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PARAMETER ESTIMATION UNDER FAILURE CENSORED CONSTANT-STRESS LIFE TESTING MODEL USING MCMC APPROACH

Ali A. Ismail*

Cairo University, Faculty of Economics and Political Science, Department of Statistics,
Giza 12613, Egypt
*Corresponding author's e-mail: aismail100@yahoo.com

In this article likelihood and Bayesian estimations for the partially accelerated constant-stress life test model are compared using Type-II censored data from the Pareto distribution of the second kind. The posterior means and posterior variances are obtained under the squared error loss function using Lindley's approximation procedure. Furthermore, the highest posterior density credible intervals of the model parameters based on Gibbs sampling technique are presented. For illustration, simulation studies are provided. It is shown with the Bayesian approach via Gibbs sampling procedure that the statistical precision of parameter estimation is improved. Consequently, the required number of failures could be reduced. That is, more savings in time and cost can be achieved through the Markov chain Monte Carlo (MCMC) technique. Reducing the total testing time and the total number of failures without sacrificing much of the statistical power in inference is often desired in industrial applications.

Keywords: Bayesian estimation; failure censoring; maximum likelihood estimation; Pareto distribution; partially accelerated constant-stress test; squared error loss function; MCMC

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1. INTRODUCTION

In many industrial fields, highly reliable specimens and materials are necessary for long-term performance. Practically, test these items in their use environment is time-consuming and expensive due to the length of time required to produce a meaningful number of failures for analysis. As indicated by Nelson (1990), in practice, most products such as lamps, semiconductors, and microelectronics, etc., run at a constant stress level. Such testing is simple and has several advantages: first, it is easier to maintain a constant stress level in most tests. Second, accelerated test models are better developed. Third, data analysis for reliability estimation is well developed and computerized.

Statistical analysis of constant-stress partially accelerated life tests (CSPALTs) is made using failure censored data from Pareto distribution of the second kind. For the two levels of CSPALTs, the test units are divided, using a pre-specified sample-proportion, into two parts. The first part runs under normal (use) condition, and the remaining units are tested under accelerated condition. The test items are run until failures occur, or the observations are censored. The purpose of partially accelerated life tests (PALTs) is to collect more failure data in a limited time without necessarily using high stresses to all test units.

For an overview of CSPALTs, interested readers can refer for example to Bai and Chung (1992), Bai et al. (1993), Ismail et al. (2011), Ismail (2014), Ismail (2015), Ismail and Al-Babtain (2015) and Ismail (2016). This paper will concentrate on the Bayesian approach with Type-II censoring for estimating the parameters under CSPALTs using MCMC technique.

From the Bayesian point of view, few of studies have been considered on PALTs. Goel (1971) used the Bayesian approach for estimating the acceleration factor and the parameters in the case of step-stress PALTs (SSPALTs) with complete sampling for items having exponential and uniform distributions. In the case of SSPALT, a test item is first run at use condition and, if it does not fail for a pre-specified time τ, then it is run at accelerated condition until it fails or the test is terminated. DeGroot and Goel (1979) investigated the optimal Bayesian design of PALTs in the case of the exponential distribution under complete sampling. Abdel-Ghani (1998) considered the Bayesian approach to estimate the parameters of Weibull distribution in SSPALT with censoring. Tahir (2003) estimated the failure rate with the Bayes estimator under the squared error loss in SSPALTs assuming the exponential distribution. Ismail (2010) considered the Bayesian approach to estimate the parameters of Gompertz distribution under SSPALT model with time censoring. In addition, Ismail (2015) considered the Bayesian approach via Lindley's approximation to estimate the parameters of Pareto distribution of the second kind using Type-II censored data.

The objective of this paper is to apply a Bayesian analysis, with a squared error (SE) loss function, using MCMC technique on CSPALTs with Type-II censoring considering the two-parameter Pareto distribution of the second kind. The Bayes estimators (BEs) of the acceleration factor and the distribution parameters are considered and compared with the maximum likelihood estimators (MLEs) counterparts by Monte Carlo simulations.

The rest of this paper is organized as follows. In Section 2, the model and test method are described. BEs of the parameters under consideration are presented in Section 3. In Section 4, MCMC approach is used to obtain BEs and a comparison between the BEs and the MLEs via simulation studies is provided. Finally, Section 5 concludes the paper.

2. THE MODEL AND TEST METHOD

2.1. The Pareto Distribution as a Lifetime Model

In this article, the two-parameter Pareto distribution of the second kind is considered as a lifetime model. The Pareto distribution was introduced by Pareto (1897) as a model for the distribution of income. In recent years, its models in several different forms have been studied by many authors; Davis and Feldstein (1979), Cohen and Whitten (1988), Grimshaw (1993) among others. The Pareto distribution of the second kind also known as Lomax or Pearson's Type VI distribution, see Johnson et al. (1994). It has been found as a good model in biomedical problems, such as survival time following a heart transplant, Bain and Engelhardt (1992). Using the Pareto distribution, Dyer (1981) studied annual wage data of production line workers in a large industrial firm. Lomax (1954) used this distribution in the analysis of business failure data. The length of wire between flaws also follows a Pareto distribution, Bain and Engelhardt (1992). Since the Pareto distribution has a decreasing hazard or failure rate, it has often been used to model incomes and survival times, Howlader and Hossain (2002).

The probability density function of the Pareto distribution of the second kind is given by

$$f_T(t;\theta,\alpha) = \frac{\alpha\theta^{\alpha}}{(\theta+t)^{\alpha+1}} \qquad ; t > 0,\ \theta > 0,\ \alpha > 0, \tag{1}$$

The survival function takes the form

$$R(t) = \frac{\theta^{\alpha}}{(\theta+t)^{\alpha}}, \tag{2}$$

The corresponding failure rate function is

$$h(t) = \frac{\alpha}{\theta+t}. \tag{3}$$

According to McCune and McCune (2000), the Pareto distribution has classically been used in economic studies of income, size of cities and firms, service time in queuing systems and so on. Also, it has been used in connection with reliability theory and survival analysis; see Davis and Feldstein (1979).

2.2. Constant-Stress PALT

The test procedure of the constant-stress PALT and its assumptions are described as follows:

Test procedure:
In constant-stress PALTs, the total sample size n of test specimens is divided into two parts such that $n\pi$ items randomly chosen among n test items sampled are allocated to accelerated condition and the remaining are allocated to use condition. Each test specimen is run until the censoring time is reached or the item fails and the test condition is not changed.

Assumptions:

1. The lifetimes $T_i, i = 1, \ldots, n(1-\pi)$ of items allocated to use condition, are i.i.d. r.v.'s.
2. The lifetimes $X_j, j = 1, \ldots, n\pi$ of items allocated to accelerated condition, are i.i.dr.v.'s.
3. Suppose that the lifetime of an item at accelerated condition is denoted by X, then the lifetime of this item at use condition T is given by the relation $T = \beta X$.

Since the lifetimes of the test items follow Pareto distribution of the second kind, the probability density function of an item tested at use condition is given by

$$f_T(t;\theta,\alpha) = \frac{\alpha\theta^{\alpha}}{(\theta+t)^{\alpha+1}} \quad ; t > 0,\ \theta > 0,\ \alpha > 0, \tag{4}$$

while for an item tested at accelerated condition, the probability density function is given by

$$f_X(x;\theta,\alpha) = \frac{\beta\alpha\theta^{\alpha}}{(\theta+\beta\, x)^{\alpha+1}} \quad ; x > 0,\ \theta > 0,\ \alpha > 0, \tag{5}$$

where $X = \beta^{-1} T$.

3. BAYESIAN ESTIMATION

3.1 Posterior Means and Posterior Variances

In this section, the SE loss function is used. Under SE loss function, the Bayes estimator of a parameter is its posterior expectation. The Bayes estimators cannot be given in explicit forms. Approximate Bayes estimators will be discussed under the assumption of non-informative priors using Lindley's approximation. Basu *et al*. (1999) showed that "in many practical situations, the information about the parameters are available in an independent manner". Thus, here it is assumed that the parameters are independent a priori and let the non-informative prior (NIP) for each parameter be represented by the limiting form of the appropriate natural conjugate prior.

Therefore, the joint NIP of the three parameters can be expressed by

$$\pi(\beta,\theta,\alpha) \propto (\beta\theta\alpha)^{-1}, \quad \beta > 1,\ \theta > 0,\ \alpha > 0. \tag{6}$$

Using Type-II censored data in CSPALTs, each test item runs at either use or accelerated condition only until a predetermined censoring time $y_{(r)}$ is reached. That is, after acquiring r failures the test is terminated. Therefore, the observed lifetimes $t_{(1)} \le \ldots \le t_{(n_u)} \le y_{(r)}$ *and* $x_{(1)} \le \ldots \le x_{(n_a)} \le y_{(r)}$ are ordered failure times at use and accelerated conditions, respectively, where n_u and n_a are the corresponding numbers of items failed in each stage. Let δ_{ui} and δ_{aj}, be indicator functions such that $\delta_{ui} \equiv \mathrm{I}\,(T_i \le y_{(r)})$ and $\delta_{aj} \equiv \mathrm{I}\,(X_j \le y_{(r)})$; where $i = 1,\ldots, n$. Then the total likelihood function for $(t_1;\delta_{u1}, \ldots, t_{n(1-\pi)};\delta_{un(1-\pi)}, x_1;\delta_{a1}, \ldots, x_{n\pi};\delta_{an\pi})$ is given by

$$\begin{aligned} L(\beta,\theta,\alpha) &= \prod_{i=1}^{n(1-\pi)} L_{ui}(t_i;\theta,\alpha) \cdot \prod_{j=1}^{n\pi} L_{aj}(x_j;\beta,\theta,\alpha) \\ &= \prod_{i=1}^{n(1-\pi)} \left[\frac{\alpha\theta^{\alpha}}{(\theta+t_i)^{\alpha+1}}\right]^{\delta_{ui}} \left[\frac{\theta^{\alpha}}{(\theta+y_{(r)})^{\alpha}}\right]^{\bar{\delta}_{ui}} \\ &\cdot \prod_{j=1}^{n\pi} \left[\frac{\beta\alpha\theta^{\alpha}}{(\theta+\beta\, x_j)^{\alpha+1}}\right]^{\delta_{aj}} \left[\frac{\theta^{\alpha}}{(\theta+\beta\, y_{(r)})^{\alpha}}\right]^{\bar{\delta}_{aj}} \end{aligned}, \tag{7}$$

where L_{ui} is the likelihood function for t_i at use condition, L_{aj} is the likelihood function for x_j at accelerated condition, π is the proportion of sample units allocated to accelerated condition, and

$$\bar{\delta}_{ui} = 1 - \delta_{ui} \quad \text{and} \quad \bar{\delta}_{aj} = 1 - \delta_{aj} \quad .$$

Obviously, the Bayes solution for estimating the parameters is extremely difficult to obtain in a closed form because the posterior density is too complicated. So, numerical approximations are necessary for finding the posterior moments of interest. For mathematical simplicity, the Bayesain analysis is introduced under the Jeffreys vague prior

distribution for the three unknown parameters. Assuming the vague prior distribution of β, θ, and α as in (3.1) and forming the product of it and the likelihood function defined in (3.2), the joint posterior distribution of β, θ, and α can be expressed as follows

$$g(\beta,\theta,\alpha|\underline{y}) \propto L(\underline{y}|\beta,\theta,\alpha)\ .\ \pi(\beta,\theta,\alpha)$$
$$\propto \frac{\beta^{n_a-1}\theta^{2n\alpha-1}\alpha^{n_u+n_a}}{(\theta+y_{(r)})^{(n\bar{\pi}-n_u)\alpha}\ (\theta+\beta\, y_{(r)})^{(n\pi-n_a)\alpha}} \left[\prod_{i=1}^{n\bar{\pi}} \frac{1}{(\theta+t_i)^{\alpha+1}}\right]^{\delta_{u_i}}$$
$$\cdot \left[\prod_{j=1}^{n\pi} \frac{1}{(\theta+\beta\, x_j)^{\alpha+1}}\right]^{\delta_{a_j}} \tag{8}$$

As mentioned earlier, under a squared error loss function, the Bayes estimator of a parameter is its posterior expectation. To obtain the posterior means and posterior variances of β, θ, and α, non-tractable integrals will be met. It is not possible to obtain them analytically. The marginal posteriors are somewhat unwieldy and require a numerical integration that may not converge. Instead, an approximation due to Lindley (1980) via an asymptotic expansion of the ratio of two non-tractable integrals is used to obtain approximate Bayes estimators. Lindley's approximation is evaluated at the ML estimates of the model parameters.

Now, let Θ be a set of parameters $\{\Theta_1, \Theta_2, \ldots, \Theta_m\}$, where m is the number of parameters, then the posterior expectation of an arbitrary function $u(\Theta)$ can be asymptotically estimated by

$$E(u(\Theta)) = \frac{\int_{\Theta} u(\Theta)\,\pi(\Theta)\, e^{ln\,\mathrm{L}(\mathrm{y}\,|\,\Theta)} d\Theta}{\int_{\Theta} \pi(\Theta)\, e^{ln\,\mathrm{L}(\mathrm{y}\,|\,\Theta)} d\Theta}$$
$$\approx [\,u + (1/2)\sum_{i,j}(u_{ij}^{(2)} + 2u_i^{(1)}\rho_j^{(1)})\,\sigma_{\mathrm{ij}} + (1/2)\sum_{i,j,k,s} L_{ijk}^{(3)}\,\sigma_{ij}\,\sigma_{ks}\,u_s^{(1)}]\downarrow \hat{\Theta}, \tag{9}$$

which is the Bayes estimator of $u(\Theta)$ under a squared error loss function, where $\pi(\Theta)$ is the prior distribution of Θ, $u \equiv u(\Theta)$, $L \equiv L(\Theta)$ is the likelihood function, $\rho \equiv \rho(\Theta) = log\ \pi(\Theta)$, σ_{ij} are the elements of the inverse of the asymptotic Fisher information matrix of β, θ and α, and

$$u_i^{(1)} = \frac{\partial u}{\partial \Theta_i},\ \ u_{ij}^{(2)} = \frac{\partial^2 u}{\partial \Theta_i \partial \Theta_j},\ \ \rho_j^{(1)} = \frac{\partial log\ \pi(\Theta)}{\partial \Theta_j} \text{ and } L_{ijk}^{(3)} = \frac{\partial^3 ln\ L(y\,|\,\Theta)}{\partial \Theta_i \partial \Theta_j \partial \Theta_k}\ .$$

Such an approximation is easy to use and does not require innovative programming and extensive computer time. According to Green (1980), the linear Bayes estimator in (3.4) is a very good and operational approximation for the ratio of multi-dimension integrals. As indicated by Sinha (1986) it has led to many useful applications. However, if the domain of the parameters is a function of the parameters, Bayes estimators using Lindley's rule are not obtainable unless the MLEs exist. The Derivation of posterior means and posterior variances is shown in the Appendix.

In the following subsection, the Bayesian interval estimators, called credible intervals, for the model parameters are obtained from their posterior distributions. We propose the following Markov Chain Monte Carlo (MCMC) method to draw samples from the posterior density function and then to compute the Bayes estimates and the highest posterior density (HPD) credible intervals. The Gibbs sampling procedure is used to compute HPD credible intervals.

3.2. Credible Intervals Using Gibbs Sampling

Assume that the priori are Gamma distributions and that they are independent. Therefore, samples of α, θ and β can be easily generated using any of the gamma generating routines. The Gibbs sampling procedure is used to generate a sample from the posterior density function and then to compute the Bayes estimates and HPD credible intervals. The Gibbs sampler is an iterative procedure of a broad class of methods commonly named MCMC. For an extensive

discussion of the MCMC approach we refer to Casella and George (1992). This approach is applicable in situations where one is not able to generate samples directly from the joint posterior density. It requires the conditional densities for generating samples. To run the Gibbs sampler algorithm, it is appropriate to start with the approximate BEs. The following algorithm is used for this purpose.

Step 1. Start with an ($\alpha^{(0)} = \alpha_{Lindley}, \theta^{(0)} = \theta_{Lindley}$ and $\beta^{(0)} = \beta_{Lindley}$) and set I = 1.

Step 2. Generate $\alpha^{(\mathrm{I})}$ from the conditional Gamma distribution $(g(\alpha|\theta^{(\mathrm{I}-1)}, \beta^{(\mathrm{I}-1)}, \underline{y}))$.

Step 3. Generate $\theta^{(\mathrm{I})}$ from the conditional Gamma distribution $(g(\theta|\alpha^{(\mathrm{I}-1)}, \beta^{(\mathrm{I}-1)}, \underline{y}))$.

Step 4. Generate $\beta^{(\mathrm{I})}$ from the conditional Gamma distribution $(g(\beta|\theta^{(\mathrm{I}-1)}, \alpha^{(\mathrm{I}-1)}, \underline{y}))$.

Step 5. Set I = I +1.

Step 6. Repeat steps 2-4 *M* times and obtain α_i, θ_i and β_i for *i*=1,. . . , *M*.

Step 7. The Bayes MCMC point estimates of α, θ and β with respect to the squared error function are then

$$\tilde{\alpha} = \tilde{\mathrm{E}}(\alpha|data) = \frac{1}{M}\sum_{k=1}^{M}\alpha_k, \ \tilde{\theta} = \tilde{\mathrm{E}}(\theta|data) = \frac{1}{M}\sum_{k=1}^{M}\theta_k \text{ and } \tilde{\beta} = \tilde{\mathrm{E}}(\beta|data) = \frac{1}{M}\sum_{k=1}^{M}\beta_k .$$

Step 8. The posterior variances of α, θ and β are

$$\tilde{V}(\alpha|data) = \frac{1}{M}\sum_{k=1}^{M}\{\alpha_k - \tilde{\mathrm{E}}(\alpha|data)\}^2, \ \tilde{V}(\theta|data) = \frac{1}{M}\sum_{k=1}^{M}\{\theta_k - \tilde{\mathrm{E}}(\theta|data)\}^2 \text{ and}$$
$$\tilde{V}(\beta|data) = \frac{1}{M}\sum_{k=1}^{M}\{\beta_k - \tilde{\mathrm{E}}(\beta|data)\}^2.$$

Step 9. To compute the credible intervals (CRIs) of φ_l $(\varphi_1 = \alpha, \varphi_2 = \theta, \varphi_3 = \beta)$, the quantiles of the sample is usually taken as the endpoints of the intervals. Order $\varphi_l^{(1)}, \varphi_l^{(2)}, ..., \varphi_l^{(M)}$ as $\varphi_{l(1)}, \varphi_{l(2)}, ..., \varphi_{l(M)}$. Then, $100(1-2\gamma)\%$ CRIs for φ_l become $(\varphi_{l(\gamma M)}, \varphi_{l((1-\gamma)M)})$.

4. SIMULATION STUDIES

In this section, the Bayesian inference procedure for CSPALTs using MCMC technique is demonstrated with Type-II censored data from Pareto distribution of the second kind. The posterior point estimates and the credible intervals for the model parameters are given in Tables 1 & 2. Moreover, the point estimate, MSEs, variance and the credible intervals of the model parameters were compared with those calculated by the ML method.

To illustrate the theoretical results of estimation, two combinations of true values of the parameters set at (β,θ,α) = (1.7, 0.5, 0.4) and (3, 1.2, 2) are used with 20,000 replications. The average of the results is taken to represent an estimate of each parameter and the mean squared errors (MSEs) are computed. The computational results of the first combination are presented in Table 1. While Table 2 summarizes the results of the second one. Tables 1 & 2 contain the results of both ML estimation and Bayesian estimation for the parameters of Pareto distribution under CSPALTs with Type-II censored data. As seen from the results, the Bayesian estimators usually have smaller MSEs and smaller variances than those of the MLEs, which is a great advantage of the use of MCMC technique. Thus, indicating that it is appropriate for practical use within this setting. Many practical aspects of MCMC are described in Gelfand & Smith (1990) and Gilks *et al*. (1996).

In addition, the Bayesian credible intervals of the model parameters were outstandingly narrower than the 95% confidence intervals of MLEs. Although these two types of intervals are not directly comparable, as the Bayesian

method treats model parameters as random variables whereas the common method treats them as unknown constants, the narrower Bayesian credible intervals demonstrate the benefit of combining prior information into statistical inference.

Table 1. Average values of the *MLEs* and *BEs* with associated *MSEs*, variances and estimated intervals lengths when β =1.7, θ = 0.5, α = 0.4, given π = 0.50 and r = 0.75 n based on varying sample sizes using Type-II censoring

n	*parameter*	*Method*	*Estimate*	*MSEs*	*variance*	*95% CIL*
25	β	*ML*	1.9481	0.0589	0.0315	0.6957
		Bayes	1.8722	0.0496	0.0255	0.5879
	θ	*ML*	0.7253	0.0326	0.0141	0.4655
		Bayes	0.5611	0.0239	0.0063	0.2901
	α	*ML*	0.4986	0.0243	0.0061	0.3062
		Bayes	0.4672	0.0182	0.0032	0.2034
50	β	*ML*	1.8375	0.0401	0.0207	0.5640
		Bayes	1.7681	0.0312	0.0129	0.4122
	θ	*ML*	0.5821	0.0216	0.0081	0.3528
		Bayes	0.5432	0.0154	0.0040	0.2253
	α	*ML*	0.4564	0.0172	0.0024	0.1920
		Bayes	0.4312	0.0122	0.0011	0.1100
75	β	*ML*	1.7966	0.0308	0.0105	0.4017
		Bayes	1.7362	0.0248	0.0034	0.1985
	θ	*ML*	0.5318	0.0161	0.0030	0.2147
		Bayes	0.5143	0.0128	0.0018	0.1402
	α	*ML*	0.4317	0.0039	0.0090	0.3719
		Bayes	0.4146	0.0014	0.0003	0.0479
100	β	*ML*	1.7434	0.0058	0.0017	0.1616
		Bayes	1.7138	0.0031	0.0013	0.1211
	θ	*ML*	0.5221	0.0043	0.0014	0.1467
		Bayes	0.5069	0.0022	0.0005	0.0545
	α	*ML*	0.4203	0.0012	0.0003	0.0679
		Bayes	0.4054	0.0009	0.0001	0.0247

Table 2. Average values of the *MLEs* and *BEs* with associated *MSEs*, variances and estimated intervals lengths when β = 3, θ =1.2, α = 2, given π = 0.50 and r = 0.75 n based on varying sample sizes using Type-II censoring

n	*parameter*	*Method*	*estimate*	*MSEs*	*variance*	*95% CIL*
25	β	*ML*	4.1591	0.0346	0.0165	0.5035
		Bayes	3.6210	0.0310	0.0123	0.4188
	θ	*ML*	1.6451	0.0211	0.0074	0.3372
		Bayes	1.4305	0.0145	0.0038	0.2261
	α	*ML*	2.4183	0.0151	0.0032	0.2217
		Bayes	2.0768	0.0106	0.0019	0.1611
50	β	*ML*	3.7954	0.0243	0.0112	0.4149
		Bayes	3.2354	0.0202	0.0067	0.3092
						0.2571
	θ	*ML*	1.5470	0.0134	0.0043	0.1434
		Bayes	1.4062	0.0089	0.0021	0.1358
	α	*ML*	2.1379	0.0108	0.0012	0.0646
		Bayes	2.0211	0.0073	0.0005	0.2630

75	β	*ML*	3.1955	0.0177	0.0045	0.1418
		Bayes	3.0262	0.0151	0.0018	0.1413
	θ	*ML*	1.3368	0.0104	0.0013	0.1001
		Bayes	1.2927	0.0071	0.0008	
	α	*ML*	2.0411	0.0022	0.0003	0.0679
		Bayes	2.0142	0.0007	0.0001	0.0352
100	β	*ML*	3.0574	0.0028	0.0007	0.1037
		Bayes	3.0104	0.0017	0.0005	0.0732
	θ	*ML*	1.2599	0.0021	0.0006	0.0960
		Bayes	1.2135	0.0011	0.0003	0.0524
	α	*ML*	2.0107	0.0005	0.0002	0.0554
		Bayes	2.0003	0.0003	0.0001	0.0371

Graphical posterior summaries such as histograms provide useful additions to the various numerical statistics for summarizing MCMC output. A histogram depicts the general shape of the marginal posterior distribution of a model parameter. It is a good practice to inspect the histogram of the marginal posterior distributions of parameters to ensure that these empirical distributions behave as expected. These plots can be used to compare the empirical posterior and the specified prior distributions to see the impact of the data.

The histograms of α, θ, and β generated by MCMC samples are shown by the following figures, respectively.

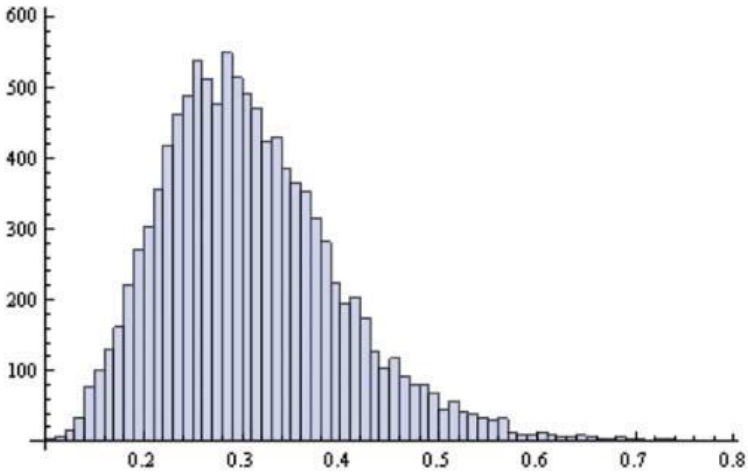

Figure 1. Histogram of α generated by the MCMC approach

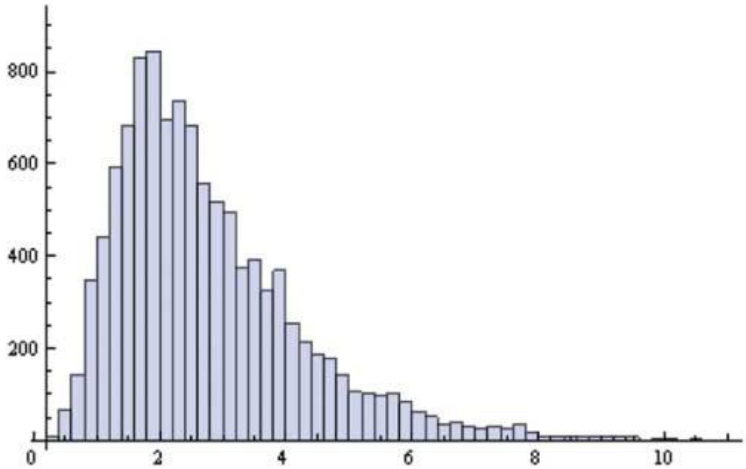

Figure 2. Histogram of θ generated by the MCMC approach

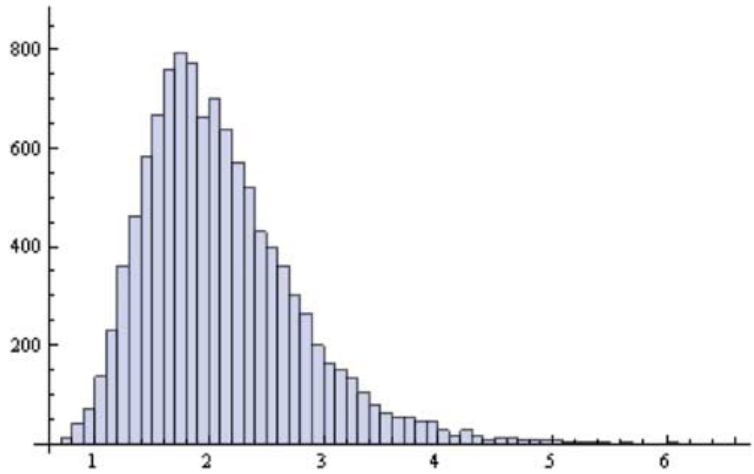

Figure 3. Histogram of β generated by the MCMC approach

It is shown from Figures 1-3 that the histograms are unimodal but are slightly skewed to the right. The unimodal histogram is a confirmation that we have obtained a good simulation of the marginal posterior distribution.

5. CONCLUSION

In this article the ML and Bayesian estimations of the parameters of Pareto distribution and the acceleration factor have been obtained under CSPALTs model using Type-II censored data. The posterior distribution and the point estimates for the model parameters are provided. The Bayes estimates have been obtained under the assumption of squared error loss functions.

It has been seen that the MCMC technique usually gives MSEs and posterior variances smaller than those of the maximum likelihood estimators. That is, it provides better estimates, which is an advantage of this method. The Bayesian inference method increases the parameter estimation precision and makes it possible to reduce the number of failures required from the test, thus shortening the testing time, without losing much of the estimation accuracy of the parameters of interest. Hence, it is worth to be noted that the MCMC technique could be of a great practical interest especially for industrial applications.

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APPENDIX – I

THE POSTERIOR MEANS AND POSTERIOR VARIANCES

Here, there are three parameters in the model. That is, m = 3. Let the subscripts 1, 2 and 3 refer to β,θ and α, respectively. It is not easy to obtain the posterior moments analytically. Therefore, using Lindley expansion, the posterior mean (i.e., Bayesian estimator under squared-error loss function) and the posterior variance of β are given, respectively, in the form

$$\begin{aligned}\beta^* &= E(\beta|\mathrm{y}) \\ &= \left[\beta - \left(\frac{\sigma_{11}}{\beta}+\frac{\sigma_{12}}{\theta}+\frac{\sigma_{13}}{\alpha}\right)+\frac{1}{2}\left(\sigma_{11}\mathrm{E}_1+\sigma_{12}\mathrm{E}_2+\sigma_{13}\mathrm{E}_3\right)\right]\downarrow\hat{\Theta}\,,\end{aligned} \tag{10}$$

and

$$\begin{aligned}Var(\beta|\mathrm{y}) &= E(\beta^2|y)-(\beta^*)^2 \\ &= \sigma_{11}-\left[\left(\frac{\sigma_{11}}{\beta}+\frac{\sigma_{12}}{\theta}+\frac{\sigma_{13}}{\alpha}\right)-\frac{1}{2}\left(\sigma_{11}\mathrm{E}_1+\sigma_{12}\mathrm{E}_2+\sigma_{13}\mathrm{E}_3\right)\right]^2\downarrow\hat{\Theta}\ .\end{aligned} \tag{11}$$

Conducting the same procedure, the posterior mean and posterior variance of the scale parameter θ take the following form

$$\begin{aligned}\theta^* &= E(\theta|\mathrm{y}) \\ &= \left[\theta - \left(\frac{\sigma_{21}}{\beta}+\frac{\sigma_{22}}{\theta}+\frac{\sigma_{23}}{\alpha}\right)+\frac{1}{2}\left(\sigma_{21}\mathrm{E}_1+\sigma_{22}\mathrm{E}_2+\sigma_{23}\mathrm{E}_3\right)\right]\downarrow\hat{\Theta}\ ,\end{aligned} \tag{12}$$

and

$$\begin{aligned}Var(\theta|\mathrm{y}) &= E(\theta^2|y)-(\theta^*)^2 \\ &= \sigma_{22}-\left[\left(\frac{\sigma_{21}}{\beta}+\frac{\sigma_{22}}{\theta}+\frac{\sigma_{23}}{\alpha}\right)-\frac{1}{2}\left(\sigma_{21}\mathrm{E}_1+\sigma_{22}\mathrm{E}_2+\sigma_{23}\mathrm{E}_3\right)\right]^2\downarrow\hat{\Theta}\,.\end{aligned} \tag{13}$$

Similarly, for the shape parameter α, the posterior mean and the posterior variance are expressed as follows

$$\begin{aligned}\alpha^* &= E(\alpha|\mathrm{y}) \\ &= \left[\alpha - \left(\frac{\sigma_{31}}{\beta}+\frac{\sigma_{32}}{\theta}+\frac{\sigma_{33}}{\alpha}\right)+\frac{1}{2}\left(\sigma_{31}\mathrm{E}_1+\sigma_{32}\mathrm{E}_2+\sigma_{33}\mathrm{E}_3\right)\right]\downarrow\hat{\Theta}\,,\end{aligned} \tag{14}$$

and

$$\begin{aligned}Var(\alpha|\mathrm{y}) &= E(\alpha^2|y)-(\alpha^*)^2 \\ &= \sigma_{33}-\left[\left(\frac{\sigma_{31}}{\beta}+\frac{\sigma_{32}}{\theta}+\frac{\sigma_{33}}{\alpha}\right)-\frac{1}{2}\left(\sigma_{31}\mathrm{E}_1+\sigma_{32}\mathrm{E}_2+\sigma_{33}\mathrm{E}_3\right)\right]^2\downarrow\hat{\Theta}\ ,\end{aligned} \tag{15}$$

where

$$\mathrm{E}_1 = \sum_{i,j}\sigma_{ij}L^{(3)}_{ij1},\ \mathrm{E}_2 = \sum_{i,j}\sigma_{ij}L^{(3)}_{ij2}\ ,\ \mathrm{E}_3 = \sum_{i,j}\sigma_{ij}L^{(3)}_{ij3}\ .$$

σ_{ij} are the elements of the inverse of the asymptotic Fisher-information matrix of the ML estimators of β, θ, and α in the case of type-II censored data and

$$L_{ijk}^{(3)}, \quad \text{For } i, j, k = 1,2,3,$$

is the third derivatives of the natural logarithm of the likelihood function.

To calculate the posterior means and the posterior variances of β, θ and α derived before, both second and third derivatives of the natural logarithm of the likelihood function in (3.2) must be obtained.

The second derivatives can be presented by the following equations.

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n_a}{\beta^2} + \frac{(n\pi - n_a)\alpha\left(y_{(r)}\right)^2}{\left(\theta + \beta\, y_{(r)}\right)^2} + (\alpha+1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j^{\,2}}{\left(\theta + \beta x_j\right)^2} \quad ,$$

$$\frac{\partial^2 \ln L}{\partial \beta\, \partial \theta} = \frac{(n\pi - n_a)\alpha\, y_{(r)}}{(\theta + \beta\, y_{(r)})^2} + (\alpha+1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{(\theta + \beta x_j)^2} \quad ,$$

$$\frac{\partial^2 \ln L}{\partial \beta\, \partial \alpha} = -\frac{(n\pi - n_a)\, y_{(r)}}{\theta + \beta\, y_{(r)}} - \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{\theta + \beta x_j} \,, \tag{18}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n\alpha}{\theta^2} + \frac{(n\pi - n_a)\alpha}{(\theta + \beta\, y_{(r)})^2} + \frac{(n\bar{\pi} - n_u)\alpha}{(\theta + y_{(r)})^2} + (\alpha+1)\left[\sum_{i=1}^{n\bar{\pi}} \frac{\delta_{ui}}{(\theta + t_i)^2} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{(\theta + \beta x_j)^2}\right] \quad , \tag{19}$$

$$\frac{\partial^2 \ln L}{\partial \theta\, \partial \alpha} = \frac{n}{\theta} - \frac{(n\pi - n_a)}{\theta + \beta\, y_{(r)}} - \frac{(n\bar{\pi} - n_u)}{\theta + y_{(r)}} \,, - \left[\sum_{i=1}^{n\bar{\pi}} \frac{\delta_{ui}}{\theta + t_i} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{\theta + \beta x_j}\right]$$

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n_u + n_a}{\alpha^2} \quad . \tag{21}$$

For the third derivatives, they are given as follows

$$L_{111}^{(3)} = \frac{\partial^3 \ln L}{\partial \beta^3} = \frac{2n_a}{\beta^3} - \frac{2(n\pi - n_a)\alpha (y_{(r)})^3}{(\theta + \beta\, y_{(r)})^3} - 2(\alpha+1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j^3}{(\theta + \beta x_j)^3} \quad , \tag{22}$$

$$L_{222}^{(3)} = \frac{\partial^3 \ln L}{\partial \theta^3} = \frac{2n\alpha}{\theta^3} - \frac{2(n\pi - n_a)\alpha}{(\theta + \beta y_{(r)})^3} - \frac{2(n\bar{\pi} - n_u)\alpha}{(\theta + y_{(r)})^3}$$
$$- 2(\alpha+1)[\sum_{i=1}^{n\bar{\pi}} \frac{\delta_{ui}}{(\theta + t_i)^3} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{(\theta + \beta x_j)^3}] \quad , \tag{23}$$

$$L_{333}^{(3)} = \frac{\partial^3 \ln L}{\partial \alpha^3} = \frac{2(n_u + n_a)}{\alpha^3} \quad , \tag{24}$$

$$L_{112}^{(3)} = \frac{\partial^3 \ln L}{\partial \beta^2 \partial \theta} = -\frac{2(n\pi - n_a)\alpha (y_{(r)})^2}{(\theta + \beta y_{(r)})^3}$$
$$- 2(\alpha+1) \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j^2}{(\theta + \beta x_j)^3} = L_{121}^{(3)} = L_{211}^{(3)} \quad , \tag{25}$$

$$L_{221}^{(3)} = \frac{\partial^3 \ln L}{\partial \theta^2 \partial \beta} = -\frac{2(n\pi - n_a)\alpha\, y_{(r)}}{(\theta + \beta y_{(r)})^3}$$
$$- 2(\alpha+1) \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{(\theta + \beta x_j)^3} = L_{212}^{(3)} = L_{122}^{(3)} \quad , \tag{26}$$

$$L_{113}^{(3)} = \frac{\partial^3 \ln L}{\partial \beta^2 \partial \alpha} = \frac{(n\pi - n_a)(y_{(r)})^2}{(\theta + \beta y_{(r)})^2} + \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j^2}{(\theta + \beta x_j)^2} = L_{131}^{(3)} = L_{311}^{(3)} \quad , \tag{27}$$

$$L_{123}^{(3)} = \frac{\partial^3 \ln L}{\partial \beta\, \partial \theta\, \partial \alpha} = \frac{(n\pi - n_a)\, y_{(r)}}{(\theta + \beta y_{(r)})^2} + \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{(\theta + \beta x_j)^2}$$
$$= L_{132}^{(3)} = L_{213}^{(3)} = L_{231}^{(3)} = L_{312}^{(3)} = L_{321}^{(3)} \quad , \tag{28}$$

$$L_{223}^{(3)} = \frac{\partial^3 \ln L}{\partial \theta^2 \partial \alpha} = -\frac{n}{\theta^2} + \frac{(n\pi - n_a)}{(\theta + \beta y_{(r)})^2} + \frac{(n\bar{\pi} - n_u)}{(\theta + y_{(r)})^2}$$
$$+ \sum_{i=1}^{n\bar{\pi}} \frac{\delta_{ui}}{(\theta + t_i)^2} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{(\theta + \beta x_j)^2}$$
$$= L_{232}^{(3)} = L_{322}^{(3)} \quad , \tag{29}$$

$$L_{331}^{(3)} = \frac{\partial^3 \ln L}{\partial \alpha^2 \partial \beta} = L_{313}^{(3)} = L_{133}^{(3)} \quad , \tag{30}$$

$$L_{332}^{(3)} = \frac{\partial^3 \ln L}{\partial \alpha^2 \partial \theta} = L_{323}^{(3)} = L_{233}^{(3)} \quad . \tag{31}$$