On Designing Time-Censored Step-Stress Life Test for Lomax Distribution
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Reference

ABSTRACT
This article presents optimum time step-stress partially accelerated life tests (SSPALTs) where a pre-specified censoring time is considered. The time to failure is assumed to have Lomax distribution. Maximum likelihood estimates (MLEs) of SSPALT model parameters are obtained. Moreover, a confidence intervals estimation for the parameters with associated coverage probabilities is obtained. In addition, optimum test plans for SSPALT are also developed. Such test plans minimize the generalized asymptotic variance (GAV) of the MLEs of the model parameters. To demonstrate the theoretical results, Monte Carlo simulations are introduced and a real life example is provided.

Keywords
reliability, Lomax distribution, step-stress partially accelerated life tests, maximum likelihood estimation, confidence intervals, coverage probabilities, optimum test plans, Type-I censoring, Monte Carlo simulation

Nomenclature
ALTs =accelerated life tests
\( f(t) \) =probability density function
\( H(t) \) =hazard (failure) rate at time \( t \) at use condition
MLEs =maximum likelihood estimations/estimators
MSE =mean square error
\( n \) =total number of test items in SSPALTs
\( n_u, n_a \) =numbers of items failed at use and accelerated conditions, respectively
\( P_u, P_a \) =probability that an item fails at accelerated
condition
PALTs =partially accelerated life tests

Introduction

For highly reliable materials or products, it is more difficult to acquire failure information quickly for products tested under normal (use) condition. In order to reduce the testing period, all or some of test units may be subjected to more severe conditions than normal ones. Such accelerated life tests (ALTs) or partially accelerated life tests (PALTs) result in shorter lives than would be observed under normal operating conditions. In ALTs, test items are run only at accelerated conditions, while in PALTs, they are run at both use and accelerated conditions.

The main assumption in ALTs is that the model or the relationship between life and stress must be known or can be assumed to obtain estimates of lifetime at design stress. If such a relationship is unknown or cannot be assumed, one cannot apply the ALTs approach. In this case, the PALTs come to be an alternative approach to study and analyze the reliability.

According to Nelson [1], the stress can be applied in various ways, commonly used method is step-stress. The step-stress scheme allows the stress setting of a unit to be changed at pre-specified times (time step) or upon the occurrence of a fixed number of failures (failure-step). In this article, we consider simple time SSPALTs (i.e., only two levels of stress) in which a test item is first run at use condition and, if it does not fail for a specified time $\tau$, then it is run at an accelerated condition until the test is terminated. The purpose of such an experiment is to collect more failure data in a limited time without necessarily using a high stress to all test units. As Bhattacharyya and Soejoeti [2] indicated, SSPALTs are practical for many problems of life testing where the test process requires a long time if the test is simply carried out under the use condition.

Concerning a literature review on such SSPALTs. Goel [3] considered the estimation problem of the acceleration factor $\beta$, which is the ratio of the hazard rate at accelerated condition to that at use condition, using the maximum likelihood method for items having exponential distribution and uniform distribution in the case of complete sampling. He also obtained the optimal PALTs plan. DeGroot and Goel [4] considered a model of PALTs in which a test item is first run at use condition and if it does not fail for a specified time $\tau$, then it is run at accelerated condition until failure. That is, $Y = T$, if $T \leq \tau$; and $Y = \tau + \beta^{-1}(T - \tau)$, if $T > \tau$, where $T$ is the lifetime of an item at use condition and $Y$ is its total lifetime. Bhattacharyya and Soejoeti [2] proposed a failure rate model in which, $h(y) = h(\tau)$ if $y \leq \tau$ and $h(y) = \beta h(\tau)$ if $y > \tau$ where $h(\tau)$ and $h(\tau)$ are failure-rate functions of $T$ and $Y$, respectively. When $T$ follows Weibull distribution, Bhattacharyya and Soejoeti [2] obtained the estimates of the model parameters by the maximum likelihood method under complete sampling.

In addition, PALTs were studied with type-I censored data by few authors. Bai and Chung [5] used the maximum likelihood method to estimate the scale parameter and the acceleration factor for exponentially distributed lifetime in the case of type-I censoring. They also considered the problem of optimally designing for the step PALT that terminate at a predetermined time. Bai et al. [6] considered the same work of Bai and Chung [5] for items having lognormal distribution. Abdel-Ghaly et al. [7] considered the maximum likelihood method for estimating the acceleration factor and the parameters of Weibull distribution in the case of type-I censoring. Ismail and Aly [8] studied the optimal design problem of failure-step PALTs under Weibull distribution with Type-II censoring. In addition, Ismail [9] developed test plans of time step PALTs under Weibull distribution under failure-censoring scheme. Furthermore, Srivastava and Mittal [10] studied optimum step-stress PALTs under truncated logistic distribution using Type-I and Type-II censored data.

This article will concentrate on designing time step PALTs under Lomax distribution with a time-censoring scheme. The rest of this article is organized as follows: in the “Model, Test Procedure and Its Assumptions” Section, the test procedure and its necessary assumptions used throughout this article are presented. The maximum likelihood estimators, average confidence intervals lengths (IL) and associated coverage probabilities (CP) are considered in the “Estimation Process” Section. In the “Optimum Test Plans” Section, statistically optimum plans for time SSPALTs are developed. Monte Carlo simulation studies are given in the Section “Simulation Studies” to illustrate the theoretical results given in this article. In addition, for more illustration, an example based on a real data set is provided. The final section is devoted to the concluding remarks.

Model, Test Procedure and its Assumptions

LOMAX DISTRIBUTION: A LIFETIME MODEL

Lomax distribution was originally proposed as a second kind of Pareto distribution by Lomax [11]. It is used to provide a good
model in biomedical problems. Lomax distribution is considered as an important lifetime model. In addition, it has been used in relation with studies of income, size of cities, and reliability modeling. Lomax distribution is being widely used for stochastic modeling of decreasing failure rate life components. It also serves as a useful model in the study of queuing theory and biological analysis.

In this article, we consider $T$ a random variable denoting the lifetime of an item at use condition with Lomax density

$$f(t; \theta, x) = \frac{2\theta x}{(\theta + t)^{x+1}}; \quad t > 0, \theta > 0, x > 0 \quad (1)$$

The reliability function of Lomax distribution takes the form

$$R(t) = \frac{\theta^x}{(\theta + t)^x}; \quad t > 0, \theta > 0, x > 0 \quad (2)$$

The corresponding hazard rate is given by

$$h(t) = \frac{x}{\theta + t}; \quad t > 0, \theta, x > 0 \quad (3)$$

which is a decreasing function as $t > 0$, representing the early failure region or the initial failure region in the bathtub shape. These types of failures may be due to initial weakness or defects, weak parts, bad assembly, or poor fits, etc. (see Martz [12]). Therefore, Lomax distribution may be used as a reliability growth model as noted by Abdel-Ghaly et al. [13].

**TEST PROCEDURE**

1. Each of the $n$ test units is first run at use condition.
2. If it does not fail at use condition by a pre-specified time $\tau$, then it is put on accelerated condition and run until either it fails or the test is terminated.

**ASSUMPTIONS**

1. The lifetimes of the $n$ test units are independent and identically distributed random variables (i.i.d. r.v.’s).
2. The lifetimes of test units are assumed to follow Lomax distribution with probability density function (pdf) given as in Eq 1.

**Estimation Process**

The method of maximum likelihood (ML) has long been widely accepted as one of the most reliable methods of estimating distribution parameters. It has been commonly used in the analysis of accelerated life tests. Although the exact sampling distribution of maximum likelihood estimators (MLE) is sometimes unknown, MLE have the desirable properties of being consistent, asymptotically normal and asymptotically efficient for large samples. As indicated by Grimshaw [14], the ML method is commonly used for most theoretical models and kinds of censored data. Moreover, Bugaighis [15] pointed out that the maximum likelihood procedure generally yields efficient estimators. However, these estimators do not always exist in closed form. So, numerical techniques are used to compute them.

**THE MLES**

The lifetime of the test item is assumed to follow Lomax distribution with scale parameter $\theta$ and shape parameter $x$. Therefore, the probability density function of total lifetime $Y$ of an item in SSPALTs is given by

$$f(y) = \begin{cases} 0 & y \leq 0 \\ \frac{2\theta x}{(\theta + y)^{x+1}} & 0 < y \leq \tau \\ \frac{\beta x \theta^x}{(\theta + \tau + \beta(y - \tau))^{x+1}} & y > \tau \end{cases} \quad (4)$$

where $\theta > 0$ and $x > 0$.

The observed values of the total lifetime $Y$ are given by

$$Y(1) \leq \cdots \leq Y(n) \leq \tau \leq Y(1+n) \leq \cdots \leq Y(n+m+n) \leq \eta$$

Let $\delta_{i1}, \delta_{i2}$ be indicator functions such that $\delta_{i1} \equiv I(Y_i \leq \tau)$, and $\delta_{i2} \equiv I(\tau < Y_i \leq \eta)$, where $i = 1, \ldots, n$. Since the total lifetimes $Y_1, \ldots, Y_n$ of $n$ items are independent and identically distributed random variables, then the total likelihood function for them is given by

$$L(\beta, \theta, x, Y_1, \delta_{11}, \delta_{21}) = \prod_{i=1}^n \left[ \frac{2\theta x}{(\theta + y_i)^{x+1}} \right]^{\delta_{i1}} \cdot \left[ \frac{\beta x \theta^x}{(\theta + \tau + \beta(y_i - \tau))^{x+1}} \right]^{\delta_{i2}} \cdot \left[ \frac{\theta^x}{(\theta + \tau + \beta(\eta - \tau))^{x+1}} \right]$$

where

$$\delta_{1i} = 1 - \delta_{i1} \quad \text{and} \quad \delta_{2i} = 1 - \delta_{i2}$$

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. The natural logarithm of the likelihood function is given by

$$\ln L = (n + n_u) \ln x + nx \ln \theta + n_u \ln \beta$$

$$- (n - n_u + n_u) \ln \theta + \tau + \beta(\eta - \tau)$$

$$- (x + 1) \sum_{i=1}^n \delta_{i1} \ln(\theta + y_i)$$

$$+ \sum_{i=1}^n \delta_{i2} \ln(\theta + \tau + \beta(y_i - \tau)) \quad (6)$$

The first derivatives of the natural logarithm of the total likelihood function in Eq 6 with respect to $\beta$, $\theta$, and $x$ are given by
the simultaneous numerically solution of these nonlinear equations is highly unlikely, iterative estimation procedures must be used to solve these equations. The Newton–Raphson procedure is applied for the simultaneous numerical solution of these nonlinear equations. Thus, once the values of $\beta$ and $\theta$ are determined, an estimate of $\alpha$ is easily obtained from Eq 10. In relation to the asymptotic variance-covariance matrix of the ML estimators of the parameters, it can be approximated by numerically inverting the Fisher-information matrix composed of the negative second derivatives of the natural logarithm of the likelihood function evaluated at the ML estimates. The asymptotic Fisher-information matrix can be written as

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n_a}{\beta} \frac{(n - n_u - n_a) x(\eta - \tau)}{(\theta + \tau + \beta(\eta - \tau))}$$

- $\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{\partial^2 \ln L}{\partial \beta \partial \theta} - \frac{\partial^2 \ln L}{\partial \beta \partial \alpha}$
- $\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} = -\frac{\partial^2 \ln L}{\partial \alpha^2}$

From Eq 9, the maximum likelihood estimate of $\alpha$ is given by

$$\hat{\alpha} = \frac{(n_a + n_u)}{(n - n_u - n_a) \ln \{\theta + \tau + \beta(\eta - \tau)\} - n \ln \theta + Q}$$

where

$$Q = \sum_{i=1}^{n} \delta_{1i} \ln(\theta + y_i) + \sum_{i=1}^{n} \delta_{2i} \ln \{\theta + \tau + \beta(y_i - \tau)\}$$

Therefore, by substituting for $\alpha$ into Eqs 7 and 8 and equating each of the resulting equations to zero, they are reduced to the following two non-linear estimating equations.

$$\begin{align*}
\frac{n_a}{\beta} & \left[ (n - n_u - n_a) \ln \{\theta + \tau + \beta(\eta - \tau)\} - n \ln \theta + Q \right] \\
& = \left\{ (n - n_u - n_a) \ln \{\theta + \tau + \beta(\eta - \tau)\} - n \ln \theta + Q \right\} + 1 \left\{ \sum_{i=1}^{n} \delta_{1i} \right\} \left\{ \frac{\theta + \tau + \beta(y_i - \tau)}{\delta_{2i}} \right\} = 0
\end{align*}$$

and

$$\begin{align*}
n(n_u + n_a) & \frac{n_u + n_a}{\theta(n - n_u - n_a) \ln \{\theta + \tau + \beta(\eta - \tau)\} - n \ln \theta + Q} \\
& = \left\{ (n - n_u - n_a) \ln \{\theta + \tau + \beta(\eta - \tau)\} - n \ln \theta + Q \right\} + 1 \left\{ \frac{n_u}{\theta + \tau + \beta(y_i - \tau)} \right\} + \sum_{i=1}^{n} \frac{\delta_{1i}}{\delta_{2i}} \left\{ \frac{\theta + \tau + \beta(y_i - \tau)}{\theta + \tau + \beta(y_i - \tau)} \right\} = 0
\end{align*}$$

The elements of the above matrix $F$ can be expressed by the following equations

$$\begin{align*}
\frac{\partial^2 \ln L}{\partial \beta^2} & = -\frac{n_a}{\beta^2} \left\{ (n - n_u - n_a) x(\eta - \tau) \right\} \\
& + \frac{(\alpha + 1) \sum_{i=1}^{n} \delta_{2i}}{\theta + \tau + \beta(y_i - \tau)}
\end{align*}$$

$$\begin{align*}
\frac{\partial^2 \ln L}{\partial \theta^2} & = -\frac{\partial^2 \ln L}{\partial \beta \partial \theta} + \frac{(\alpha + 1) \sum_{i=1}^{n} \delta_{2i}}{\theta + \tau + \beta(y_i - \tau)} + \left\{ \frac{n_u}{\theta + \tau + \beta(y_i - \tau)} \right\}^2
\end{align*}$$

$$\begin{align*}
\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & = -\frac{\partial^2 \ln L}{\partial \alpha^2} \left\{ \frac{n_a + n_u}{\alpha^2} \right\}
\end{align*}$$

$$\begin{align*}
\frac{\partial^2 \ln L}{\partial \alpha^2} & = -\frac{(n - n_u - n_a) x(\eta - \tau)}{(\theta + \tau + \beta(\eta - \tau))^2} \\
& + \frac{(\alpha + 1) \sum_{i=1}^{n} \delta_{2i}}{(\theta + \tau + \beta(y_i - \tau))^2}
\end{align*}$$

$$\begin{align*}
\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & = -\frac{(n - n_u - n_a) x(\eta - \tau)}{(\theta + \tau + \beta(\eta - \tau))^2} \\
& + \frac{(\alpha + 1) \sum_{i=1}^{n} \delta_{2i}}{(\theta + \tau + \beta(y_i - \tau))^2}
\end{align*}$$
\[
\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} = -\frac{(n - n_u - n_a)(\eta - \tau)}{\{\theta + \tau + \beta(\eta - \tau)\}} - \sum_{i=1}^{n} \frac{\delta_{ij} (y_i - \tau)}{\{\theta + \tau + \beta(y_i - \tau)\}}
\]

(18)

\[
\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \frac{n - (n - n_u - n_a)}{\theta + \tau + \beta(\eta - \tau)} - \left[ \sum_{i=1}^{n} \frac{\delta_{ij} (\theta + y_i)}{\{\theta + \tau + \beta(y_i - \tau)\}} + \sum_{i=1}^{n} \frac{\delta_{ij} (\theta + \tau + \beta(\eta - \tau))}{\{\theta + \tau + \beta(y_i - \tau)\}} \right]
\]

(19)

Therefore, the maximum likelihood estimators of \(\beta, \theta,\) and \(\alpha\) have an asymptotic variance-covariance matrix defined by inverting the above Fisher information matrix.

**ASYMPTOTIC CONFIDENCE BOUNDS**

The maximum likelihood method provides a single point estimate for a population parameter. A confidence interval indicates the uncertainty in an estimate calculated from sample data, it enclouses the population parameter with a specified high probability. The most common method to set confidence bounds for the parameters is to use the large-sample normal distribution of the ML estimates (Vander Wiel and Meeker [16]).

To define a confidence interval for a population parameter \(\lambda\), suppose \(L_\lambda = L_\lambda (y_1, ..., y_n)\) and \(U_\lambda = U_\lambda (y_1, ..., y_n)\) are functions of the sample data \(y_1, ..., y_n\) such that

\[P(L_\lambda \leq \lambda \leq U_\lambda) = \gamma \quad (20)\]

where the interval \([L_\lambda, U_\lambda]\) is called a two-sided \(\gamma\) 100% confidence interval for \(\lambda\). \(L_\lambda\) and \(U_\lambda\) are the lower and upper confidence limits for \(\lambda\), respectively. The confidence limits \(L_\lambda\) and \(U_\lambda\) enclose \(\lambda\) with probability \(\gamma\).

For large sample size the maximum likelihood estimators, are consistent and asymptotically normally distributed. Therefore, the two-sided approximate \(\gamma\) 100% confidence limits for a population parameter \(\lambda\) can be obtained such that

\[P\left[-z \leq \frac{\hat{\lambda} - \lambda}{\sigma(\lambda)} \leq z\right] \geq \gamma \quad (21)\]

where \(z\) is the \([100(1 - \gamma)/2]\)th standard normal percentile. Therefore, the two-sided approximate \(\gamma\) 100% confidence limits for \(\beta, \theta\) and \(\alpha\) are given, respectively, as follows

\[
L_\beta = \hat{\beta} - z\sigma(\hat{\beta}) \quad U_\beta = \hat{\beta} + z\sigma(\hat{\beta})
\]

\[
L_\theta = \hat{\theta} - z\sigma(\hat{\theta}) \quad U_\theta = \hat{\theta} + z\sigma(\hat{\theta})
\]

\[
L_\alpha = \hat{\alpha} - z\sigma(\hat{\alpha}) \quad U_\alpha = \hat{\alpha} + z\sigma(\hat{\alpha})
\]

(22)

**OPTIMUM TEST PLANS**

**Optimal Stress Change-Time \(\tau^*\)**

In this section, the problem of optimally designing simple step-stress PALTs, which terminates at a predetermined time, is considered. The optimum test plan for products having Lomax lifetime distribution is developed in which the choice of \(\tau^*\) will be determined such that the GAV of the MLEs of the model parameters at use stress is minimized.

**GENERALIZED ASYMPOTIC VARIANCE OF THE MLES OF THE MODEL PARAMETERS: AN OPTIMALITY CRITERION**

The GAV of the MLEs of the model Parameters is the reciprocal of the determinant of \(F\). That is

\[GAV(\hat{\beta}, \hat{\theta}, \hat{\alpha}) = |F|^{-1}\]

The Newton–Raphson method is applied to obtain the optimal stress-change time \(\tau^*\), which minimize the GAV as defined above. Therefore, the corresponding optimal expected numbers of items failed at use and accelerated conditions are, respectively, given by

\[nP_u = n \left[1 - \frac{(\hat{\beta})^z}{(\hat{\theta} + \tau^*)^z}\right]^{(\hat{\theta} + \tau^*)^z}
\]

and

\[nP_a = n \left[\frac{(\hat{\beta})^z}{(\hat{\theta} + \tau^*)^z}\right]^{(\hat{\theta} + \tau^*)^z} \left[1 - \frac{(\hat{\beta})^z}{(\hat{\theta} + \tau^*)^z}\right]^{(\hat{\theta} + \tau^*)^z}
\]

**Simulation Studies**

Simulation results are given in which the parameters are estimated from simulated partially accelerated life test data drawn from Lomax distribution. Newton-Raphson iterative method is used to find the MLEs of \(\beta\) and \(\theta\) assuming initial values for these parameters. The derived nonlinear logarithmic likelihood estimating Eqs 11 and 12 are solved iteratively. Once the values of \(\beta\) and \(\theta\) are determined, an estimate of the shape parameter \(\alpha\) is easily obtained from Eq 10. To explore the effect of sample size on the MLEs, different sized samples were simulated ranging in size from \(n = 30\) to \(n = 100\) with results presented in Tables 1 and 2 using true parameters values set at \(\beta = 3, \theta = 2,\) and \(\alpha = 1.5,\) given \(\tau = 2\) and \(\eta = 7.\) For each sample size, the experiment has 10 000 replications. Another set of data are generated with true parameters values set at \(\beta = 2, \theta = 3,\) and \(\alpha = 0.6,\) given \(\tau = 2\) and \(\eta = 7.\) In addition, 10 000 repeated samples are obtained randomly for each sample size with results shown in Tables 3 and 4.

For different sample sizes and true values of the parameters, the ML estimates, mean square error (MSE), average confidence intervals lengths (IL), and the associated coverage probabilities (CP) are obtained with results reported in Tables 1 and 3. Results of simulation studies offer perception into the sampling behavior of the estimators. The numerical results indicate that the ML estimates are close to the true values of the parameters as the
From the results, the optimal GAV of the MLEs of the model parameters, usually approaches the censoring time at the mal stress change-time, minimizing the GAV of the MLEs of the model parameters. It is numerically obtained with the MLEs of the model parameters. It is numerically obtained as the sample size $n$ increases. The equations in Eq. 22 are used to construct approximate confidence limits for the three parameters $\beta$, $\theta$, and $\alpha$ based on a 95% confidence degree. It is shown from the numerical results that the IL decrease as the sample size $n$ increases. Moreover, it is observed that the CP for each parameter is close to the nominal confidence level $1 - \alpha$. That is, the procedure is working very well.

For the two sets of data stated before, optimum test plans are also developed numerically. Both Tables 2 and 4 give the optimal stress change-time $\tau^*$ for the considered different sized samples. In addition, the corresponding optimal expected numbers of items failed at use and accelerated conditions, $nP_u$ and $nP_a$, respectively, are presented in these tables. The numerical results shown in both Tables 2 and 4 demonstrate that the optimal stress change-time, minimizing the GAV of the MLEs of the model parameters, usually approaches the censoring time at which the life test is terminated. That is, all units tend to fail at use condition and no items appear to fail at accelerated conditions. Furthermore, Tables 2 and 4 show the optimal GAV of the MLEs of the model parameters. It is numerically obtained with $\tau^*$ in place of $\tau$ for different sized samples. As indicated from the results, the optimal GAV of the MLEs of the model parameters decreases as the sample size $n$ increases. In practice, the optimum test plans are important for improving precision in parameter estimation and thus improving the quality of statistical inference. As such, these optimum test plans are more useful and more efficient for estimating the life distribution at design stress.

### Table 1

<table>
<thead>
<tr>
<th>$n$</th>
<th>Parameter</th>
<th>Estimate</th>
<th>MSE</th>
<th>IL95 %</th>
<th>95% CP</th>
</tr>
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<tbody>
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<td>3.4261</td>
<td>0.0486</td>
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<td>0.9472</td>
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</table>

### Table 2

The results of optimal design of the life test, given $\tau = 2$ and $\eta = 7$, for different sized samples under Type-I censoring.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\tau^*$</th>
<th>$nP_u$</th>
<th>$nP_a$</th>
<th>Optimal GAV</th>
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<td>6.9986</td>
<td>27</td>
<td>0</td>
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<tr>
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<td>0.0022</td>
</tr>
<tr>
<td>100</td>
<td>6.9999</td>
<td>90</td>
<td>0</td>
<td>0.0007</td>
</tr>
</tbody>
</table>

### Table 3

The average values of MLEs, MSE, IL95 % of the parameters $(\beta, \theta, \alpha)$ set at $(3, 2, 1.5)$, respectively, given $\tau = 2$ and $\eta = 7$ for different sized samples under Type-I censoring.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Parameter</th>
<th>Estimate</th>
<th>MSE</th>
<th>IL95 %</th>
<th>95% CP</th>
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### Table 4

The results of optimal design of the life test, given $\tau = 2$ and $\eta = 7$, for different sized samples under Type-I censoring.

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<tr>
<th>$n$</th>
<th>$\tau^*$</th>
<th>$nP_u$</th>
<th>$nP_a$</th>
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A REAL LIFE EXAMPLE

To illustrate the use of the methodology introduced in this article, a real data set from Nelson [17] is investigated. Nelson [17] reported an accelerated life test with 76 times (in minutes) to breakdown of an insulating fluid at voltage stress (kv). The stress $x_i$ is defined as the natural logarithm of the ratio of voltage to insulation thickness. That is, the design (use) stress $x_1 = \ln[20] = 2.9957$, and the high (accelerated) stress level $x_2 = \ln[28] = 3.3322$. The Lomax model is used to fit this data set. To check the validity of the model, we compute the Kolmogorov–Smirnov (K–S) distance between the empirical distribution function and the fitted distribution function when the parameters are obtained by the method of maximum likelihood. The result of K–S test is $D = 0.0872$ with $p$-value = 0.635. This result obviously shows that the Lomax model provides
excellent fit to the data set. Thus, it can be used successfully to analyze this data set.

Therefore, assuming the Lomax distribution with type-I censoring, we use \( n = 76 \), \( \beta = 2 \), \( \alpha = 0.5 \), \( \tau = 2 \), and \( \eta = 7 \). The MLEs of the model parameters \( \beta \), \( \alpha \), and \( \eta \) are, respectively, 2.04, 1.57, and 2.53. The MSE associated with the MLEs of the parameters \( \beta \), \( \alpha \), and \( \eta \) are 0.0141, 0.0172, and 0.0003, respectively. In addition, the associated 95% CP of the model parameters \( \beta \), \( \alpha \), and \( \eta \) are, respectively, 0.0027, 0.2146, and 0.0479, respectively. In addition, the 95% confidence interval of the model parameters \( \beta \), \( \alpha \), and \( \eta \) are, respectively, 0.9533, 0.9487, and 0.9521. Moreover, the optimal stress-change time \( \tau^* \) is 6.9897 with \( nP_a = 75 \), \( nP_a = 0 \) and optimal GAV = 0.0027. It can be observed that the same findings recorded using the simulation studies are obtained.

Concluding Remarks

In this article, time-censored step-stress partially accelerated life test plans have been addressed assuming Lomax distribution as a lifetime model. The maximum likelihood method has been applied to estimate the distribution parameters and the acceleration factor. The Newton--Raphson method as an iterative procedure has been used to obtain the estimates. The performance of the estimators has been assessed for different parameter values and different sample sizes. In addition, the average confidence intervals’ lengths and the associated coverage percentages have been obtained. It is observed that the coverage probabilities of the confidence intervals are close to the nominal confidence level 1−α. That is, the procedure is working very well.

Moreover, statistically optimum step-stress partially accelerated life test plans have been developed. The optimality criterion adopted is the minimization of the GAV of the MLEs of the model parameters. That is, the optimal stress-change time \( \tau^* \) is obtained such that the GAV of the MLEs of the model parameters is minimized. As shown from the numerical results, at different sized samples, \( \tau^* \) always approaches the censoring time \( \eta \), indicating that all items tend to fail at use condition. That is, testing only at normal condition. It is worth noting that the optimum test plan provides the most accurate estimate of the model parameters. Thus, the optimal design of the life tests can be considered as a technique to improve the quality of the statistical inference.

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References


