mathcal{F}(s, \psi_1(s), h(\psi_1(s))) - \mathcal{F}(s, \psi_2(s), h(\psi_2(s)))\| \, \Delta s.
\end{aligned}$$

By (H$_2$), we get

$$\begin{aligned}
&\|T\psi_1(t) - T\psi_2(t)\| \\
&\quad \le \int_a^t |e_{\wp}(t, \sigma(s))|\beta \left[\|\psi_1(s) - \psi_2(s)\| + \|h(\psi_1(s)) - h(\psi_2(s))\|\right] \Delta s \\
&\quad \le \int_a^t |e_{\wp}(t, \sigma(s))|\beta \left[\|\psi_1(s) - \psi_2(s)\| + \gamma \, \|\psi_1(s) - \psi_2(s)\|\right] \Delta s \\
&\quad \le \int_a^t |e_{\wp}(t, \sigma(s))|\beta(1+\gamma) \, \|\psi_1(s) - \psi_2(s)\| \, \Delta s
\end{aligned}$$

$$\begin{aligned}
&\leq \beta(1+\gamma)\left\|\psi_1-\psi_2\right\|_\infty \int_a^t |e_{\wp}(t,\sigma(s))|\Delta s \\
&\leq \theta\beta(1+\gamma)\left\|\psi_1-\psi_2\right\|_\infty .
\end{aligned}$$

This implies that T is a contraction. Therefore, T has a unique fixed point ψ, which is the unique solution of (2.1) satisfying $\psi(a) = a_0$. Thus, by Theorem 2.1, ψ is the unique solution of (1.1). □

Corollary 2.1 *Assume* (H_1), (H_2), *and* (H_5). *If* $a_0 \in \mathbb{X}$, *then* (1.1) *has a unique solution.*

3 Hyers–Ulam Stability

In this section, we investigate Hyers–Ulam stability of (1.1). For a function $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$, we denote

$$H_\psi(t) := \mathcal{F}(t, \psi(t), h(\psi(t))) + f(t) \tag{3.1}$$

and

$$g_\psi(t) := \psi^\Delta(t) - \wp(t)\psi(t) - H_\psi(t). \tag{3.2}$$

Definition 3.1 (Hyers–Ulam stability) We say that (1.1) has Hyers–Ulam stability if there exists a constant $L > 0$, a so-called HUS constant, with the following property: For any $\varepsilon > 0$, if $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$ is such that

$$\left\|g_\psi(t)\right\| \leq \varepsilon \quad \text{for all} \quad t \in \mathcal{I}^\kappa, \tag{3.3}$$

then there exists a solution $\phi : \mathcal{I} \to \mathbb{X}$ of (1.1) such that

$$\|\psi(t) - \phi(t)\| \leq L\varepsilon \quad \text{for all} \quad t \in \mathcal{I}. \tag{3.4}$$

The following result establishes a new sufficient condition for Hyers–Ulam stability of (1.1).

Theorem 3.1 *If* (H_1), (H_2), *and* (H_3) *hold, then* (1.1) *has Hyers–Ulam stability with HUS constant*

$$L := (b-a)e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a). \tag{3.5}$$

Proof Let $\varepsilon > 0$. Suppose $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$ is such that (3.3) holds. Then

$$\begin{aligned}
\psi^\Delta(t) &= \wp(t)\psi(t) + H_\psi(t) + \psi^\Delta(t) - \wp(t)\psi(t) - H_\psi(t) \\
&= \wp(t)\psi(t) + H_\psi(t) + g_\psi(t).
\end{aligned}$$

Set $a_0 = \psi(a)$. By Theorem 2.1,

$$\psi(t) = e_{\wp}(t,a)a_0 + \int_a^t e_{\wp}(t,\sigma(s))\left[H_\psi(s) + g_\psi(s)\right]\Delta s. \tag{3.6}$$

By (H$_3$), there exists a solution ϕ of (1.1) with $\phi(a) = a_0$, that is, by Theorem 2.1,

$$\phi(t) = e_{\wp}(t,a)a_0 + \int_a^t e_{\wp}(t,\sigma(s))H_\phi(s)\Delta s, \quad t \in \mathcal{I}. \tag{3.7}$$

Subtracting (3.7) from (3.6), we find, for all $t \in \mathcal{I}$,

$$\begin{aligned}\|\psi(t) - \phi(t)\| \le& \left\|\int_a^t e_{\wp}(t,\sigma(s))g_\psi(s)\Delta s\right\| \\ &+ \left\|\int_a^t e_{\wp}(t,\sigma(s))\left[\mathcal{F}(s,\psi(s),h(\psi(s))) - \mathcal{F}(s,\phi(s),h(\phi(s)))\right]\Delta s\right\|.\end{aligned}$$

Since $\left\|g_\psi(t)\right\| \le \varepsilon$ holds for $t \in \mathcal{I}$ and taking into account (H$_2$), we get

$$\begin{aligned}\|\psi(t) - \phi(t)\| \le& \ \varepsilon\int_a^t \left|e_{\wp}(t,\sigma(s))\right|\Delta s \\ &+ \int_a^t \left|e_{\wp}(t,\sigma(s))\right|\beta\left[\|\psi(s)-\phi(s)\| + \|h(\psi(s)) - h(\phi(s))\|\right]\Delta s \\ \le& \ \varepsilon\int_a^t \left|e_{\wp}(t,\sigma(s))\right|\Delta s \\ &+ \int_a^t \left|e_{\wp}(t,\sigma(s))\right|\beta\left[\|\psi(s)-\phi(s)\| + \gamma\|\psi(s)-\phi(s)\|\right]\Delta s \\ \le& \ \varepsilon\int_a^t \left|e_{\wp}(t,\sigma(s))\right|\Delta s \\ &+ \int_a^t \left|e_{\wp}(t,\sigma(s))\right|\beta(1+\gamma)\|\psi(s)-\phi(s)\|\Delta s \\ \le& \ \varepsilon(b-a)e_{|\wp|}(b,a) + \beta(1+\gamma)e_{|\wp|}(b,a)\int_a^t \|\psi(s)-\phi(s)\|\Delta s.\end{aligned}$$

Thus, by Gronwall's inequality, Theorem 1.2, we deduce that

$$\|\psi(t) - \phi(t)\| \le \varepsilon(b-a)e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a) = L\varepsilon.$$

Therefore, (1.1) has Hyers–Ulam stability with HUS constant L given in (3.5). □

Remark 3.1 If $\mathcal{F} = 0$, then (1.1) is a first-order linear dynamic equation. In this case, we obtain the HUS constant $L = (b-a)e_{|\wp|}(b,a)$ given in [1].

As a direct consequence of Theorems 2.2 and Corollary 2.1, we deduce the following result.

Corollary 3.1 *Assume* (H_1) *and* (H_2)*. In addition, assume* (H_4) *or* (H_5)*. Then* (1.1) *has Hyers–Ulam stability with constant* L*.*

Example 3.1 We now give an example such that (H_1), (H_2), and (H_5) are statisfied, so that, for example, Corollary 3.1 applies. Consider

$$\mathbb{T} = \mathbb{P}_{1,1} := \bigcup_{k=0}^{\infty} [2k, 2k+1]$$

and let

$$m \in \mathbb{N}, \quad a = 0, \quad b = 2m+1, \quad \beta \in \left(0, \frac{1}{(2m+1)(2e)^{m+1}}\right).$$

Moreover, we let $f \in \mathrm{C}_{\mathrm{rd}}$, $\wp(t) \equiv 1$, and

$$\mathcal{F}(t, x, y) = \beta(\sin x + y), \quad h(x) = \cos x.$$

Equation (1.1) then takes the form

$$\psi^{\Delta}(t) = \psi(t) + \beta\left(\sin(\psi(t)) + \cos(\psi(t))\right) + f(t).$$

We note that (H_1) is satisfied because $\wp \in \mathcal{R}$ and $f \in \mathrm{C}_{\mathrm{rd}}$. We also note that ($\mathrm{H}_2$) is satisfied because $\mathcal{F}$ is Lipschitz continuous with Lipschitz constant β and h is Lipschitz continuous with Lipschitz constant $\gamma = 1$. Finally, according to [5, Example 2.58],

$$e_1(b, a) = e_1(2m+1, 0) = 2^m e^{m+1}.$$

Hence,

$$(b-a)e_{|\wp|}(b, a) = (2m+1)2^m e^{m+1} < \frac{1}{2\beta} = \frac{1}{\beta(1+\gamma)},$$

and thus (H_5) is satisfied as well.

4 Hyers–Ulam–Rassias Stability

In this section, we investigate Hyers–Ulam–Rassias stability of (1.1).

Definition 4.1 (Hyers–Ulam–Rassias stability) Let $\mathcal{M}$ be a family of positive rd-continuous functions defined on $\mathcal{I}$. We say that (1.1) has Hyers–Ulam–Rassias stability of type $\mathcal{M}$ if there exists a constant $L > 0$, a so-called $\mathrm{HURS}_{\mathcal{M}}$ constant, with the following property: For any $\omega \in \mathcal{M}$, if $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$ is such that

$$\left\| g_{\psi}(t) \right\| \leq \omega(t) \quad \text{for all} \quad t \in \mathcal{I}^{\kappa}, \tag{4.1}$$

then there exists a solution $\phi : \mathcal{I} \to \mathbb{X}$ of (1.1) such that

$$\|\psi(t) - \phi(t)\| \le L\omega(t) \quad \text{for all} \quad t \in \mathcal{I}. \tag{4.2}$$

We use the notations (3.1), (3.2),

$$\mathcal{M}^* := \{\omega \in \mathrm{C_{rd}}(\mathcal{I}, (0, \infty)) : \ \omega \text{ is nondecreasing}\},$$

and for $p \ge 1$,

$$\mathcal{M}_p := \left\{\omega \in \mathrm{C_{rd}}(\mathcal{I}, (0, \infty)) : \int_a^t \omega^p(s)\Delta s \le \omega^p(t) \text{ for all } t \in \mathcal{I}\right\}.$$

The following results are concerned with Hyers–Ulam–Rassias stability.

Theorem 4.1 *If* (H_1), (H_2), *and* (H_3) *hold, then* (1.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{M}^*$ *with* $HURS_{\mathcal{M}^*}$ *constant*

$$L := (b-a)e_{|\wp|}(b,a)\left(1 + (b-a)\beta(1+\gamma)e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\right). \tag{4.3}$$

Proof Let $\omega \in \mathcal{M}^*$ and $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{X})$ be such that (4.1) holds. Let $a_0 = \psi(a)$. By Theorem 2.1, (3.6) holds. By (H_3), there exists a solution ϕ of (1.1) that satisfies $\phi(a) = a_0$. By Theorem 2.1, (3.7) holds. Subtracting (3.7) from (3.6), we obtain, for all $t \in \mathcal{I}$,

$$\begin{aligned}
\|\psi(t) - \phi(t)\| &\le \int_a^t \left|e_\wp(t, \sigma(s))\right| \left\|g_\psi(s)\right\| \Delta s \\
&\quad + \int_a^t \left|e_\wp(t, \sigma(s))\right| \|\mathcal{F}(s, \psi(s), h(\psi(s))) - \mathcal{F}(s, \phi(s), h(\phi(s)))\| \Delta s \\
&\le e_{|\wp|}(b,a) \int_a^t \omega(s)\Delta s + e_{|\wp|}(b,a)\beta(1+\gamma) \int_a^t \|\psi(s) - \phi(s)\| \Delta s \\
&\le (b-a)e_{|\wp|}(b,a)\omega(t) + \beta(1+\gamma)e_{|\wp|}(b,a) \int_a^t \|\psi(s) - \phi(s)\| \Delta s.
\end{aligned}$$

Applying Gronwall's inequality, Theorem 1.2, we get, for all $t \in \mathcal{I}$,

$$\begin{aligned}
&\|\psi(t) - \phi(t)\| \le (b-a)e_{|\wp|}(b,a)\omega(t) \\
&\quad + \int_a^t e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(t, \sigma(s))(b-a)e_{|\wp|}(b,a)\omega(s)\beta(1+\gamma)e_{|\wp|}(b,a)\Delta s \\
&= (b-a)e_{|\wp|}(b,a)\omega(t) \\
&\quad + (b-a)\left(e_{|\wp|}(b,a)\right)^2 \beta(1+\gamma) \int_a^t e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(t, \sigma(s))\omega(s)\Delta s \\
&\le (b-a)e_{|\wp|}(b,a)\omega(t)
\end{aligned}$$

$$
\begin{aligned}
&+(b-a)^2 \left(e_{|\wp|}(b,a)\right)^2 \beta(1+\gamma)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\omega(t) \\
&= L\omega(t).
\end{aligned}
$$

Therefore, (1.1) is Hyers–Ulam-Rassias stable of type $\mathcal{M}^*$ with constant L given in (4.3). □

If we consider $\omega \in \mathcal{M}^* \cap \mathcal{M}_1$, then we can improve the HURS constant (if $b > a+1$) as follows.

Theorem 4.2 *If* (H_1), (H_2), *and* (H_3) *hold, then* (1.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{M}^* \cap \mathcal{M}_1$ *with* $\mathrm{HURS}_{\mathcal{M}^*\cap\mathcal{M}_1}$ *constant*

$$
L := e_{|\wp|}(b,a)\left(1+(b-a)\beta(1+\gamma)e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\right). \tag{4.4}
$$

Proof Let $\omega \in \mathcal{M}^* \cap \mathcal{M}_1$ and $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I},\mathbb{X})$ be such that (4.1) holds. Let $a_0 := \psi(a)$. By Theorem 2.1, (3.6) holds. By (H_3), there exists a solution ϕ of (1.1) such that $\phi(a) = a_0$. By Theorem 2.1, (3.7) holds. Subtracting (3.7) from (3.6), we obtain, for all $t \in \mathcal{I}$,

$$
\begin{aligned}
\|\psi(t)-\phi(t)\| &\le \int_a^t \left|e_\wp(t,\sigma(s))\right| \left\|g_\psi(s)\right\| \Delta s \\
&\quad + \int_a^t \left|e_\wp(t,\sigma(s))\right| \|\mathcal{F}(s,\psi(s),h(\psi(s))) - \mathcal{F}(s,\phi(s),h(\phi(s)))\| \Delta s \\
&\le e_{|\wp|}(b,a)\int_a^t \omega(s)\Delta s + e_{|\wp|}(b,a)\beta(1+\gamma)\int_a^t \|\psi(s)-\phi(s)\|\Delta s \\
&\le e_{|\wp|}(b,a)\omega(t) + \beta(1+\gamma)e_{|\wp|}(b,a)\int_a^t \|\psi(s)-\phi(s)\|\Delta s.
\end{aligned}
$$

Applying Gronwall's inequality, Theorem 1.2, we get, for all $t \in \mathcal{I}$,

$$
\begin{aligned}
\|\psi(t)-\phi(t)\| &\le e_{|\wp|}(b,a)\omega(t) \\
&\quad + \int_a^t e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(t,\sigma(s))e_{|\wp|}(b,a)\omega(s)\beta(1+\gamma)e_{|\wp|}(b,a)\Delta s \\
&= e_{|\wp|}(b,a)\omega(t) \\
&\quad + \left(e_{|\wp|}(b,a)\right)^2 \beta(1+\gamma)\int_a^t e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(t,\sigma(s))\omega(s)\Delta s \\
&\le e_{|\wp|}(b,a)\omega(t) \\
&\quad +(b-a)\left(e_{|\wp|}(b,a)\right)^2 \beta(1+\gamma)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\omega(t) \\
&= L\omega(t).
\end{aligned}
$$

Therefore, (1.1) is Hyers–Ulam–Rassias stable of type $\mathcal{M}^* \cap \mathcal{M}_1$ with constant L given in (4.4). □

Theorem 4.3 *If* (H_1)*,* (H_2)*, and* (H_3) *hold, then* (1.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{M}_1$ *with HURS*$_{\mathcal{M}_1}$ *constant*

$$L := e_{|\wp|}(b,a)\left(1+\beta(1+\gamma)e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\right). \tag{4.5}$$

Proof Let $\omega \in \mathcal{M}_1$ and $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I},\mathbb{X})$ be such that (4.1) holds. Let $a_0 := \psi(a)$. By Theorem 2.1, (3.6) holds. By (H_3), there exists a solution ϕ of (1.1). By Theorem 2.1, (3.7) holds. Subtracting (3.7) from (3.6), we obtain, for all $t \in \mathcal{I}$,

$$\begin{aligned}
\|\psi(t)-\phi(t)\| &\le \int_a^t \left|e_{\wp}(t,\sigma(s))\right| \left\|g_\psi(s)\right\| \Delta s \\
&\quad + \int_a^t |e_{\wp}(t,\sigma(s))| \|\mathcal{F}(s,\psi(s),h(\psi(s))) - \mathcal{F}(s,\phi(s),h(\phi(s)))\| \Delta s \\
&\le e_{|\wp|}(b,a)\int_a^t \omega(s)\Delta s + e_{|\wp|}(b,a)\beta(1+\gamma)\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s \\
&\le e_{|\wp|}(b,a)\omega(t) + \beta(1+\gamma)e_{|\wp|}(b,a)\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s.
\end{aligned}$$

Applying Gronwall's inequality, Theorem 1.2, we get, for all $t \in \mathcal{I}$,

$$\begin{aligned}
&\|\psi(t)-\phi(t)\| \\
&\le e_{|\wp|}(b,a)\omega(t) + \int_a^t e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(t,\sigma(s))e_{|\wp|}(b,a)\omega(s)\beta(1+\gamma)e_{|\wp|}(b,a)\Delta s \\
&\le e_{|\wp|}(b,a)\omega(t) + \left(e_{|\wp|}(b,a)\right)^2 \beta(1+\gamma)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\int_a^t \omega(s)\Delta s \\
&\le e_{|\wp|}(b,a)\omega(t) + \left(e_{|\wp|}(b,a)\right)^2 \beta(1+\gamma)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\omega(t) \\
&= L\omega(t).
\end{aligned}$$

Therefore, (1.1) is Hyers–Ulam–Rassias stable of type $\mathcal{M}_1$ with constant L given in (4.5). □

Theorem 4.4 *Let* $p > 1$ *and* $q := p/(p-1)$*. If* (H_1)*,* (H_2)*, and* (H_3) *hold, then* (1.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{M}_p$ *with HURS*$_{\mathcal{M}_p}$ *constant*

$$L := e_{|\wp|}(b,a)\sqrt[q]{b-a}\left(1+\beta(1+\gamma)e_{|\wp|}(b,a)\sqrt[q]{b-a}\,e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\right). \tag{4.6}$$

Proof Let $\omega \in \mathcal{M}_p$ and $\psi \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I},\mathbb{X})$ be such that (4.1) holds. Let $a_0 := \psi(a)$. By Theorem 2.1, (3.6) holds. By (H_3), there exists a solution ϕ of (1.1) such that $\phi(a) = a_0$. By Theorem 2.1, (3.7) holds. Subtracting (3.7) from (3.6), we obtain, for all $t \in \mathcal{I}$,

$$\|\psi(t)-\phi(t)\| \le \int_a^t \left|e_{\wp}(t,\sigma(s))\right| \left\|g_\psi(s)\right\| \Delta s$$

$$
\begin{aligned}
&+\int_a^t \left|e_{\wp}(t,\sigma(s))\right| \|\mathcal{F}(s,\psi(s),h(\psi(s)))-\mathcal{F}(s,\phi(s),h(\phi(s)))\|\,\Delta s\\
&\le e_{|\wp|}(b,a)\int_a^t \omega(s)\Delta s + e_{|\wp|}(b,a)\beta(1+\gamma)\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s\\
&\le e_{|\wp|}(b,a)\sqrt[q]{t-a}\sqrt[p]{\int_a^t \omega^p(s)\Delta s}\\
&\quad+\beta(1+\gamma)e_{|\wp|}(b,a)\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s\\
&\le e_{|\wp|}(b,a)\sqrt[q]{b-a}\sqrt[p]{\omega^p(t)}+\beta(1+\gamma)e_{|\wp|}(b,a)\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s\\
&\le e_{|\wp|}(b,a)\sqrt[q]{b-a}\omega(t)+\beta(1+\gamma)e_{|\wp|}(b,a)\int_a^t \|\psi(s)-\phi(s)\|\,\Delta s,
\end{aligned}
$$

where we have used Hölder's inequality, Theorem 1.3. Thus, by applying Gronwall's inequality, Theorem 1.2, we get, for all $t\in\mathcal{I}$,

$$
\begin{aligned}
\|\psi(t)-\phi(t)\| &\le e_{|\wp|}(b,a)\sqrt[q]{b-a}\omega(t)\\
&\quad+\int_a^t e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(t,\sigma(s))e_{|\wp|}(b,a)\sqrt[q]{b-a}\omega(s)\beta(1+\gamma)e_{|\wp|}(b,a)\Delta s\\
&\le e_{|\wp|}(b,a)\sqrt[q]{b-a}\omega(t)\\
&\quad+\left(e_{|\wp|}(b,a)\right)^2 e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\sqrt[q]{b-a}\beta(1+\gamma)\int_a^t \omega(s)\Delta s\\
&\le L\omega(t).
\end{aligned}
$$

Therefore, (1.1) has Hyers–Ulam–Rassias stability of type $\mathcal{M}_p$ with constant L given in (4.6). □

For the last result, we denote, for $r\ge 0$, by $\mathcal{M}_p^r$ the family

$$
\mathcal{M}_p^r := \left\{\omega\in \mathrm{C}_{\mathrm{rd}}(\mathcal{I},(0,\infty)) : \int_a^t \omega^p(s)\Delta s \le r\omega^p(t) \text{ for all } t\in\mathcal{I}\right\}.
$$

Theorem 4.5 *If* (H_1), (H_2), *and* (H_3) *hold, then* (1.1) *has Hyers–Ulam–Rassias stability of type* $\mathcal{M}_p^r$ *with HURS*$_{\mathcal{M}_p^r}$ *constant*

$$
L := e_{|\wp|}(b,a)\sqrt[q]{b-a}\sqrt[p]{r}\left(1+\beta(1+\gamma)\sqrt[q]{b-a}\sqrt[p]{r}e_{|\wp|}(b,a)e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\right). \tag{4.7}
$$

Proof Let $\omega\in\mathcal{M}_p^r$ and $\psi\in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I},\mathbb{X})$ be such that (4.1) holds. Let $a_0 := \psi(a)$. By Theorem 2.1, (3.6) holds. By (H_3), there exists a solution ϕ of (1.1) such that $\phi(a)=a_0$. By Theorem 2.1, (3.7) holds. Subtracting (3.7) from (3.6), we obtain, for all $t \in \mathcal{I}$,

$$\begin{aligned}
\|\psi(t)-\phi(t)\| &\le \int_a^t \left|e_{\wp}(t,\sigma(s))\right| \left\|g_\psi(s)\right\| \Delta s \\
&\quad + \int_a^t \left|e_{\wp}(t,\sigma(s))\right| \|\mathcal{F}(s,\psi(s),h(\psi(s)))-\mathcal{F}(s,\phi(s),h(\phi(s)))\| \Delta s \\
&\le e_{|\wp|}(b,a)\int_a^t \omega(s)\Delta s + e_{|\wp|}(b,a)\beta(1+\gamma)\int_a^t \|\psi(s)-\phi(s)\| \Delta s \\
&\le e_{|\wp|}(b,a)\sqrt[q]{t-a}\sqrt[p]{\int_a^t \omega^p(s)\Delta s} \\
&\quad + e_{|\wp|}(b,a)\beta(1+\gamma)\int_a^t \|\psi(s)-\phi(s)\| \Delta s \\
&\le e_{|\wp|}(b,a)\sqrt[q]{b-a}\sqrt[p]{r}\omega(t) + e_{|\wp|}(b,a)\beta(1+\gamma)\int_a^t \|\psi(s)-\phi(s)\| \Delta s,
\end{aligned}$$

where we have used Hölder's inequality, Theorem 1.3. Thus, by applying Gronwall's inequality, Theorem 1.2, we get, for all $t \in \mathcal{I}$,

$$\begin{aligned}
\|\psi(t)-\phi(t)\| &\le e_{|\wp|}(b,a)\sqrt[q]{b-a}\sqrt[p]{r}\omega(t) \\
&\quad + \int_a^t e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(t,\sigma(s))e_{|\wp|}(b,a)\beta(1+\gamma)\sqrt[q]{b-a}\sqrt[p]{r}\omega(s)e_{|\wp|}(b,a)\Delta s \\
&\le e_{|\wp|}(b,a)\sqrt[q]{b-a}\sqrt[p]{r}\omega(t) \\
&\quad + \left(e_{|\wp|}(b,a)\right)^2 e_{\beta(1+\gamma)e_{|\wp|}(b,a)}(b,a)\beta(1+\gamma)\sqrt[q]{b-a}\sqrt[p]{r}\int_a^t \omega(s)\Delta s \\
&\le L\omega(t).
\end{aligned}$$

Therefore, (1.1) has Hyers–Ulam-Rassias stability of type $\mathcal{M}_p^r$ with constant L given in (4.7). ◻

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