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Optimum Partially Accelerated Life Test Plans with Progressively Type I Interval-Censored Data

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Abstract: Because of continual improvement in manufacturing design, one often deals with high quality products that are highly reliable with a substantially long life span. This article discusses k-level step-stress partially accelerated tests under type I progressive interval censoring with equal inspection intervals of length τ. It is assumed that the lifetime of a testing unit follows a Weibull distribution. The problem of choosing the optimal τ is considered according to a certain optimality criterion. Two selection criteria that enable us to obtain the optimum test plans are investigated. Monte Carlo simulations are presented to illustrate the proposed methods.

Keywords: Design; Inspection; Partially accelerated step-stress test; Reliability analysis; Type I progressive censoring; Weibull distribution.

Subject Classifications: 62N01; 62N05.

1. INTRODUCTION

The purpose of inspecting materials, goods, or semifinished goods is to accept or reject a batch of products. Practically, an important quality characteristic is the lifetime of a product. In many situations, standard or traditional life testing methods may require time-consuming and prohibitively expensive testing time to obtain enough failure data necessary to make the desired inference. In order to assure rapid failure and then to shorten the testing period, all or some of the test units may be subjected to stress conditions more severe than normal ones. Such accelerated life

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tests (ALTs) or partially accelerated life tests (PALTs) result in shorter lives than would be observed under normal operating conditions. In ALTs, the items are run only at accelerated conditions (stress), whereas in PALTs they are run at both use and accelerated conditions.

As indicated by Nelson (1990), stress can be applied in various ways; commonly used methods are step-stress and constant-stress. Under step-stress PALTs, a test item is first run at use condition and, if it does not fail for a specified time, then it is run at accelerated condition until failure occurs or the observation is censored. But the constant-stress PALTs run each item at either use condition or accelerated condition only; that is, each unit is run at a constant-stress level until the test is terminated. Accelerated test stresses involve higher than usual temperature, voltage, pressure, load, humidity, etc., or some combination of these. The objective of PALTs is to collect more failure data in a limited time without necessarily using high stresses to all test units.

ALTs are often used for reliability analysis. According to step-stress ALTs scheme, a test unit is subjected to successively higher levels of stress. ALTs can be applied only if the relation that relates between the life and stress is known or can be assumed; if not, ALTs cannot be applied and PALTs are a good alternative method to use in reliability analysis via a tampered random variable model proposed by DeGroot and Goel (1979). This model is described as $Y = T$, if $T \leq \tau$; and $Y = \tau + \beta^{-1}(T - \tau)$, if $T > \tau$; where T is the lifetime of an item at use condition, Y is its total lifetime, and τ is the stress change time. The intent of such experiments is to collect more failure data in a limited time without necessarily using a high stress to all test units. As Bhattacharyya and Soejoeti (1989) indicated, step-stress PALTs are practical for many problems of life testing where the test process requires a long time if the test is simply carried out under the use condition.

In practice, step-stress PALTs are easier to implement and have many advantages, including

1. Time saving: Step-stress PALTs can substantially shorten the duration of the test without affecting the accuracy of lifetime distribution estimates.
2. Economical: Testing units under step-stress PALTs can reduce the costs of experiments because not all test units are run at higher stresses.
3. Adaptable: Step-stress PALTs are flexible test strategy, especially for new products when one presumably has little information regarding appropriate test stresses. In such situations, it may not be easy for the experimenter to determine suitable test stress levels. In simple PALTs, the second stress level, as well as the transition time, could be dynamically adjusted as failure information is being gathered under the first stress level.

To save more time and cost, ALTs or PALTs are used under censored sampling. Censoring is very common in life tests. According to Wu et al. (2006), censoring schemes arises in a life test whenever the experimenter does not observe the lifetimes of all test units. The traditional and most common censoring schemes are type I censoring and type II censoring. They do not allow for units to be removed from the test at any point other than the final termination point. However, this allowance may be needed when a compromise between reduced time of experiment and the observation of some extreme lifetimes is sought. These reasons lead us into the area of progressive censoring (PC). PC is a method that enables an efficient exploitation of the available resources by continual removal of a prespecified number of unfailed test units at the end of testing time at each stage.

The primary focus of this article is to combine PALTs with PC and then to concentrate on the optimal choice of change points of the stress levels. Note, however, that we will consider the equal-spaced case with a single τ denoting the duration of the testing stage at each step. That is, the lengths of inspection intervals in k stages are all equal. This type of constant interval inspection plan is usually used for administrative convenience in practice.

In practice, it is often impossible to continuously observe the testing process. We can only record whether a test unit fails in an interval instead of measuring failure time exactly. Hence, the test units are inspected intermittently. Recently, progressive type I interval censoring has received the attention of many authors. Some important literature can be found, for example see (Aggarwala, 2001; Ding et al., 2010; Gouno et al., 2004; Huang and Wu, 2008; Shen et al., 2011; Wu et al., 2006, 2008; Xiang and Tse, 2005; Yang and Tse, 2005).

All of the past works on step-stress PALTs (SSPALTs) had been considered under traditional type I and type II censoring, type II progressive censoring, and hybrid censoring; for example, see Aly and Ismail (2008) and Ismail (2010, 2012a,b, 2013). The present work will concentrate on a censoring scheme more general than traditional type I censoring, namely, type I progressive interval censoring. The idea of planning SSPALTs under progressive type I interval censoring scheme is a new one.

The main objective of this article is to explore the choice of length of the inspection interval τ based on results of samples from Weibull distribution. We investigate the selection of τ according to two competing criteria of optimality: variance (Var) optimality and determinant (D) optimality. The rest of this article is organized as follows: In Section 2 the experiment method and its assumptions used throughout the article are presented. In Section 3 the model is described. Point and interval estimations of the parameters are considered in Section 4. Section 5 presents optimum SSPALTs plans under progressive type I interval censoring scheme. Section 6 contains the main findings and presents a discussion via simulation studies to illustrate the theoretical results. Finally, Section 7 is devoted to the concluding remarks and future works needed in this direction.

2. THE EXPERIMENT METHOD

Let us consider the following life-testing scheme with type I progressive interval censoring. Suppose that n identical and independent units are simultaneously placed on a life test at time 0 under design stress or use condition x_0 and run until time τ, at which point the number of failed units n_1 are counted, r_1 surviving units are arbitrarily withdrawn from the test, and the stress is changed to a higher level of stress x_1. The test is continued on $n - n_1 - r_1$ units until time 2τ, at which point the stress is changed to x_2, and r_2 units are withdrawn from the test, and so on. At time $k\tau$, the surviving $r_k = n - \sum_{i=1}^{k} n_i - \sum_{j=1}^{k-1} r_j$ units are withdrawn, thereby terminating the test. The objective here is to choose the optimal length of τ according to a certain optimality criterion.

3. THE MODEL

This article is concerned with two-parameter Weibull distribution, which is widely employed as a model in life testing because of the many shapes it attains for various values of shape parameter. It can therefore model a great variety of data and life characteristics (Dimitri, 1991). The probability density function (p.d.f.) of a two-parameter Weibull distribution is given by

$$f_Y(y;\alpha,\theta) = \frac{\alpha}{\theta}\left(\frac{y}{\theta}\right)^{\alpha-1}\exp\{-(y/\theta)^\alpha\};\ y>0,\alpha>0,\theta>0. \tag{3.1}$$

The Weibull reliability function takes the form

$$S(y) = \exp\{-(y/\theta)^\alpha\}. \tag{3.2}$$

The corresponding failure rate function is given by

$$h(y) = \frac{\alpha}{\theta}\left(\frac{y}{\theta}\right)^{\alpha-1} \tag{3.3}$$

Therefore, using k-level step-stress partially accelerated tests under type I progressive interval censoring with equal inspection intervals of length τ, the cumulative distribution function is given by

$$F(y) = \begin{cases} 1-\exp\{-(y/\theta)^\alpha\}, & \text{if } 0<y\le\tau,\\ 1-\exp\{-([\tau+\beta_1(y-\tau)]/\theta)^\alpha\}, & \text{if } \tau<\mathrm{y}\le 2\tau,\\ . & \\ . & \\ . & \\ 1-\exp\{-([(k-1)\tau+\beta_{k-1}(y-(k-1)\tau)]/\theta)^\alpha\}, & \text{if } (k-1)\tau<\mathrm{y}<\infty, \end{cases} \tag{3.4}$$

The matching p.d.f. of Y is given by

$$f(y) = \begin{cases} \frac{\alpha}{\theta}\left(\frac{y}{\theta}\right)^{\alpha-1}\exp\{-(y/\theta)^\alpha\}, & \text{if } 0<\mathrm{y}\le\tau,\\ \beta_1\frac{\alpha}{\theta}\left(\frac{[\tau+\beta_1(y-\tau)]}{\theta}\right)^{\alpha-1}\exp\{-([\tau+\beta_1(y-\tau)]/\theta)^\alpha\}, & \text{if } \tau<\mathrm{y}\le 2\tau,\\ . & \\ . & \\ . & \\ \beta_{k-1}\frac{\alpha}{\theta}\left(\frac{[(k-1)\tau+\beta_{k-1}(y-(k-1)\tau)]}{\theta}\right)^{\alpha-1} & \\ \times\exp\{-([(k-1)\tau+\beta_{k-1}(y-(k-1)\tau)]/\theta)^\alpha\}, & \text{if } (k-1)\tau<\mathrm{y}<\infty. \end{cases} \tag{3.5}$$

4. PARAMETER ESTIMATION

The idea behind maximum likelihood parameter estimation is to determine the parameters that maximize the probability (likelihood) of the sample data. From a statistical point of view, the method of maximum likelihood is considered to be more robust and yields estimators with good statistical properties. In other words, maximum likelihood methods are versatile and apply to most models and to different types of data. In addition, they provide efficient methods for quantifying uncertainty through confidence bounds. Because these estimators do not always exist in closed form, numerical techniques are used to compute them.

This section discusses the process of obtaining the maximum likelihood estimates of the parameters α, θ, and β based on progressively type I interval censored data. Both point and interval estimations of the parameters are derived.

4.1. Point Estimation

Let $n_1, n_2, \ldots, n_k$ be a progressively type I group-censored sample with censoring scheme $(r_1, r_2, \ldots, r_k)$ from a k-stage SSPALT. That is, the number of failed units n_i are observed while testing in the interval $((i-1)\tau, i\tau]$ at stress x_i, $i = 1, 2, \ldots, k$. Thus, $n_i, n_{i-1}, \ldots, n_1 \sim$ binomial $(m_i, F_i(\tau))$, $i = 1, 2, \ldots, k$, where $m_i = n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i-1} r_j$ is the number of nonremoved surviving units at the beginning of the ith stage and

$$F_i(\tau) = \frac{F(i\tau) - F((i-1)\tau)}{1 - F((i-1)\tau)}.$$

The likelihood function is then given by

$$\prod_{i=1}^{k} [F(i\tau) - F((i-1)\tau)]^{n_i} [1 - F(i\tau)]^{r_i}.$$

The natural logarithm of the likelihood function can be written as

$$\ln L \propto \sum_{i=1}^{k} \{n_i \ln\,[\exp\{-(\psi_{i-1}/\theta)^{\alpha}\} - \exp\{-(\psi_i/\theta)^{\alpha}\}] - (m_i - n_i)(\psi_i/\theta)^{\alpha}\}, \tag{4.1}$$

where $\psi_{i-1} = (i-1)\tau + \beta_{i-1}(y - (i-1)\tau)$ and $\psi_i = i\tau + \beta_i(y - i\tau)$.

The maximum likelihood estimations (MLEs) of α, θ, and β can be found by solving the following equations:

$$\begin{aligned}\frac{\partial lnL}{\partial \alpha} = \sum_{i=1}^{k} \Bigg\{ &\frac{n_i}{\omega_{i-1} - \omega_i} [-(\psi_{i-1}/\theta)^{\alpha} \ln\,(\psi_{i-1}/\theta)\omega_{i-1} + (\psi_i/\theta)^{\alpha} \ln\,(\psi_i/\theta)\omega_i] \\ &- (m_i - n_i)(\psi_i/\theta)^{\alpha} \ln\,(\psi_i/\theta) \Bigg\} = 0,\end{aligned} \tag{4.2}$$

where $\omega_{i-1} = \exp\{-(\psi_{i-1}/\theta)^{\alpha}\} and \omega_i = \exp\{-(\psi_i/\theta)^{\alpha}\}$.

$$\frac{\partial lnL}{\partial \theta} = \frac{\alpha}{\theta^{\alpha+1}} \sum_{i=1}^{k} \left\{ \frac{n_i}{\omega_{i-1} - \omega_i} [\psi_{i-1}^{\alpha}\omega_{i-1} - \psi_i^{\alpha}\omega_i] + (m_i - n_i)\psi_i^{\alpha} \right\} = 0, \tag{4.3}$$

$$\frac{\partial lnL}{\partial \beta_i} = \frac{\alpha}{\theta^\alpha}\sum_{i=1}^{k}\left\{\frac{n_i}{\omega_{i-1}-\omega_i}[\omega_i\psi_i^{\alpha-1}(y-i\tau)-\omega_{i-1}\psi_{i-1}^{\alpha-1}(y-(i-1)\tau)]\right.$$
$$\left. -(m_i-n_i)\psi_i^{\alpha-1}(y-i\tau)\right\}=0. \quad (4.4)$$

Now, we have a system of nonlinear equations. It is clear that a closed-form solution is very difficult to obtain. Therefore, an iterative procedure should be used to find a numerical solution of the above system.

4.2. Interval Estimation

Here, the approximate confidence intervals of the parameters are derived based on the asymptotic distributions of the MLEs of the elements of the vector of unknown parameters $\Omega = (\alpha, \theta, \underline{\beta})$. It is known that the asymptotic distribution of the MLEs of Ω is given by (see Miller, 1981)

$$((\hat{\alpha}-\alpha), (\hat{\theta}-\theta), (\underline{\hat{\beta}}-\underline{\beta})) \rightarrow N(0, \mathbf{F}^{-1}(\alpha, \theta, \underline{\beta})), \quad (4.5)$$

where $\mathbf{F}^{-1}(\alpha, \theta, \underline{\beta})$ is the variance–covariance matrix of the unknown parameters $\Omega = (\alpha, \theta, \underline{\beta})$. The elements of the matrix $\mathbf{F}^{-1}$; $F_{ij}(\Omega)$, i, $j = 1, 2, \ldots, k$; can be approximated by $\mathbf{F}_{ij}(\hat{\Omega})$, where

$$\mathbf{F}_{ij}(\hat{\Omega}) = -\frac{\partial^2 \ln L(\Omega)}{\partial\Omega_i\partial\Omega_j}\mid_{\Omega=\hat{\Omega}}. \quad (4.6)$$

Now, the elements of the observed information matrix F can be expressed as follows:

$$\frac{\partial^2 \ln L}{\partial\alpha^2} = \sum_{i=1}^{k}\left\{\frac{n_i}{(\omega_{i-1}-\omega_i)^2}[-(\psi_{i-1}/\theta)^\alpha(\ln(\psi_{i-1}/\theta)^2\omega_{i-1}+(\psi_{i-1}/\theta)^{2\alpha}(\ln(\psi_{i-1}/\theta))^2\omega_{i-1}\right.$$
$$+(\psi_i/\theta)^\alpha(\ln(\psi_i/\theta))^2\omega_i-(\psi_i/\theta)^{2\alpha}(\ln(\psi_i/\theta))^2\omega_i](\omega_{i-1}-\omega_i)$$
$$-[-(\psi_{i-1}/\theta)^\alpha\ln(\psi_{i-1}/\theta)\omega_{i-1}+(\psi_i/\theta)^\alpha\ln(\psi_i/\theta)\omega_i]$$
$$\times[-(\psi_{i-1}/\theta)^\alpha\ln(\psi_{i-1}/\theta)\omega_{i-1}$$
$$\left.+(\psi_i/\theta)^\alpha\ln(\psi_i/\theta)\omega_i]-(m_i-n_i)(\psi_i/\theta)^\alpha(\ln(\psi_i/\theta))^2\right\}, \quad (4.7)$$

$$\frac{\partial^2 \ln L}{\partial\theta^2} = \frac{\alpha}{\theta^{2(\alpha+1)}}\left[\alpha\sum_{i=1}^{k}\left\{\frac{n_i}{(\omega_{i-1}-\omega_i)^2}\{(\omega_{i-1}-\omega_i)[\psi_{i-1}^{2\alpha}\omega_{i-1}-\psi_i^{2\alpha}\omega_i]\right.\right.$$
$$\left.-(\psi_{i-1}^\alpha\omega_{i-1}-\psi_i^\alpha\omega_i)^2\}\right\}$$
$$\left.-(\alpha+1)\theta^\alpha\sum_{i=1}^{k}\left\{\frac{n_i}{\omega_{i-1}-\omega_i}[\psi_{i-1}^\alpha\omega_{i-1}-\psi_i^\alpha\omega_i]+(m_i-n_i)\psi_i^\alpha\right\}\right], \quad (4.8)$$

$$\frac{\partial^2 \ln L}{\partial \beta_i^2} = \frac{\alpha}{\theta^\alpha} \sum_{i=1}^{k} \Bigg\{ \frac{n_i}{(\omega_{i-1} - \omega_i)^2} [(\omega_{i-1} - \omega_i)\Bigg[- \frac{\alpha}{\theta^\alpha} \omega_i \psi_i^{2(\alpha-1)} (y - i\tau)$$
$$+ (\alpha - 1)\omega_i \psi_i^{\alpha-2} (y - i\tau)$$
$$+ \frac{\alpha}{\theta^\alpha} \omega_{i-1} \psi_{i-1}^{2(\alpha-1)} (y - (i-1)\tau) - (\alpha - 1)\omega_{i-1} \psi_{i-1}^{\alpha-2} (y - (i-1)\tau)\Bigg]$$
$$+ [\omega_i \psi_i^{\alpha-1} (y - i\tau) - \omega_{i-1} \psi_{i-1}^{\alpha-1} (y - (i-1)\tau)]$$
$$\times \left[\frac{\alpha}{\theta^\alpha} \omega_{i-1} \psi_{i-1}^{\alpha-1} (y - (i-1)\tau) + \frac{\alpha}{\theta^\alpha} \omega_i \psi_i^{\alpha-1} (y - i\tau) \right]\Bigg]$$
$$- (m_i - n_i)(\alpha - 1)\psi_i^{\alpha-2} (y - i\tau)^2 \Bigg\}, \tag{4.9}$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} = \sum_{i=1}^{k} \Bigg\{ \frac{n_i}{(\omega_{i-1} - \omega_i)^2} \Bigg\{ \Bigg[\frac{\alpha \psi_{i-1}^{\alpha}}{\theta^{\alpha+1}} \ln\ (\psi_{i-1}/\theta)\omega_{i-1}$$
$$+ (\psi_{i-1}/\theta)^\alpha \omega_{i-1} \left(\frac{1}{\theta^2} - \frac{\alpha ln(\psi_{i-1}/\theta)\psi_{i-1}^\alpha}{\theta^{\alpha+1}} \right)$$
$$+ \frac{\alpha \psi_i^\alpha}{\theta^{\alpha+1}} \ln\ (\psi_i/\theta)\omega_i + (\psi_i/\theta)^\alpha \omega_i \left(\frac{1}{\theta^2} - \frac{\alpha \ln(\psi_i/\theta)\psi_i^\alpha}{\theta^{\alpha+1}} \right) \Bigg] (\omega_{i-1} - \omega_i)$$
$$- \frac{\alpha}{\theta^{\alpha+1}} [-(\psi_{i-1}/\theta)^\alpha \ln\ (\psi_{i-1}/\theta)\omega_{i-1} + (\psi_i/\theta)^\alpha ln(\psi_i/\theta)\omega_i][\omega_{i-1}\psi_{i-1}^\alpha - \omega_i \psi_i^\alpha] \Bigg\}$$
$$- \frac{(m_i - n_i)\psi_i^\alpha}{\theta^{\alpha+1}} [\alpha \ln\ (\psi_i/\theta) + 1] \Bigg\}, \tag{4.10}$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta_i} = \sum_{i=1}^{k} \frac{n_i}{(\omega_{i-1} - \omega_i)} \Bigg\{ \frac{-\omega_{i-1}(y - (i-1)\ \tau)(\psi_{i-1}/\theta)^{\alpha-1}}{\theta} [\alpha \ln\ (\psi_{i-1}/\theta) + 1]$$
$$+ \frac{\omega_{i-1}\alpha(y - (i-1)\tau)}{\theta} (\psi_{i-1}/\theta)^{2\alpha-1} \ln\ (\psi_{i-1}/\theta)$$
$$+ \frac{\omega_i (y - i\tau)(\psi_i/\theta)^{\alpha-1}}{\theta} [\alpha \ln\ (\psi_i/\theta) + 1]$$
$$- \frac{\omega_i \alpha(y - i\tau)}{\theta} (\psi_i/\theta)^{2\alpha-1} \ln\ (\psi_i/\theta) \Bigg\}$$
$$+ \frac{\alpha n_i [-(\psi_{i-1}/\theta)^\alpha \ln\ (\psi_{i-1}/\theta)\omega_{i-1} + (\psi_i/\theta)^\alpha \ln\ (\psi_i/\theta)\omega_i]}{\theta(\omega_{i-1} - \omega_i)^2}$$
$$\times [\omega_{i-1}(y - (i-1)\tau)(\psi_{i-1}/\theta)^{\alpha-1} + \omega_i (y - i\tau)(\psi_i/\theta)^{\alpha-1}]$$
$$- (m_i - n_i) \frac{(\psi_i/\theta)^{\alpha-1}(y - i\tau)}{\theta} [\alpha \ln\ (\psi_i/\theta) + 1], \tag{4.11}$$

and

$$\frac{\partial^2 \ln L}{\partial\theta\partial\beta_i} = \frac{\alpha}{\theta^{2\alpha}}\Bigg[\Bigg\{\sum_{i=1}^{k}\Bigg\{\frac{n_i}{(\omega_{i-1}-\omega_i)^2}\Bigg[\frac{\alpha(\omega_{i-1}-\omega_i)}{\theta^{\alpha+1}}[\omega_i\psi_i^{2\alpha-1}(y-i\tau)$$
$$-\omega_{i-1}\psi_{i-1}^{2\alpha-1}(y-(i-1)\tau)]$$
$$-[\omega_i\psi_i^{\alpha-1}(y-i\tau)-\omega_{i-1}\psi_{i-1}^{\alpha-1}(y-(i-1)\tau)]\frac{\alpha}{\theta^{\alpha+1}}[\omega_i\psi_i^{\alpha}-\omega_{i-1}\psi_{i-1}^{\alpha}]\Bigg]\Bigg\}\theta^{\alpha}$$
$$-\Bigg[\sum_{i=1}^{k}\Bigg\{\frac{n_i}{\omega_{i-1}-\omega_i}[\omega_i\psi_i^{\alpha-1}(y-i\tau)-\omega_{i-1}\psi_{i-1}^{\alpha-1}(y-(i-1)\tau)]$$
$$-(m_i-n_i)\psi_i^{\alpha-1}(y-i\tau)\Bigg\}\Bigg]\alpha\theta^{\alpha-1}\Bigg]. \tag{4.12}$$

Thus, the approximate $100(1-\gamma)\%$ two-sided confidence intervals for α, θ, and β are, respectively, given by

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{\mathbf{F}_{11}^{-1}(\hat{\alpha})}, \quad \hat{\theta} \pm Z_{\gamma/2}\sqrt{\mathbf{F}_{22}^{-1}(\hat{\theta})} \quad \text{and} \quad \hat{\beta}_i \pm Z_{\gamma/2}\sqrt{\mathbf{F}_{33}^{-1}(\hat{\beta}_i)}, \quad i = 1, 2, \ldots, k, \tag{4.13}$$

where $Z_{\gamma/2}$ is the upper $(\gamma/2)$th percentile of a standard normal distribution.

5. OPTIMUM TEST PLAN

The main purpose of this study is to explore the choice of τ, length of the inspection interval, in k-stage SSPALTs with type I progressive interval censoring. Two selection criteria are proposed that enable one to choose the optimal value of τ.

5.1. Var-Optimality

The mean lifetime is an important characteristic in reliability analysis. In a step-stress setting, the experimenter is often interested in estimating the mean life at use condition with maximum precision. Let $\hat{\bar{T}}$ be the MLE of mean lifetime at use condition. Then, the criterion function is defined by

$$\phi(\tau) = A\mathrm{Var}\,(\ln \hat{\bar{T}}) = n[1, 1, x_0]F^{-1}(\alpha, \theta, \beta)[1, 1, x_0]', \tag{5.1}$$

where x_0 is the design stress and $\bar{T}$ is the mean time to failure of the Weibull distribution. The variance-optimal τ is then obtained by minimizing $\phi(\tau)$.

5.2. D-Optimality

Yet another optimality criterion is based on the determinant of Fisher's information matrix F. It has been extensively used in the context of planning life tests. If one is more interested in estimation with high precision, a more reasonable criterion should be D-optimality, which takes into account the overall parameter space. It can be constructed in terms of the generalized asymptotic variance (GAV) of the MLEs of the model parameters. This GAV is proportional to reciprocal of the determinant of Fisher information matrix; see Bai et al. (1993), so that maximizing this determinant is equivalent to minimizing GAV. The criterion function is then defined by

$$GAV(\hat{\alpha}, \hat{\theta}, \underline{\hat{\beta}}) = \frac{1}{|\mathbf{F}|}. \tag{5.2}$$

Hence, the optimal length of inspection interval τ is chosen so that GAV is minimized.

It is noted that both variance-optimality and D-optimality criteria are based on information matrix F. These criteria have been extensively used in the design selection process for designed experiments.

6. SIMULATION RESULTS AND DISCUSSION

In this section, to explore the optimal choice of τ, length of the inspection interval, in k-stage SSPALTs with type I progressive interval censoring, simulation studies are performed using different values of sample size and number of stress levels. The optimal value of τ according to two optimality criteria, variance-optimality and D-optimality, is investigated. Let τ_V^* and τ_D^* be optimal lengths of inspection intervals according to variance-optimality and D-optimality, respectively. Tables 1, 2, and 3 present τ_V^* and τ_D^* values for $k = 2, 3, 4$ when n equals 30, 50, 100, 200, 300, 400, 500 and π equals 0.05, 0.10, 0.15.

It is assumed that the proportions to be removed at different stages are all equal. That is, $\pi_1 = \pi_2 = \cdots = \pi_k = \pi$. We assume that the lengths of inspection intervals are all equal for simplicity of discussion. The equilength assumption is also convenient for practitioners. The proposed optimality criteria can lead to better designs for conducting life tests. It provides the most efficient use of experimenter's resources.

From Tables 1–3, we can record the following findings:

1. For fixed π and n, both τ_V^* and τ_D^* decrease as k increases. That is, the larger number of stress levels, the more likely a short length of inspection interval.
2. For fixed π and k both τ_V^* and τ_D^* increase as n increases. That is, the larger number of test units n, the larger the optimal length of the inspection interval.
3. When k and n are fixed, both τ_V^* and τ_D^* decrease as π increases. That is, the larger the proportion to be removed at each stage, the shorter the optimal length of the inspection interval.
4. The D-optimal length of inspection interval τ_D^* is always smaller than the variance-optimal length of inspection interval τ_V^*.
5. It is shown that the second proposed criterion (D-optimality) can reduce the required number of failures and, consequently, reduce the total testing time without losing much precision.
6. When π increases, the experiment is terminated more quickly. However, it is important to note that with a much larger π the experiments will be less informative and lead to larger standard errors in estimates. These results coincide

with the note of Wu et al. (2006). That is, smaller or moderate, not larger, values of the proportion to be removed at each stage are recommended.

The design of an optimal life test already enables us to obtain estimations with a high degree of precision. This coincides with the note of Wu and Huang (2010). They said that in order to obtain a precise estimate of mean life, one needs to design an optimal life test.

Table 1. Optimal lengths τ_V^* and τ_D^* according to Var-optimality and D-optimality under k-stage SSPALTs and progressive type I interval censoring with proportion of removals $\pi = 0.05$ when $\underline{\beta} = 2; 2.5; 3$, $\theta = 2.5$, and $\alpha = 0.5$.

	K = 2		K = 3		K = 4	
n	τ_D^*	τ_V^*	τ_D^*	τ_V^*	τ_D^*	τ_V^*
30	24	27	21	25	19	22
50	43	48	37	42	33	38
100	87	93	64	72	37	41
200	169	182	127	154	71	83
300	224	253	146	159	83	103
400	320	341	210	246	113	123
500	411	449	268	308	132	162

Table 2. Optimal lengths τ_V^* and τ_D^* according to Var-optimality and D-optimality under k-stage SSPALTs and progressive type I interval censoring with proportion of removals $\pi = 0.10$ when $\underline{\beta} = 2; 2.5; 3$, $\theta = 2.5$, and $\alpha = 1$.

	$K = 2$		$K = 3$		$K = 4$	
n	τ_D^*	τ_V^*	τ_D^*	τ_V^*	τ_D^*	τ_V^*
30	20	23	18	21	15	17
50	39	42	33	36	21	26
100	64	71	49	57	24	31
200	126	138	96	117	57	64
300	167	192	105	121	61	76
400	231	256	162	186	85	99
500	312	338	194	238	104	123

Table 3. Optimal lengths τ_V^* and τ_D^* according to Var-optimality and D-optimality under k-stage SSPALTs and progressive type I interval censoring with proportion of removals $\pi = 0.15$ when $\underline{\beta} = 2; 2.5; 3$, $\theta = 2.5$, and $\alpha = 1.5$.

	$K = 2$		$K = 3$		$K = 4$	
n	τ_D^*	τ_V^*	τ_D^*	τ_V^*	τ_D^*	τ_V^*
30	17	21	14	19	8	10
50	22	29	18	20	11	14
100	24	32	20	22	13	17
200	52	65	37	45	22	25
300	68	86	41	48	23	30
400	94	116	64	74	32	38
500	124	156	77	97	40	47

7. CONCLUSION

In reliability analysis of progressively interval censored life test data, for given stress levels and number of test units, determining the appropriate length of the inspection interval is an important issue for experimenters. Two optimality criteria—D-optimality and variance-optimality—for choosing the optimal length of the inspection interval are used for comparison purpose. This article has discussed a union of three ideas of progressive censoring, interval data, and SSPALTs, developing k-stage SSPALTs with progressively type I interval censored samples. In the case of certain life tests, some test units need to be removed at points other than the final termination point of the experiment and it is not practical to screen the test units constantly. Here, the progressive interval-censoring scheme permits units removed early and inspected from time to time.

Based on the fourth finding, the optimal length of the inspection interval is shorter in the case of the D-optimality criterion. Therefore, the D-optimality criterion is recommended for obtaining the optimal life test plan. In conclusion, these results provide valuable insight for practitioners to set up optimal life test plans under progressive type I interval censoring. As a future work, designing k-stage SSPALTs with progressively type I interval-censored samples can be extended to deal with unequal proportions to be removed at different stages and to optimize the sample size.

NOMENCLATURE

k	number of inspections
m_i	number of nonremoved surviving units at the beginning of the ith stage
n	number of test units
n_i	number of failures at the ith stage
r_i	number of removals at the ith stage
x_i	stress level, $i = 1, 2, \ldots, k$; $x_1 < x_2 < \cdots < x_k$
x_0	design stress
y_i	observed lifetime of unit i tested under SSPALTs
α	Weibull distribution shape parameter ($\alpha > 0$)
$\underline{\beta} \equiv \beta_i$	acceleration factors ($\beta_i > 1$, $i = 1, \ldots, k$)
δ_{1i}, δ_{2i}	indicator functions: $\delta_{1i} \equiv I(y_i \leq \tau)$, $\delta_{2i} \equiv I(y_i > \tau)$
θ	Weibull distribution scale parameter ($\theta > 0$)
π_i	proportion of removals at the ith stage
τ	length of the inspection interval
τ_D^*	optimal τ according to the D-optimality criterion
τ_V^*	optimal τ according to the Var-optimality criterion
$\wedge$	denotes maximum likelihood estimate

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REFERENCES

Aggarwala, R. (2001). Progressive Interval Censoring: Some Mathematical Results with Applications to Inference, *Communication in Statistics - Theory and Methods* 30: 1921–1935.

Aly, H. M. and Ismail, A. A. (2008). Optimum Simple Time-Step Stress Plans for Partially Accelerated Life Testing with Censoring, *Far East Journal of Theoretical Statistics* 24: 175–200.

Bai, D. S., Kim, J. G., and Chun, Y. R. (1993). Design of Failure-Censored Accelerated Life-Test Sampling Plans for Lognormal and Weibull Distributions, *Engineering Optimality* 21: 197–212.

Bhattacharyya, G. K. and Soejoeti, Z. (1989). A Tampered Failure Rate Model for Step Stress Accelerated Life Test, *Communication in Statistics - Theory & Methods* 18: 1627–1643.

DeGroot, M. H. and Goel, P. K. (1979). Bayesian Estimation and Optimal Designs in Partially Accelerated Life Testing, *Naval Research Logistics Quarterly* 16: 223–235.

Dimitri, K. (1991). *Reliability Engineering Handbook*, vol. 1, Englewood Cliffs: Prentice Hall.

Ding, C., Yang, C., and Tse, S.-K. (2010). Accelerated Life Test Sampling Plans for the Weibull Distribution Under Type I Progressive Interval Censoring with Random Removals, *Journal of Statistical Computation and Simulation* 80: 903–914.

Gouno, E., Sen, A., and Balakrishnan, N. (2004). Optimal Step-Stress Test Under Progressive Type I Censoring, *IEEE Transactions on Reliability* 53: 388–393.

Huang, S.-R. and Wu, S.-J. (2008). Reliability Sampling Plans Under Progressive Type I Interval Censoring Using Cost Functions, *IEEE Transactions on Reliability* 57: 445–451.

Ismail, A. A. (2010). Bayes Estimation of Gompertz Distribution Parameters and Acceleration Factor Under Partially Accelerated Life Tests with Type-I Censoring, *Journal of Statistical Computation and Simulation* 80: 1253–1264.

Ismail, A. A. (2012a). Estimating the Parameters of Weibull Distribution and the Acceleration Factor from Hybrid Partially Accelerated Life Test, *Applied Mathematical Modeling* 36: 2920–2925.

Ismail, A. A. (2012b). Inference in the Generalized Exponential Distribution Under Partially Accelerated Tests with Progressive Type-II Censoring, *Theoretical and Applied Fracture Mechanics* 59: 49–56.

Ismail, A. A. (2013). On Designing Step-Stress Partially Accelerated Life Tests Under Failure-Censoring Scheme, *Proceedings of the Institution of Mechanical Engineers - Part O: Journal of Risk and Reliability* 227: 662–670.

Miller, R. G. (1981). *Survival Analysis*, New York: Wiley.

Nelson, W. (1990). *Accelerated Testing: Statistical Models, Data Analysis and Test Plans*, New York: Wiley.

Shen, K.-F., Shen, Y.-J., and Leu, L.-Y. (2011). Design of Optimal Step Stress Accelerated Life Tests Under Progressive Type I Censoring with Random Removals, *Quality and Quantity* 45: 587–597.

Wu, S.-J. and Huang, S.-R. (2010). Optimal Progressive Group-Censoring Plans for Exponential Distribution in Presence of Cost Constraint, *Statistical Papers* 51: 431–443.

Wu, S.-J., Lin, Y.-P., and Chen, Y.-J. (2006). Planning Step-Stress Life Test with Progressively Type I Group-Censored Exponential Data, *Statistica Neerlandica* 60: 46–56.

Wu, S.-J., Lin, Y.-P., and Chen, Y.-J. (2008). Optimal Step-Stress Test Under Type I Progressive Group-Censoring with Random Removals, *Journal of Statistical Planning and Inference* 138: 817–826.

Xiang, L. and Tse, S.-K. (2005). Maximum Likelihood Estimation in Survival Studies Under Progressive Interval Censoring with Random Removals, *Journal of Biopharmacy and Statistics* 15: 981–991.

Yang, C. and Tse, S.-K. (2005). Planning Accelerated Life Tests Under Progressive Type I Interval Censoring with Random Removals, *Communication in Statistics - Simulations and Computation* 34: 1001–1025.