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Parameter Estimation Under Failure-Censored Constant-Stress Life Testing Model: A Bayesian Approach

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Abstract: This article compares likelihood and Bayesian estimations for partially accelerated constant-stress life test model under type II censoring assuming Pareto distribution of the second kind. Both maximum likelihood and Bayesian estimators of the model parameters are derived. The posterior means and posterior variances are obtained under the squared error loss function using Lindley's approximation procedure. The advantages of this proposed procedure are shown. Monte Carlo simulations are conducted under different samples sizes and different parameter values to assess and compare the proposed methods of estimation. A noninformative prior on the model parameters is used to make the comparison more meaningful. It has been observed that Lindley's method usually provides posterior variances and mean squared errors smaller than those of the maximum likelihood estimators. That is, Lindley's method produces improved estimates, which is an advantage of this method.

Keywords: Bayesian estimation; Failure censoring; Maximum likelihood estimation; Pareto distribution; Partially accelerated constant-stress test; Squared error loss function.

Subject Classifications: 62N01; 62N05.

1. INTRODUCTION

Based on failure censored data, reliability analysis for constant-stress partially accelerated life tests (CSPALT) with product lifetime following Pareto distribution of the second kind is investigated. In practice, most products such as lamps, semiconductors, microelectronics, etc., run at a constant stress level. Such testing is

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simple and has several advantages: first, it is easier to maintain a constant stress level in most tests. Second, accelerated test models are better developed. Third, data analysis for reliability estimation is well developed and computerized Nelson (1990).

In the case of simple CSPALT, the sample size of the test units is divided, using a prespecified sample proportion, into two parts. The first part runs under use/normal condition and the remaining units are tested under accelerated condition. The test items are run until failures occur or the observations are censored. The object of partially accelerated life tests (PALT) is to collect more failure data in a limited time without necessarily using high stresses to all test units.

For an overview of CSPALT, there are few studies on constant stress, such as Bai and Chung (1992); Bai et al. (1993); Ismail et al. (2011), and Ismail (2014). All of these studies were performed based on classical methods under type I censoring. This article deals with the Bayesian approach with type II censoring for estimating the parameters under CSPALT considering the Lindley method for approximation of integrals. As indicated by Sinha (1986), such an approximation has led to many useful applications. In addition, as pointed out by Achcar (1994), the use of approximate Bayesian methods could be a good alternative for the usual existing classical asymptotic methods used in accelerated life testing (ALT). With this approach, simple expressions easy to use for the marginal posterior moments are obtained, which is a result that could be of great practical interest, especially for industrial applications.

From the Bayesian point of view, few of studies have been considered on PALT. Goel (1971) used the Bayesian approach for estimating the acceleration factor and the parameters in the case of step-stress PALT (SSPALT) with complete sampling for items having exponential and uniform distributions. In the case of SSPALT, a test item is first run at use condition and, if it does not fail for a prespecified time τ, then it is run at accelerated condition until it fails or the test is terminated. DeGroot and Goel (1979) investigated the optimal Bayesian design of PALT in the case of the exponential distribution under complete sampling. Abdel-Ghani (1998) considered the Bayesian approach to estimate the parameters of Weibull distribution in SSPALT with censoring. Tahir (2003) estimated the failure rate with the Bayes estimator under the squared error loss in SSPALT assuming the exponential distribution. Ismail (2010) considered the Bayesian approach to estimate the parameters of Gompertz distribution under SSPALT model with time censoring.

The objective of this article is to apply a Bayesian analysis, with a squared error loss function, on CSPALT with type II censoring considering a two-parameter Pareto distribution. The Bayes estimators (BEs) of the acceleration factor and the distribution parameters are derived and compared with the maximum likelihood estimator (MLE) counterparts by Monte Carlo simulations. To make the comparison more meaningful, the noninformative priors (NIPs) on both shape and scale parameters are considered.

The rest of this article is organized as follows. In Section 2, the model and test method are described. Approximate BEs of the parameters under consideration are derived in Section 3. In Section 4, BEs derived in Section 3 are obtained numerically using Lindley's approximation and compared with the MLEs. Finally, Section 5 concludes the article.

2. THE MODEL AND TEST METHOD

2.1. The Pareto Distribution as a Lifetime Model

In this article, the two-parameter Pareto distribution of the second kind is considered as a lifetime model. The Pareto distribution was introduced by Pareto (1897) as a model for the distribution of income. In recent years, its models in several different forms have been studied by many authors (Cohen and Whitten, 1988; Davis and Feldstein, 1979; Grimshaw, 1993; among others). The Pareto distribution of the second kind is also known as Lomax's or Pearson's type VI distribution; see Johnson et al. (1994). It has been found to be a good model in biomedical problems, such as survival time following a heart transplant Bain and Engelhardt (1992). Using the Pareto distribution, Dyer (1981) studied annual wage data of production line workers in a large industrial firm. Lomax (1954) used this distribution in the analysis of business failure data. The length of wire between flaws also follows a Pareto distribution (Bain and Engelhardt, 1992). Because the Pareto distribution has a decreasing hazard or failure rate, it has often been used to model incomes and survival times (Howlader and Hossain, 2002).

The probability density function of the Pareto distribution of the second kind is given by

$$f_T(t; \theta, \alpha) = \frac{\alpha\theta^\alpha}{(\theta + t)^{\alpha+1}}; t > 0, \theta > 0, \alpha > 0. \tag{2.1}$$

The survival function takes the form

$$R(t) = \frac{\theta^\alpha}{(\theta + t)^\alpha}. \tag{2.2}$$

The corresponding failure rate function is

$$h(t) = \frac{\alpha}{\theta + t}. \tag{2.3}$$

According to McCune and McCune (2000), the Pareto distribution has classically been used in economic studies of income, size of cities and firms, service time in queuing systems, and so on. In addition, it has been used in connection with reliability theory and survival analysis; see Davis and Feldstein (1979).

2.2. Constant-Stress PALT

The test procedure of the constant-stress PALT and its assumptions are described as follows:

2.2.1. Test Procedure

In a constant-stress PALT, the total sample size n of test units is divided into two parts such that $n\pi$ items randomly chosen among n test items sampled are allocated to accelerated condition and the remaining are allocated to use condition. Each test item is run until the censoring time is reached or the item fails and the test condition is not changed.

2.2.2. Assumptions

1. The lifetimes T_i, $i = 1, \ldots, n(1-\pi)$ of items allocated to use condition are independent and identically distributed (i.i.d.) random variables (r.v.s).
2. The lifetimes X_j, $j = 1, \ldots, n\pi$ of items allocated to accelerated condition are i.i.d. r.v.s.
3. Suppose that the lifetime of an item at accelerated condition is denoted by X, then the lifetime of this item at use condition T is given by the relation $T = \beta X$.

Because the lifetimes of the test items follow a Pareto distribution of the second kind, the probability density function of an item tested at use condition is given by

$$f_T(t;\theta,\alpha) = \frac{\alpha\theta^{\alpha}}{(\theta+t)^{\alpha+1}};\ t>0, \theta>0, \alpha>0, \tag{2.4}$$

whereas for an item tested at accelerated condition, the probability density function is given by

$$f_X(x;\theta,\alpha) = \frac{\beta\alpha\theta^{\alpha}}{(\theta+\beta x)^{\alpha+1}};\ x>0, \theta>0, \alpha>0, \tag{2.5}$$

where $X = \beta^{-1}T$.

3. BAYESIAN ESTIMATION

In this section, the squared error (SE) loss function is considered. Under SE loss function, the Bayes estimator of a parameter is its posterior expectation. The Bayes estimators cannot be expressed in explicit forms. Approximate Bayes estimators will be obtained under the assumption of NIPs using Lindley's approximation. In most applied problems, information about the parameters is available in an independent manner' see Basu et al. (1999). Thus, here it is assumed that the parameters are independent a priori and let the NIP for each parameter be represented by the limiting form of the appropriate natural conjugate prior.

It follows that an NIP for the acceleration factor β is given by $\pi_1(\beta) \propto \beta^{-1}$, $\beta > 1$.

In addition, the NIPs for the scale parameter θ and the shape parameter α are respectively

$$\pi_2(\theta) \propto \theta^{-1}, \theta>0, \text{ and } \pi_3(\alpha) \propto \alpha^{-1}, \alpha>0.$$

Therefore, the joint NIP of the three parameters can be expressed by

$$\pi(\beta,\theta,\alpha) \propto (\beta\theta\alpha)^{-1}, \beta>1, \theta>0, \alpha>0. \tag{3.1}$$

Using type II censored data in CSPALT, each test item runs at either use or accelerated condition only until a predetermined censoring time $y_{(r)}$ is reached. That is, after acquiring r failures the test is terminated. Therefore, the observed lifetimes $t_(1) \leq \cdots \leq t_{(nu)} \leq y_(r)$, and $x_{(1)} \leq \cdots = x_{(na)} \leq y_{(r)}$ are ordered failure times at use and accelerated conditions, respectively, where n_u and n_a are the corresponding numbers of items failed in each stage. Let δ_{ui} and δ_{aj} be indicator functions such that $\delta_{ui} \equiv I\ (T_i = y_{(r)})$ and $\delta_{aj} \equiv \mathrm{I}\ (X_j \leq y_{(r)})$, where $i = 1, \ldots, n$. Then the total likelihood function for $(t_1;\ \delta_{u1}, \ldots, t_{n(1-\pi)};\ \delta_{un(1-\pi)}, x_1;\ \delta_{a1}, \ldots, x_{n_\pi};\ \delta_{an_\pi})$ is given by

$$\begin{aligned} L(\beta, \theta, \alpha) &= \prod_{i=1}^{n(1-\pi)} L_{ui}(t_i; \theta, \alpha) \cdot \prod_{j=1}^{n\pi} L_{aj}(x_j; \beta, \theta, \alpha) \\ &= \prod_{i=1}^{n(1-\pi)} \left[\frac{\alpha\theta^{\alpha}}{(\theta + t_i)^{\alpha+1}}\right]^{\delta_{ui}} \left[\frac{\theta^{\alpha}}{(\theta + y_{(r)})^{\alpha}}\right]^{\bar{\delta}_{ui}}, \\ &\cdot \prod_{j=1}^{n\pi} \left[\frac{\beta\alpha\theta^{\alpha}}{(\theta + \beta\ x_j)^{\alpha+1}}\right]^{\delta_{aj}} \left[\frac{\theta^{\alpha}}{(\theta + \beta\ y_{(r)})^{\alpha}}\right]^{\bar{\delta}_{aj}} \end{aligned} \tag{3.2}$$

where L_{ui} is the likelihood function for t_i at use condition, L_{aj} is the likelihood function for x_j at accelerated condition, π is the proportion of sample units allocated to accelerated condition, and

$$\bar{\delta}_{ui} = 1 - \delta_{ui}$$

and

$$\bar{\delta}_{aj} = 1 - \delta_{aj}.$$

Obviously, the Bayes solution for estimating the parameters is extremely difficult to obtain in a closed form because the posterior density is too complicated. Therefore, numerical approximations are necessary for finding the posterior moments of interest. For mathematical simplicity, the Bayesian analysis is introduced under the Jeffreys vague prior distribution for the three unknown parameters. Assuming the vague prior distribution of β, θ, and α as in (3.1) and forming the product of it and the likelihood function defined in (3.3), the joint posterior distribution of β, θ, and α can be expressed as follows:

$$\begin{aligned} g(\beta, \theta, \alpha\,|\underline{y}) &\propto L(\underline{y}\,|\beta, \theta, \alpha)\ .\ \pi(\beta, \theta, \alpha) \\ &\propto \frac{\beta^{n_a-1}\theta^{2n\alpha-1}\alpha^{n_u+n_a}}{\left(\theta + y_{(r)}\right)^{(n\bar{\pi}-n_u)\alpha}\left(\theta + \beta\ y_{(r)}\right)^{(n\pi-n_a)\alpha}} \left[\prod_{i=1}^{n\bar{\pi}} \frac{1}{(\theta + t_i)^{\alpha+1}}\right]^{\delta_{u_i}} \\ &\cdot \left[\prod_{j=1}^{n\pi} \frac{1}{(\theta + \beta\ x_j)^{\alpha+1}}\right]^{\delta_{a_j}} \end{aligned} \tag{3.3}$$

As mentioned earlier, under a squared error loss function, the Bayes estimator of a parameter is its posterior expectation. To obtain the posterior means and posterior variances of β, θ, and α, nontractable integrals will be met. It is not possible to obtain them analytically. The marginal posteriors are somewhat unwieldy and require a numerical integration that may not converge. Instead, an approximation due to Lindley (1980) via an asymptotic expansion of the ratio of two nontractable integrals is used to obtain approximate Bayes estimators. Lindley's approximation is evaluated at the MLEs of the model parameters.

Now, let Θ be a set of parameters $\{\Theta_1, \Theta_2, \ldots, \Theta_m\}$, where m is the number of parameters, then the posterior expectation of an arbitrary function $u(\Theta)$ can be asymptotically estimated by

$$E(u(\Theta)) = \frac{\int_{\Theta} u(\Theta)\pi(\Theta)e^{ln\ \mathrm{L}(y|\Theta)}d\Theta}{\int_{\Theta} \pi(\Theta)e^{ln\ \mathrm{L}(y|\Theta)d\Theta}}$$
$$\approx \left[u + (1/2)\sum_{i,j}(u_{ij}^{(2)} + 2u_i^{(1)}\rho_j^{(1)})\sigma_{ij} + (1/2)\sum_{i,j,k,s} L_{ijk}^{(3)}\sigma_{ij}\sigma_{ks}u_s^{(1)}\right] \downarrow \hat{\Theta}, \qquad (3.4)$$

which is the Bayes estimator of $u(\Theta)$ under a squared error loss function, where $\pi(\Theta)$ is the prior distribution of Θ, $u \equiv u(\Theta)$, $L \equiv L(\Theta)$ is the likelihood function, $\rho \equiv \rho(\Theta) = \log \pi(\Theta)$, σ_{ij} are the elements of the inverse of the asymptotic Fisher information matrix of β, θ, and α, and

$$u_i^{(1)} = \frac{\partial u}{\partial \Theta_i},\ u_{ij}^{(2)} = \frac{\partial^2 u}{\partial \Theta_i \partial \Theta_j},\ \rho_j^{(1)} = \frac{\partial log\ \pi(\Theta)}{\partial \Theta_j} \text{ and } L_{ijk}^{(3)} = \frac{\partial^3 ln\ L(y|\Theta)}{\partial \Theta_i \partial \Theta_j \partial \Theta_k}$$

Such an approximation is easy to use and does not require innovative programming and extensive computer time. According to Green (1980), the linear Bayes estimator in (3.4) is a very good and operational approximation for the ratio of multidimension integrals. As indicated by Sinha (1986), it has led to many useful applications. However, if the domain of the parameters is a function of the parameters, Bayes estimators using Lindley's rule are not obtainable unless the MLEs exist. The derivation of posterior means and posterior variances is shown in the Appendix.

4. SIMULATION STUDIES

In this section, the objective is to illustrate the use of Bayesian approach via Lindley's method for approximation of integrals to derive the marginal posterior moments of interest in the case of CSPALT under type II censoring. The data are generated from a Pareto distribution with different sample sizes. For each sample size, 5,000 samples are obtained randomly. The posterior means and posterior variances of the three parameters are obtained numerically. In addition, the MLEs and Bayes estimators are compared with respect to the mean squared errors (MSEs) and variability.

Two numerical examples are presented to illustrate the theoretical results of estimation. The true values of the parameters are set at $(\beta, \theta, \alpha) = (1.5, 0.4, 0.3)$ as a first example with computational results presented in Table 1. In the second example, the true values of the parameters are (4, 2, 3) with numerical results summarized in Table 2. Tables 1 and 2 contain the results of both MLE and Bayesian estimation for the parameters of Pareto distribution applied to CSPALT with type II censoring. As seen from the numerical results, the Bayesian estimators usually have smaller MSEs and smaller variances than those of the MLEs, which is a great advantage of the use of Lindley's method. Accordingly, the posterior means or approximate BEs as shown from the computational results using Lindley's method are more accurate and more efficient than MLEs.

Table 1. Average values of the MLEs and approximate BEs with associated estimated variances and MSEs when $\beta = 1.5$, $\theta = 0.4$, $\alpha = 0.3$, given $\pi = 0.50$ and $r = 0.70\ n$ based on varying sample sizes using type II censoring

n	Parameter	Method	Estimate	MSE	Variance
25	β	MLE	1.7647	0.0602	0.0327
		Bayes	1.7009	0.0510	0.0261
	θ	MLE	0.6091	0.0345	0.0146
		Bayes	0.4593	0.0259	0.0072
	α	MLE	0.4023	0.0249	0.0065
		Bayes	0.3849	0.0186	0.0036
50	β	MLE	1.6592	0.0443	0.0213
		Bayes	1.5818	0.0338	0.0134
	θ	MLE	0.4776	0.0238	0.0084
		Bayes	0.4366	0.0161	0.0045
	α	MLE	0.3699	0.0184	0.0029
		Bayes	0.3442	0.0131	0.0013
75	β	MLE	1.6126	0.0314	0.0109
		Bayes	1.5375	0.0263	0.0038
	θ	MLE	0.4323	0.0175	0.0033
		Bayes	0.4154	0.0135	0.0020
	α	MLE	0.3305	0.0042	0.0010
		Bayes	0.3151	0.0017	0.0005
100	β	MLE	1.5292	0.0061	0.0022
		Bayes	1.5198	0.0033	0.0020
	θ	MLE	0.4092	0.0047	0.0019
		Bayes	0.3982	0.0026	0.0007
	α	MLE	0.3137	0.0013	0.0004
		Bayes	0.3023	0.0011	0.0001

Table 2. Average values of the MLEs and approximate BEs with associated estimated variances and MSEs when $\beta = 4$, $\theta = 2$, $\alpha = 3$, given $\pi = 0.50$ and $r = 0.70\ n$ based on varying sample sizes using type II censoring

n	Parameter	Method	Estimate	MSE	Variance
25	β	MLE	5.0466	0.0375	0.0182
		Bayes	4.8144	0.0317	0.0143
	θ	MLE	2.8742	0.0214	0.0081
		Bayes	2.7284	0.0162	0.0042
	α	MLE	3.5296	0.0155	0.0036
		Bayes	3.2854	0.0116	0.0023
50	β	MLE	4.7954	0.0276	0.0117
		Bayes	4.6208	0.0211	0.0071
	θ	MLE	2.5788	0.0149	0.0047
		Bayes	2.4762	0.0101	0.0025
	α	MLE	3.1752	0.0114	0.0016
		Bayes	3.0784	0.0082	0.0008

(*continued*)

Table 2. (*continued*)

75	β	MLE	4.2494	0.0195	0.0060
		Bayes	4.0972	0.0165	0.0021
	θ	MLE	2.4864	0.0109	0.0019
		Bayes	2.4422	0.0084	0.0011
	α	MLE	3.0666	0.0028	0.0005
		Bayes	3.0208	0.0010	0.0002
100	β	MLE	4.0788	0.0037	0.0012
		Bayes	4.0226	0.0021	0.0009
	θ	MLE	2.4234	0.0029	0.0011
		Bayes	2.4042	0.0016	0.0004
	α	MLE	3.0204	0.0008	0.0003
		Bayes	3.0004	0.0007	0.0001

5. CONCLUSION

In this article the MLEs and Bayesian estimations of the parameters of Pareto distribution and the acceleration factor have been obtained under CSPALT model with type II censoring. The Bayes estimators have been obtained under the assumptions of squared error loss functions and noninformative priors. It has been observed that the Bayesian estimators cannot be obtained analytically. Instead, Lindley's approximation has been used to obtain the Bayesian estimates numerically.

It is seen that the approximation works very well even for small sample sizes. In addition, it is noted that Lindley's method usually gives posterior variances smaller than the variances of the MLEs. That is, it provides better estimates, which is an advantage of this method. It can be said that the intrinsic appeal of this method can be expressed in is its being a sort of adjustment to the maximum likelihood approach to reduce variability.

APPENDIX: THE POSTERIOR MEANS AND POSTERIOR VARIANCES

There are three parameters in the model. That is, $m = 3$. Let the subscripts 1, 2, and 3 refer to β, θ, and α, respectively. It is not easy to obtain the posterior moments analytically. Therefore, using Lindley's expansion, the posterior mean (i.e., Bayesian estimator under squared error loss function) and the posterior variance of β are given, respectively, in the form

$$\begin{aligned}\beta^* &= E(\beta\,|\mathrm{y}) \\ &= \left[\beta - \left(\frac{\sigma_{11}}{\beta} + \frac{\sigma_{12}}{\theta} + \frac{\sigma_{13}}{\alpha}\right) + \frac{1}{2}(\sigma_{11}E_1 + \sigma_{12}E_2 + \sigma_{13}E_3)\right] \downarrow \hat{\Theta}, \qquad \text{(A.1)}\end{aligned}$$

and

$$\begin{aligned}Var\,(\beta\,|\mathrm{y}) &= E(\beta^2\,|y) - (\beta^*)^2 \\ &= \sigma_{11} - \left[\left(\frac{\sigma_{11}}{\beta} + \frac{\sigma_{12}}{\theta} + \frac{\sigma_{13}}{\alpha}\right) - \frac{1}{2}(\sigma_{11}E_1 + \sigma_{12}E_2 + \sigma_{13}E_3)\right]^2 \downarrow \hat{\Theta}. \quad \text{(A.2)}\end{aligned}$$

Conducting the same procedure, the posterior mean and posterior variance of the scale parameter θ take the following form:

$$\begin{aligned}\theta^* &= E(\theta\,|\mathrm{y}) \\ &= \left[\theta - \left(\frac{\sigma_{21}}{\beta} + \frac{\sigma_{22}}{\theta} + \frac{\sigma_{23}}{\alpha}\right) + \frac{1}{2}(\sigma_{21}E_1 + \sigma_{22}E_2 + \sigma_{23}E_3)\right] \downarrow \hat{\Theta}, \end{aligned} \tag{A.3}$$

and

$$\begin{aligned}Var(\theta\,|\mathrm{y}) &= E(\theta^2\,|y) - (\theta^*)^2 \\ &= \sigma_{22} - \left[\left(\frac{\sigma_{21}}{\beta} + \frac{\sigma_{22}}{\theta} + \frac{\sigma_{23}}{\alpha}\right) - \frac{1}{2}(\sigma_{21}E_1 + \sigma_{22}E_2 + \sigma_{23}E_3)\right]^2 \downarrow \hat{\Theta}. \end{aligned} \tag{A.4}$$

Similarly, for the shape parameter α, the posterior mean and the posterior variance are expressed as follows:

$$\begin{aligned}\alpha^* &= E(\alpha\,|\mathrm{y}) \\ &= \left[\alpha - \left(\frac{\sigma_{31}}{\beta} + \frac{\sigma_{32}}{\theta} + \frac{\sigma_{33}}{\alpha}\right) + \frac{1}{2}(\sigma_{31}E_1 + \sigma_{32}E_2 + \sigma_{33}E_3)\right] \downarrow \hat{\Theta}, \end{aligned} \tag{A.5}$$

and

$$\begin{aligned}Var(\alpha\,|\mathrm{y}) &= E(\alpha^2\,|y) - (\alpha^*)^2 \\ &= \sigma_{33} - \left[\left(\frac{\sigma_{31}}{\beta} + \frac{\sigma_{32}}{\theta} + \frac{\sigma_{33}}{\alpha}\right) - \frac{1}{2}(\sigma_{31}E_1 + \sigma_{32}E_2 + \sigma_{33}E_3)\right]^2 \downarrow \hat{\Theta}, \end{aligned} \tag{A.6}$$

$$E_1 = \sum_{i,j} \sigma_{ij} L^{(3)}_{ij1}, \quad E_2 = \sum_{i,j} \sigma_{ij} L^{(3)}_{ij2}, \quad E_3 = \sum_{i,j} \sigma_{ij} L^{(3)}_{ij3}.$$

where

For $i, j = 1, 2, 3$,

σ_{ij} are the elements of the inverse of the asymptotic Fisher information $L^{(3)}_{ijk}$, $k = 1, 2, 3$ matrix of the MLEs of β, θ, and α in the case of type II censored data and is the third derivative of the natural logarithm of the likelihood function.

To calculate the posterior means and the posterior variances of β, θ, and α derived before, both second and third derivatives of the natural logarithm of the likelihood function in (3.3) must be obtained.

The second derivatives can be presented by the following equations:

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n_a}{\beta^2} + \frac{(n\pi - n_a)\,\alpha\big(y_{(r)}\big)^2}{\big(\theta + \beta\ y_{(r)}\big)^2} + (\alpha+1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{{x_j}^2}{\big(\theta + \beta x_j\big)^2}, \tag{A.7}$$

$$\frac{\partial^2 \ln L}{\partial \beta\, \partial \theta} = \frac{(n\pi - n_a)\alpha\, y_{(r)}}{\big(\theta + \beta\ y_{(r)}\big)^2} + (\alpha+1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{\big(\theta + \beta x_j\big)^2}, \tag{A.8}$$

$$\frac{\partial^2 \ln L}{\partial \beta\ \partial \alpha} = -\frac{(n\pi - n_a)\, y_{(r)}}{\theta + \beta\, y_{(r)}} - \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{\theta + \beta x_j}, \tag{A.9}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n\alpha}{\theta^2} + \frac{(n\pi - n_a)\alpha}{\left(\theta + \beta\ y_{(r)}\right)^2} + \frac{(n\bar{\pi} - n_u)\alpha}{\left(\theta + y_{(r)}\right)^2} + (\alpha + 1)\left[\sum_{i=1}^{n\bar{\pi}} \frac{\delta_{ui}}{(\theta + t_i)^2} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{\left(\theta + \beta x_j\right)^2}\right] \tag{A.10}$$

$$\frac{\partial^2 \ln L}{\partial \theta\ \partial \alpha} = \frac{n}{\theta} - \frac{(n\pi - n_a)}{\theta + \beta\, y_{(r)}} - \frac{(n\bar{\pi} - n_u)}{\theta + y_{(r)}} - \left[\sum_{i=1}^{n\bar{\pi}} \frac{\delta_{ui}}{\theta + t_i} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{\theta + \beta x_j}\right], \tag{A.11}$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n_u + n_a}{\alpha^2} \tag{A.12}$$

For the third derivatives, they are given as follows:

$$L_{111}^{(3)} = \frac{\partial^3 \ln L}{\partial \beta^3} = \frac{2n_a}{\beta^3} - \frac{2(n\pi - n_a)\alpha(y_{(r)})^3}{\left(\theta + \beta\ y_{(r)}\right)^3} - 2(\alpha + 1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j^3}{(\theta + \beta x_j)^3}, \tag{A.13}$$

$$L_{222}^{(3)} = \frac{\partial^3 \ln L}{\partial \theta^3} = \frac{2n\alpha}{\theta^3} - \frac{2(n\pi - n_a)\alpha}{\left(\theta + \beta\ y_{(r)}\right)^3} - \frac{2(n\bar{\pi} - n_u)\alpha}{\left(\theta + y_{(r)}\right)^3} - 2(\alpha + 1)\left[\sum_{i=1}^{n\bar{\pi}} \frac{\delta_{ui}}{(\theta + t_i)^3} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{(\theta + \beta x_j)^3}\right], \tag{A.14}$$

$$L_{333}^{(3)} = \frac{\partial^3 \ln L}{\partial \alpha^3} = \frac{2(n_u + n_a)}{\alpha^3}, \tag{A.15}$$

$$L_{112}^{(3)} = \frac{\partial^3 \ln L}{\partial \beta^2 \partial \theta} = -\frac{2(n\pi - n_a)\alpha(y_{(r)})^2}{\left(\theta + \beta\ y_{(r)}\right)^3} - 2(\alpha + 1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j^2}{(\theta + \beta x_j)^3} = L_{121}^{(3)} = L_{211}^{(3)}, \tag{A.16}$$

$$L_{221}^{(3)} = \frac{\partial^3 \ln L}{\partial \theta^2 \partial \beta} = -\frac{2(n\pi - n_a)\alpha\, y_{(r)}}{\left(\theta + \beta\ y_{(r)}\right)^3} - 2(\alpha + 1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{(\theta + \beta x_j)^3} = L_{212}^{(3)} = L_{122}^{(3)}, \tag{A.17}$$

$$L_{113}^{(3)} = \frac{\partial^3 \ln L}{\partial \beta^2 \partial \alpha} = \frac{(n\pi - n_a)\,(y_{(r)})^2}{\left(\theta + \beta\ \, y_{(r)}\right)^2} + \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j^2}{(\theta + \beta x_j)^2} = L_{131}^{(3)} = L_{311}^{(3)}, \tag{A.18}$$

$$L_{123}^{(3)} = \frac{\partial^3 \ln L}{\partial \beta\ \partial \theta\ \partial \alpha} = \frac{(n\pi - n_a)\, y_{(r)}}{\left(\theta + \beta\ \, y_{(r)}\right)^2} + \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{(\theta + \beta x_j)^2}$$
$$= L_{132}^{(3)} = L_{213}^{(3)} = L_{231}^{(3)} = L_{312}^{(3)} = L_{321}^{(3)}, \tag{A.19}$$

$$L_{223}^{(3)} = \frac{\partial^3 \ln L}{\partial \theta^2 \partial \alpha} = -\frac{n}{\theta^2} + \frac{(n\pi - n_a)}{\left(\theta + \beta\ \, y_{(r)}\right)^2} + \frac{(n\bar{\pi} - n_u)}{\left(\theta + y_{(r)}\right)^2}$$
$$+ \sum_{i=1}^{n\bar{\pi}} \frac{\delta_{ui}}{(\theta + t_i)^2} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{(\theta + \beta x_j)^2}$$
$$= L_{232}^{(3)} = L_{322}^{(3)}, \tag{A.20}$$

$$L_{331}^{(3)} = \frac{\partial^3 \ln L}{\partial \alpha^2 \partial \beta} = L_{313}^{(3)} = L_{133}^{(3)}, \tag{A.21}$$

$$L_{332}^{(3)} = \frac{\partial^3 \ln L}{\partial \alpha^2 \partial \theta} = L_{323}^{(3)} = L_{233}^{(3)}. \tag{A.22}$$

NOMENCLATURE

n	number of step-stress test units (total sample size)
n_u, n_a	number of test units failed at use and accelerated conditions, respectively
T	lifetime of an item at use condition
X	lifetime of an item at accelerated condition
$y_{(r)}$	censoring time in CSPALT (the time of the rth failure at which the test is terminated)
α	shape parameter ($\alpha > 0$)
β	acceleration factor ($\beta > 1$)
θ	scale parameter ($\theta > 0$)
π	proportion of tested items that allocated to accelerated condition
$\wedge$	implies a maximum likelihood estimator
$\downarrow$ (.)	evaluated at (.)

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