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BAYESIAN ESTIMATION UNDER CONSTANT-STRESS PARTIALLY ACCELERATED LIFE TEST FOR PARETO DISTRIBUTION WITH TYPE-I CENSORING

Ali A. Ismail[a,b]

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This article discusses likelihood and Bayesian estimations under constant-stress partially accelerated life test model with type-I censoring assuming Pareto distribution of the second kind. Both maximum likelihood and Bayesian estimators of the model parameters are derived. The posterior means and posterior variances are obtained under the squared error loss function using Lindley's approximation procedure. The advantages of this approximation are shown. Monte Carlo simulations are made under different samples sizes and different parameter values for evaluating and comparing the proposed methods of estimation.

Keywords: reliability, testing, maximum likelihood estimation, Bayesian estimation, squared error loss function.

Introduction. Accelerated testing ensures that specimens are exposed to elevated environmental conditions for fixed periods of time. Overstress testing consists of running a product at higher than normal levels of some accelerating stress(es) to shorting product life or to degrade product performance faster. Overstress constant stress testing is the most common form of accelerated testing of specimens. According to Nelson [1], "each specimen is tested under a constant stress level. Such testing scheme is easy and has numerous advantages."

As indicated by Ismail [2], "accelerated life testing and partially accelerated life testing (PALT) are frequently used in modern reliability engineering to save time and cost."

Constant-stress PALT with type-I censoring were studied by some authors such as [3–6]. These studies had been made based on classical methods. This paper considers Lindley technique for estimating the parameters in constant-stress PALT. According to [7], "such an approximation has numerous valuable applications especially for industrial fields." Also, in this respect, Achcar [8] indicated that "the use of approximate Bayesian methods could be a good alternative for the usual existing classical asymptotic methods used in accelerated life testing (ALT)."

There were some works on PALT in the context of Bayesian approach. For example, see Goel [9], DeGroot and Goel [10], Abdel-Ghani [11], Ismail [12]. The objective of this article is to use the Lindley method to make a Bayesian analysis with a squared error loss function under time-censoring CSPALT. The Bayes estimators (BEs) of the acceleration factor and the distribution parameters are derived and compared with the maximum likelihood estimators (MLEs) counterparts by Monte Carlo simulations.

The present paper is arranged as follows. In Section 1, the model and test method are described. Approximate BEs of the parameters under consideration are derived in Section 2. In Section 3, BEs derived in Section 2 are obtained numerically using the Lindley approximation and compared with the MLEs. Finally, Section 4 concludes the paper.

[a]King Saud University, College of Science, Department of Statistics and Operations Research, P.O. BOX 2455, Riyadh 11451, Saudi Arabia. [b]Cairo University, Faculty of Economics & Political Science, Department of Statistics, Giza 12613, Egypt [aismail100@yahoo.com (Ali A. Ismail)]. Translated from Problemy Prochnosti, No. 4, pp. 147 – 156, July – August, 2015. Original article submitted Septemer 9, 2014.

1. The Model and Test Method. The probability density function (PDF) of the Pareto distribution of the second kind is given by

$$f_T(t;\theta,\alpha)=\frac{\alpha\theta^{\alpha}}{(\theta+t)^{\alpha+1}},\qquad t>0,\ \theta>0,\ \alpha>0. \tag{1}$$

The survival function takes the form

$$R(t)=\frac{\theta^{\alpha}}{(\theta+t)^{\alpha}}, \tag{2}$$

and the corresponding failure rate function is

$$h(t)=\frac{\alpha}{\theta+t}. \tag{3}$$

In a constant-stress PALT, $n\pi$ units randomly selected among n test units sampled are allocated to severe condition and the remaining are allocated to normal condition. Each test item is tested until the censoring time is reached or the item fails.

The following assumptions are considered:

1. The lifetimes T_i, $i=1, \ldots, n(1-\pi)$ of items allocated to use condition, are i.i.d. r.v.'s.
2. The lifetimes X_j, $j=1, \ldots, n\pi$ of items allocated to accelerated condition, are i.i.d r.v.'s.
3. Suppose that the lifetime of an item at accelerated condition is denoted by X, then the lifetime of this item at use condition T is given by the relation $T=\beta X$.

Since the lifetimes of the test items follow Pareto distribution of the second kind, the probability density function of an item tested at normal condition is given by (1).

The PDF under severe condition is expressed by

$$f_X(x;\theta,\alpha)=\frac{\beta\alpha\theta^{\alpha}}{(\theta+\beta x)^{\alpha+1}},\qquad x>0,\ \theta>0,\ \alpha>0, \tag{4}$$

where $X=\beta^{-1}T$.

2. Bayesian Estimation. Here, Bayesian estimates are considered using non-informative priors via the technique of Lindley and the squared error (SE) loss function. The non-informative prior (NIP) for each parameter be represented by the limiting form of the appropriate natural conjugate prior.

It follows that a NIP for the acceleration factor β is given by

$$\pi_1(\beta)\propto\beta^{-1},\qquad \beta>1.$$

Also, the NIP's for the scale parameter θ and the shape parameter α are, respectively, as

$$\pi_2(\theta)\propto\theta^{-1},\qquad \theta>0\qquad\text{and}\qquad \pi_3(\alpha)\propto\alpha^{-1},\qquad \alpha>0. \tag{5}$$

Therefore, the joint NIP of the three parameters can be expressed by

$$\pi(\beta,\theta,\alpha)\propto(\beta\theta\alpha)^{-1},\qquad \beta>1,\ \theta>0,\ \alpha>0. \tag{6}$$

Via time-censored data, any unit can be tested at one condition only until a pre-fixed censoring time η is attained. Therefore, the observed lifetimes $t_{(1)}\le\ldots\le t_{(n_u)}\le\eta$ and $x_{(1)}\le\ldots\le x_{(n_a)}\le\eta$ are ordered failure times at normal use and accelerated conditions, respectively, where n_u and n_a are the corresponding numbers of items failed in each stage. Let δ_{ui} and δ_{aj}, be indicator functions such that $\delta_{ui} \equiv I(T_i \le \eta)$ and $\delta_{aj} \equiv I(X_j \le \eta)$, where $i = 1, \ldots, n$. Then, the overall likelihood function can be expressed by

$$(\beta, \theta, \alpha) = \prod_{i=1}^{n(1-\pi)} L_{ui}(t_i; \theta, \alpha) \prod_{j=1}^{n\pi} L_{aj}(x_j; \beta, \theta, \alpha)$$

$$= \prod_{i=1}^{n(1-\pi)} \left[\frac{\alpha\theta^{\alpha}}{(\theta + t_i)^{\alpha+1}}\right]^{\delta_{ui}} \left[\frac{\theta^{\alpha}}{(\theta+\eta)^{\alpha}}\right]^{\bar{\delta}_{ui}} \prod_{j=1}^{n\pi} \left[\frac{\beta\alpha\theta^{\alpha}}{(\theta+\beta x_j)^{\alpha+1}}\right]^{\delta_{aj}} \left[\frac{\theta^{\alpha}}{(\theta+\beta\eta)^{\alpha}}\right]^{\bar{\delta}_{aj}}, \tag{7}$$

where L_{ui} is the likelihood function for t_i at use condition, L_{aj} is the likelihood function for x_j at accelerated condition, π is the proportion of sample units allocated to accelerated condition, and

$$\bar{\delta}_{ui} = 1 - \delta_{ui} \qquad \text{and} \qquad \bar{\delta}_{aj} = 1 - \delta_{aj}.$$

Using (6) and (7), the joint posterior distribution can be given by

$$g(\beta, \theta, \alpha \mid \underline{y}) \propto L(\underline{y} \mid \beta, \theta, \alpha)\pi(\beta, \theta, \alpha)$$

$$\propto \frac{\beta^{n_a - 1}\theta^{2n\alpha - 1}\alpha^{n_u + n_a}}{(\theta+\eta)^{(n\bar{\pi} - n_u)\alpha}(\theta+\beta\eta)^{(n\pi - n_a)\alpha}} \left[\prod_{i=1}^{n\bar{\pi}} \frac{1}{(\theta+t_i)^{\alpha+1}}\right]^{\delta_{ui}} \left[\prod_{j=1}^{n\pi} \frac{1}{(\theta+\beta x_j)^{\alpha+1}}\right]^{\delta_{aj}}. \tag{8}$$

To obtain the posterior means and posterior variances of β, θ, and α, an approximation due to Lindley [13] is used.

Now, let Θ be a set of parameters $\{\Theta_1, \Theta_2, \ldots, \Theta_m\}$, where m is the number of parameters, then the posterior expectation of an arbitrary function $u(\Theta)$ can be asymptotically estimated by

$$E(u(\Theta)) = \frac{\int_{\Theta} u(\Theta)\pi(\Theta)e^{\ln L(y|\Theta)}d\Theta}{\int_{\Theta} \pi(\Theta)e^{\ln L(y|\Theta)}d\Theta} \approx \left[u + (1/2)\sum_{i,j}(u_{ij}^{(2)} + 2u_i^{(1)}\rho_j^{(1)})\sigma_{ij} + (1/2)\sum_{i,j,k,s} L_{ijk}^{(3)}\sigma_{ij}\sigma_{ks}u_s^{(1)}\right]\downarrow \hat{\Theta}, \tag{9}$$

which is the Bayes estimator of $u(\Theta)$ under a squared error loss function, where $\pi(\Theta)$ is the prior distribution of Θ, $u \equiv u(\Theta)$, $L \equiv L(\Theta)$ is the likelihood function, $\rho \equiv \rho(\Theta) = \log \pi(\Theta)$, σ_{ij} are the elements of the inverse of the asymptotic Fisher's information matrix of β, θ, and α, and

$$u_i^{(1)} = \frac{\partial u}{\partial \Theta_i}, \quad u_{ij}^{(2)} = \frac{\partial^2 u}{\partial\Theta_i \partial\Theta_j}, \quad \rho_j^{(1)} = \frac{\partial \log \pi(\Theta)}{\partial \Theta_j}, \quad \text{and} \quad L_{ijk}^{(3)} = \frac{\partial^3 \ln L(y|\Theta)}{\partial\Theta_i\partial\Theta_j\partial\Theta_k}.$$

According to Green [14], the above posterior expectation is "very good and operational approximation for the ratio of multi-dimension integrals." Also, according to [7], "it has an important applied aspect." Some mathematical details are given at the end of this paper.

3. Monte Carlo Simulation Studies. In this section, we illustrate the use of Bayesian approach via Lindley method for approximation of integrals to derive the marginal posterior moments of interest in the case of constant-stress PALT under type-I censoring. The data are generated from the Pareto distribution with different sample sizes. For each sample size, 5,000 samples are obtained randomly. The posterior means and posterior variances of the three parameters are obtained numerically. In addition, the ML estimators and Bayes estimators are compared with respect to the mean squared errors (MSEs) and variability.

TABLE 1. Results of MLEs and Approximate BEs with Corresponding Estimated Variances and MSEs ($\beta = 3$, $\theta = 0.8$, $\alpha = 0.5$, $\pi = 0.50$, and $\eta = 10$) Using Different Time-Censored Sample Sizes

n	Parameter	Method	Estimate	MSE	Variance
25	β	ML	3.6014	0.0692	0.0372
		Bayes	3.4712	0.0586	0.0295
	θ	ML	1.2431	0.0396	0.0166
		Bayes	0.9374	0.0298	0.0082
	α	ML	0.8211	0.0286	0.0074
		Bayes	0.7855	0.0214	0.0041
50	β	ML	3.3862	0.0509	0.0242
		Bayes	3.2281	0.0389	0.0148
	θ	ML	0.9747	0.0274	0.0096
		Bayes	0.8911	0.0185	0.0051
	α	ML	0.6733	0.0211	0.0033
		Bayes	0.6209	0.0150	0.0015
75	β	ML	3.2911	0.0361	0.0124
		Bayes	3.0766	0.0302	0.0043
	θ	ML	0.8823	0.0201	0.0038
		Bayes	0.8477	0.0155	0.0023
	α	ML	0.5725	0.0048	0.0011
		Bayes	0.5410	0.0019	0.0006
100	β	ML	3.1208	0.0069	0.0025
		Bayes	3.0241	0.0038	0.0023
	θ	ML	0.8351	0.0054	0.0022
		Bayes	0.8126	0.0030	0.0008
	α	ML	0.5219	0.0015	0.0005
		Bayes	0.5046	0.0013	0.0003

To assess and compare the performance of the MLEs and proposed estimators with the Lindley method, we perform simulation comparisons with data generated via various scenarios. Four numerical examples are provided with equal and unequal proportions of allocation for illustration. One of the considered populations is set the combination of (β, θ, α) at (3, 0.8, 0.5) with equal proportion of allocation $\pi = 0.50$ with results reported in Table 1. A second combination is set at (2, 1.2, 1.5) using also equal proportion of allocation ($\pi = 0.50$) with results shown in Table 2. While the third combination is taken as (3, 0.8, 0.5) based on unequal proportion of allocation ($\pi = 0.30$) with numerical results reported in Table 3. Concerning the fourth scenario, the combination is (2, 1.2, 1.5) based on proportion of allocation set at $\pi = 0.70$ with numerical results displayed in Table 4.

Also, concerning the comparison between Bayesian estimators and the likelihood ones, the results have the same trend when unequal proportions of allocation are used. But, with larger proportion of allocation to the accelerated condition, it is noticed that Lindley method is much better than the likelihood-based method.

4. Some Main Remarks and Further Studies. In this paper both ML and Bayes estimations of the CSPALT model parameters have been presented using time-censored samples from Pareto distribution. The Bayes estimators have been considered under the assumptions of squared error loss functions and non-informative priors. Lindley's technique has been used to obtain the Bayesian estimates numerically. It has been found that the technique works very well even for small sample sizes. Also, it has been noted that Lindley's technique frequently produces posterior variances smaller than the variances of the maximum likelihood estimators. So, it gives efficient estimates. As a future work, a Bayesian analysis via another approximation such as the Laplace approximation method or the Markov chain Monte Carlo (MCMC) algorithm will be discussed.

TABLE 2. Results of MLEs and Approximate BEs with Corresponding Estimated Variances and MSEs ($\beta = 2$, $\theta = 1.2$, $\alpha = 1.5$, $\pi = 0.50$, and $\eta = 10$) Using Different Time-Censored Sample Sizes

n	Parameter	Method	Estimate	MSE	Variance
25	β	ML	2.5233	0.0436	0.0212
		Bayes	2.4072	0.0369	0.0168
	θ	ML	1.4371	0.0249	0.0095
		Bayes	1.3642	0.0188	0.0047
	α	ML	1.7648	0.0180	0.0042
		Bayes	1.6427	0.0135	0.0023
50	β	ML	2.3977	0.0321	0.0138
		Bayes	2.3104	0.0245	0.0084
	θ	ML	1.2894	0.0173	0.0055
		Bayes	1.2380	0.0117	0.0029
	α	ML	1.5876	0.0133	0.0019
		Bayes	1.5392	0.0095	0.0009
75	β	ML	2.1247	0.0227	0.0071
		Bayes	2.0486	0.0192	0.0025
	θ	ML	1.2432	0.0127	0.0022
		Bayes	1.2211	0.0098	0.0013
	α	ML	1.5333	0.0032	0.0006
		Bayes	1.5104	0.0012	0.0002
100	β	ML	2.0394	0.0043	0.0014
		Bayes	2.0113	0.0024	0.0011
	θ	ML	1.2117	0.0034	0.0013
		Bayes	1.2021	0.0019	0.0005
	α	ML	1.5102	0.0009	0.0003
		Bayes	1.5002	0.0008	0.0001

TABLE 3. Results of MLEs and Approximate BEs with Corresponding Estimated Variances and MSEs ($\beta = 3$, $\theta = 0.8$, $\alpha = 0.5$, $\pi = 0.30$, and $\eta = 10$) Using Different Time-Censored Sample Sizes

n	Parameter	Method	Estimate	MSE	Variance
1	2	3	4	5	6
25	β	ML	3.9125	0.0985	0.0502
		Bayes	3.5824	0.0779	0.0398
	θ	ML	1.4729	0.0675	0.0224
		Bayes	1.2366	0.0533	0.0111
	α	ML	1.0781	0.0492	0.0102
		Bayes	1.0262	0.0346	0.0055
50	β	ML	3.4521	0.0665	0.0327
		Bayes	3.3217	0.0492	0.0203
	θ	ML	1.2107	0.0477	0.0131
		Bayes	1.1638	0.0314	0.0069
	α	ML	0.9658	0.0311	0.0045
		Bayes	0.9104	0.0294	0.0026

1	2	3	4	5	6
75	β	ML	3.2982	0.0431	0.0167
		Bayes	3.2290	0.0378	0.0058
	θ	ML	1.0726	0.0287	0.0051
		Bayes	1.0179	0.0212	0.0031
	α	ML	0.8721	0.0113	0.0015
		Bayes	0.7913	0.0102	0.0008
100	β	ML	3.1876	0.0094	0.0034
		Bayes	3.1155	0.0057	0.0031
	θ	ML	0.9857	0.0088	0.0030
		Bayes	0.9274	0.0067	0.0011
	α	ML	0.7119	0.0052	0.0007
		Bayes	0.6781	0.0034	0.0004

TABLE 4. Results of MLEs and Approximate BEs with Corresponding Estimated Variances and MSEs ($\beta = 2,\ \theta = 1.2,\ \alpha = 1.5,\ \pi = 0.70$, and $\eta = 10$) Using Different Time-Censored Sample Sizes

n	Parameter	Method	Estimate	MSE	Variance
25	β	ML	2.4113	0.0379	0.0187
		Bayes	2.3271	0.0321	0.0148
	θ	ML	1.3570	0.0217	0.0084
		Bayes	1.3111	0.0164	0.0041
	α	ML	1.7142	0.0157	0.0037
		Bayes	1.6281	0.0117	0.0021
50	β	ML	2.2915	0.0279	0.0121
		Bayes	2.2681	0.0213	0.0074
	θ	ML	1.2270	0.0151	0.0048
		Bayes	1.1860	0.0102	0.0026
	α	ML	1.5852	0.0116	0.0017
		Bayes	1.5472	0.0083	0.0008
75	β	ML	2.0844	0.0197	0.0062
		Bayes	2.0352	0.0167	0.0022
	θ	ML	1.2130	0.0112	0.0019
		Bayes	1.1941	0.0085	0.0012
	α	ML	1.5318	0.0028	0.0005
		Bayes	1.5009	0.0014	0.0002
100	β	ML	2.0102	0.0037	0.0012
		Bayes	2.0087	0.0021	0.0010
	θ	ML	1.2024	0.0032	0.0011
		Bayes	1.2007	0.0017	0.0004
	α	ML	1.5001	0.0007	0.0002
		Bayes	1.5000	0.0005	0.0001

APPENDIX (Derivation of Posterior Means and Posterior Variances):

Here, there are three parameters in the model. That is, $m = 3$. Let the subscripts 1, 2, and 3 refer to β, θ, and α, respectively. It is not easy to obtain the posterior moments analytically. Therefore, using the Lindley expansion, the posterior mean (i.e., Bayesian estimator under squared-error loss function) and the posterior variance of β are given, respectively, in the form

$$\beta^* = E(\beta | y) = \left[\beta - \left(\frac{\sigma_{11}}{\beta} + \frac{\sigma_{12}}{\theta} + \frac{\sigma_{13}}{\alpha}\right) + \frac{1}{2}(\sigma_{11}E_1 + \sigma_{12}E_2 + \sigma_{13}E_3)\right] \downarrow \hat{\Theta}, \tag{A1}$$

and

$$\operatorname{var}(\beta | y) = E(\beta^2 | y) - (\beta^*)^2 = \sigma_{11} - \left[\left(\frac{\sigma_{11}}{\beta} + \frac{\sigma_{12}}{\theta} + \frac{\sigma_{13}}{\alpha}\right) - \frac{1}{2}(\sigma_{11}E_1 + \sigma_{12}E_2 + \sigma_{13}E_3)\right]^2 \downarrow \hat{\Theta}. \tag{A2}$$

Applying the same technique, the posterior mean and posterior variance of the scale parameter θ take the following form:

$$\theta^* = E(\theta | y) = \left[\theta - \left(\frac{\sigma_{21}}{\beta} + \frac{\sigma_{22}}{\theta} + \frac{\sigma_{23}}{\alpha}\right) + \frac{1}{2}(\sigma_{21}E_1 + \sigma_{22}E_2 + \sigma_{23}E_3)\right] \downarrow \hat{\Theta}, \tag{A3}$$

and

$$\operatorname{var}(\theta | y) = \sigma_{22} - \left[\left(\frac{\sigma_{21}}{\beta} + \frac{\sigma_{22}}{\theta} + \frac{\sigma_{23}}{\alpha}\right) - \frac{1}{2}(\sigma_{21}E_1 + \sigma_{22}E_2 + \sigma_{23}E_3)\right]^2 \downarrow \hat{\Theta}. \tag{A4}$$

Similarly, for the shape parameter α, the posterior mean and the posterior variance are given by

$$\alpha^* = E(\alpha | y) = \left[\alpha - \left(\frac{\sigma_{31}}{\beta} + \frac{\sigma_{32}}{\theta} + \frac{\sigma_{33}}{\alpha}\right) + \frac{1}{2}(\sigma_{31}E_1 + \sigma_{32}E_2 + \sigma_{33}E_3)\right] \downarrow \hat{\Theta}, \tag{A5}$$

and

$$\operatorname{var}(\alpha | y) = \sigma_{33} - \left[\left(\frac{\sigma_{31}}{\beta} + \frac{\sigma_{32}}{\theta} + \frac{\sigma_{33}}{\alpha}\right) - \frac{1}{2}(\sigma_{31}E_1 + \sigma_{32}E_2 + \sigma_{33}E_3)\right]^2 \downarrow \hat{\Theta}, \tag{A6}$$

where

$$E_1 = \sum_{i,j} \sigma_{ij} L_{ij1}^{(3)}, \qquad E_2 = \sum_{i,j} \sigma_{ij} L_{ij2}^{(3)}, \qquad E_3 = \sum_{i,j} \sigma_{ij} L_{ij3}^{(3)},$$

for $i, j = 1, 2, 3$, σ_{ij} are the elements of the inverse of the asymptotic Fisher-information matrix of the ML estimators of β, θ, and α in the case of type-I censored data and $i, j = 1, 2, 3$, is the third derivatives of the $L_{ijk}^{(3)}$ natural logarithm of the likelihood function in type-I censoring.

To compute the posterior means and the posterior variances of β, θ, and α derived before, both second and third derivatives of the natural logarithm of the likelihood function in (7) must be got.

The second derivatives can be given by the following equations:

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{n_a}{\beta^2} + \frac{(n\pi - n_a)\alpha\eta^2}{(\theta + \beta\eta)^2} + (\alpha + 1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j^2}{(\theta + \beta x_j)^2}, \tag{A7}$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial \theta} = \frac{(n\pi - n_a)\alpha\eta}{(\theta + \beta\eta)^2} + (\alpha + 1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{(\theta + \beta x_j)^2}, \tag{A8}$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial\alpha} = -\frac{(n\pi - n_a)\eta}{\theta + \beta\eta} - \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{\theta + \beta x_j}, \tag{A9}$$

$$\frac{\partial^2 \ln L}{\partial\theta^2} = -\frac{n\alpha}{\theta^2} + \frac{(n\pi - n_a)\alpha}{(\theta + \beta\eta)^2} + \frac{(n\bar{\pi} - n_u)\alpha}{(\theta + \eta)^2} + (\alpha + 1)\left[\sum_{i=1}^{n\bar{\pi}} \frac{\partial_{ui}}{(\theta + t_i)^2} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{(\theta + \beta x_j)^2}\right], \tag{A10}$$

$$\frac{\partial^2 \ln L}{\partial\theta\partial\alpha} = \frac{n}{\theta} - \frac{(n\pi - n_a)}{\theta + \beta\eta} - \frac{(n\bar{\pi} - n_u)}{\theta + \eta} - \left[\sum_{i=1}^{n\bar{\pi}} \frac{\delta_{ui}}{\theta + t_i} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{\theta + \beta x_j}\right], \tag{A11}$$

$$\frac{\partial^2 \ln L}{\partial\alpha^2} = -\frac{n_u + n_a}{\alpha^2}. \tag{A12}$$

For the third derivatives, they are given as follows:

$$L_{111}^{(3)} = \frac{\partial^3 \ln L}{\partial\beta^3} = \frac{2n_a}{\beta^3} - \frac{2(n\pi - n_a)\alpha\eta^3}{(\theta + \beta\eta)^3} - 2(\alpha + 1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j^3}{(\theta + \beta x_j)^3}, \tag{A13}$$

$$L_{222}^{(3)} = \frac{\partial^3 \ln L}{\partial\theta^3} = \frac{2n\alpha}{\theta^3} - \frac{2(n\pi - n_a)\alpha}{(\theta + \beta\eta)^3} - \frac{2(n\bar{\pi} - n_u)\alpha}{(\theta + \eta)^3} - 2(\alpha + 1)\left[\sum_{i=1}^{n\bar{\pi}} \frac{\delta_{ui}}{(\theta + t_i)^3} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{(\theta + \beta x_j)^3}\right], \tag{A14}$$

$$L_{333}^{(3)} = \frac{\partial^3 \ln L}{\partial\alpha^3} = \frac{2(n_u + n_a)}{\alpha^3}, \tag{A15}$$

$$L_{112}^{(3)} = \frac{\partial^3 \ln L}{\partial\beta^2\partial\theta} = -\frac{2(n\pi - n_a)\alpha\eta^2}{(\theta + \beta\eta)^3} - 2(\alpha + 1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j^2}{(\theta + \beta x_j)^3} = L_{121}^{(3)} = L_{211}^{(3)}, \tag{A16}$$

$$L_{221}^{(3)} = \frac{\partial^3 \ln L}{\partial\theta^2\partial\beta} = -\frac{2(n\pi - n_a)\alpha\eta}{(\theta + \beta\eta)^3} - 2(\alpha + 1)\sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{(\theta + \beta x_j)^3} = L_{212}^{(3)} = L_{122}^{(3)}, \tag{A17}$$

$$L_{113}^{(3)} = \frac{\partial^3 \ln L}{\partial\beta^2\partial\alpha} = \frac{(n\pi - n_a)\eta^2}{(\theta + \beta\eta)^2} + \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j^2}{(\theta + \beta x_j)^2} = L_{131}^{(3)} = L_{311}^{(3)}, \tag{A18}$$

$$L_{123}^{(3)} = \frac{\partial^3 \ln L}{\partial\beta\partial\theta\partial\alpha} = \frac{(n\pi - n_a)\eta}{(\theta + \beta\eta)^2} + \sum_{j=1}^{n\pi} \delta_{aj} \frac{x_j}{(\theta + \beta x_j)^2} = L_{132}^{(3)} = L_{213}^{(3)} = L_{231}^{(3)} = L_{312}^{(3)} = L_{321}^{(3)}, \tag{A19}$$

$$L_{223}^{(3)} = \frac{\partial^3 \ln L}{\partial\theta^2\partial\alpha} = -\frac{n}{\theta^2} + \frac{(n\pi - n_a)}{(\theta + \beta\eta)^2} + \frac{(n\bar{\pi} - n_u)}{(\theta + \eta)^2} + \sum_{i=1}^{n\bar{\pi}} \frac{\delta_{ui}}{(\theta + t_i)^2} + \sum_{j=1}^{n\pi} \frac{\delta_{aj}}{(\theta + \beta x_j)^2} = L_{232}^{(3)} = L_{322}^{(3)}, \tag{A20}$$

$$L_{331}^{(3)} = \frac{\partial^3 \ln L}{\partial\alpha^2\partial\beta} = L_{313}^{(3)} = L_{133}^{(3)}, \tag{A21}$$

$$L_{332}^{(3)} = \frac{\partial^3 \ln L}{\partial\alpha^2\partial\theta} = L_{323}^{(3)} = L_{233}^{(3)}. \tag{A22}$$

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