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ABSTRACT

For highly reliable products, a progressive stress accelerated life test has been proposed to obtain timely information of the product's lifetime distribution. This article considers a progressive stress partially accelerated life test model when the lifetime of a product under use condition follows Weibull distribution. It is assumed that the progressive stress is directly proportional to time. The statistical properties of the maximum likelihood (ML) estimators of the model parameters such as existence, uniqueness, and invariance are studied. The biases and mean square errors of the maximum likelihood estimators are computed to assess their performances in the presence of the stress method developed in this article through a Monte Carlo simulation study.

Keywords

partially accelerated life tests, progressive stress, Weibull distribution, exponential distribution, maximum likelihood estimator, Type-I censoring, existence, uniqueness, invariance

Nomenclature

a, b = parameters of function that relates rate parameter and stress
$ALTs$ = accelerated life tests
K = slope of linear progressive stress (stress rate, a some pre-assigned positive constant)
K_1, K_2 = design and high stress rates
$MLEs$ = maximum likelihood estimates/estimators
$MTTF$ = mean time to failure
n = number of test units (sample size)
n_a = number of failures at accelerated condition
n_c = number of non-observed (censored) units

n_u = number of failures at use condition
$PALTs$ = partially accelerated life tests
PS = progressive stress
q = number of stress levels
S = stress
S_i = stress of level i, $i = 1, 2, \ldots, q$
$SSPALTs$ = step-stress partially accelerated life tests
t_i = observed lifetime of unit i tested under PSPALTs
WD = Weibull distribution
α = WD shape parameter
β = acceleration factor ($\beta > 1$)
$\varepsilon = \max_i (t_i - \tau_i)$
λ = WD rate parameter (inverse of scale parameter)
η = censoring time
θ = WD scale parameter ($\theta = 1/\lambda$)
τ = stress change time
τ_i = time at which the stress goes from S_i to S_{i+1}
$\hat{}$ = implies a maximum likelihood estimate

Introduction

Accelerated life tests of a product with high reliability under severer than design conditions involving high temperature, voltage, vibration, cycle rate, load, etc., are commonly used to reduce test time and cost. Practitioners using accelerated life tests (ALTs) generally assume a relationship between the stress level and the corresponding life distribution according to the failure mechanism of the particular test. This relationship is usually referred to as the *time transformation function*. Interested readers can refer to Meeker and Escobar [1] and Nelson [2–4] which are comprehensible sources for ALTs. When such a function is not known or cannot be assumed, ALTs cannot be applied and other tests, namely partially accelerated life tests (PALTs), come to be used instead of ALTs. In ALTs, the test items are run only under high stress. However, in PALTs, the test items are run under both design- and high-stresses.

The stress can be applied in various ways. According to Yin and Sheng [5], usually there are three kinds of stress: constant, step, and progressive. Under constant-stress PALTs, each item is run at either use condition or accelerated condition only. That is, each unit is run at a steady stress (constant-stress level) until the test is terminated or the unit fails. However, in the case of step-stress PALTs, a test item is first run at use condition and, if it does not fail for a pre-specified time τ, then it is run at accelerated condition until it fails or the test is terminated. Progressive stress (PS) is similar to step stress, but the stress on items is a progressive (continuous) function. Statistical theory for progressive stress ALTs has been studied by some authors. For example, Mann et al. [6] and Yin and Sheng [5] discussed this kind of testing theoretically when the underlying life distributions are exponential. A statistical model for items having exponential lives under progressive stress was proposed former by Allen [7]. Abdel-Hamid and Al-Hussaini [8] presented progressive stress accelerated life tests under finite mixture models. Moreover, Abdel-Hamid and Al-Hussaini [9] considered progressive stress accelerated life tests when the lifetime of an item under use condition follows the Weibull distribution (WD) with a scale parameter satisfying the inverse power law.

This article is concerned with discussing PALTs by PS when the PS is directly proportional to time. That is, the stress is a linearly increasing function in time. The objective of PALTs is to collect more failure data in a limited time without necessarily using high stresses to all test units. According to Lin and Fei [10], testing time can be further shortened by progressive stress. They considered a nonparametric approach to progressive stress accelerated life tests (PSALTs). Specifically, all published works on PALTs had been considered under the two traditional types of stress: constant and step. For example, see Goel [11], DeGroot and Goel [12], Bai and Chung [13], Bai et al. [14], Ismail [15], Abdel-Hamid and Al-Hussaini [16], Aly and Ismail [17], Ismail [18], Ismail and Aly [19], Ismail [20–23], Srivastava and Mittal [24], Tahir [25], and Bhattacharyya and Soejoeti [26]. Now, the present work will concentrate on PALTs with PS. The idea of using PALTs under PS is a new one. The maximum likelihood (ML) estimators of the model parameters are derived and their statistical properties such as existence, uniqueness and invariance are investigated.

The rest of this article is organized as follows: in the second section, the test procedure and its assumptions used throughout the article are presented and the model is also described. In the third section, ML equations under PSPALTs are outlined to get the ML estimates of the model parameters. The fourth section discusses some statistical properties of the estimates. The fifth section provides some simulation results about the performance of the ML estimators. In the final section, we conclude the article and present some future ideas.

Test Assumptions and Model

BASIC ASSUMPTIONS AND TEST PROCEDURE

The following assumptions are used throughout the paper in the framework of PSPALTs:

1. For design stress, the lifetime distribution is assumed to be Weibull distribution. Under high stresses, the lifetime distribution is of the same type.
2. The failure mechanisms of a test unit are the same at any level of stress.
3. Progressive stress is directly proportional to time (the stress is a linearly increasing function of time).
4. The testing is type-i censored sample testing.

5. The rate parameter of Weibull distribution (inverse of the scale parameter) and stress are related as $\lambda = aS^b$. The stress can be as $S(t) = Kt$, where $a > 0$, $b > 0$, and $K > 0$. Progressive stress during stress level i ($i = 1, 2$) is expressed by $S_i(t) = K_i t$, with K_i pre-assigned positive constants, $K_1 < K_2$. Simple progressive stress test is assumed, i.e., we have only two stress levels which are design stress and high stress. Let n units be tested under the progressive stress $S_i(t) = K_i t$, $i = 1,2$ for a pre-assigned censoring time η. Often the over-stressing level at which the failure mechanism changes can be roughly estimated. Thus, we can avoid using failure data obtained at overstressing level by choosing proper censoring times.

The test procedure is as follows. n test units are tested under a linearly increasing stress condition. Each of the n test units is first run under use condition with stress rate K_1. If it does not fail by a pre-specified timeτ, it is run under accelerated condition with stress rate K_2 until it fails or it is censored.

THE WEIBULL DISTRIBUTION AS A FAILURE TIME MODEL

This article is concerned with two-parameter Weibull distribution, which is widely employed as a model in life testing because of the many shapes it attains for various values of the shape parameter. It can therefore model a great variety of data and life characteristics. The probability density function (pdf) of a two-parameter Weibull distribution is given by:

$$f_T(t;\alpha,\theta) = \frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1}\exp\{-(t/\theta)^{\alpha}\}; \quad t > 0, \alpha > 0, \theta > 0 \tag{1}$$

The Weibull reliability function takes the form

$$R(t) = \exp\{-(t/\theta)^{\alpha}\} \tag{2}$$

and the corresponding failure rate function is given by:

$$h(t) = \frac{\alpha}{\theta}\left(\frac{t}{\theta}\right)^{\alpha-1} \tag{3}$$

Therefore, the cumulative distribution function (CDF) is given by

$$F(t) = 1 - \exp[-(\lambda t)^{\alpha}], \quad t \geq 0 \tag{4}$$

where λ is the rate parameter of Weibull distribution (inverse of the scale parameter).

Then, according to Yin and Sheng [5] the CDF with design stress rate k_1 based on assumption 5 is given by

$$F(t) = 1 - \exp\{-[a\,k_1^b\,t^{b+1}]^{\alpha}\}, \quad t \geq 0 \tag{5}$$

which can be easily rewritten as

$$F(t) = 1 - \exp\{-[(a\,k_1^b)^{1/(b+1)}t]^{(b+1)\alpha}\}, \quad t \geq 0 \tag{6}$$

The CDF in Eq 6 under linear PS is the Weibull distribution with new rate and shape parameters:

$$\breve{\lambda} = (a\,k_1^b)^{1/(b+1)} \quad \text{and} \quad \breve{\alpha} = (b+1)\alpha$$

As a special case of Weibull distribution, when $\alpha = 1$, the exponential distribution is obtained.

Parameters Estimation

This section discusses the process of obtaining the ML estimates of the model parameters based on PSPALTs with Type-I censored data. According to DeGroot and Goel [12], the lifetime of a unit under step-stress PALT (SSPALT) can be written as

$$Y = \begin{cases} T & \text{if } T \leq \tau \\ \tau + \beta^{-1}(T - \tau) & \text{if } T > \tau \end{cases} \tag{7}$$

where:

T = the lifetime of the unit under use condition,

τ = the stress change time, and

β = the acceleration factor, $\beta > 1$.

This model is called the tampered random variable (TRV) model.

As mentioned by Yin and Sheng [5], progressive stress is similar to step stress, but the stress on specimens is a progressive (continuous) function. In addition, they said that constant stress and step stress are particular cases of progressive stress.

According to Yin and Sheng [5], for any progressive stress $s(t)$, there is a step stress

$$\tilde{s}(t) = s(\tau_i) \quad \tau_{i-1} \leq t < \tau_i, \quad \tau_0 = 0, \quad i = 1, 2, \ldots, q$$

where $\tau_i, i = 1, 2, \ldots, q$ are points in the domain of definition of $s(t)$ and represent the times at which the stress goes from S_i to S_{i+1}.

$\tilde{s}(t)$ is an approximation of $s(t)$. Thus, $s(t) = \lim_{\varepsilon\to 0}\tilde{s}(t)$ where $\varepsilon \equiv \max_i(t_i - \tau_i)$. That is, the maximum time difference between the needed time, t_i, to increase the stress and the time τ_i at which the stress goes from S_i to S_{i+1} will be very small or tends to zero.

Since the step stress $\tilde{s}(t)$ becomes progressive stress $s(t)$ when $\varepsilon \to 0$, the CDF, $\tilde{F}(t)$, under step stress converges to the CDF, $F(t)$, under progressive stress when $\varepsilon \to 0$. That is, $F(t) = \lim_{\varepsilon\to 0}\tilde{F}(t)$.

Progressive stress during stress level i ($i = 1, 2$) is expressed by $S_i(t) = K_i t$, with K_i pre-assigned positive constants, $K_1 < K_2$. That is, based on the relationship $S_i(t) = K_i t$, $K_1 < K_2$, the units under use condition have stress rate (K_1) different from that (K_2) of the units under accelerated condition. Therefore, the pdf of Y under PSPALTs model can be given by

$$f_Y(t)=\begin{cases}0, & t\le 0\\ f_1(t)=\alpha a\,k_1^b\,t^b\left(a\,k_1^b\,t^{b+1}\right)^{\alpha-1}\exp\{-[a\,k_1^b\,t^{b+1}]^\alpha\}, & 0<t\le\tau\\ f_2(t)=\alpha\beta a\,k_2^b(\beta(t-\tau)+\tau)^b\left(a\,k_2^b(\beta(t-\tau)+\tau)^{b+1}\right)^{\alpha-1}\exp\{-[a\,k_2^b(\beta(t-\tau)+\tau)^{b+1}]^\alpha\}, & t>\tau\end{cases} \quad (8)$$

where $f_1(t)$ is the pdf under use condition and $f_2(t)$ obtained by the transformation-variable technique using $f_1(t)$ and the model given in Eq 7 is the pdf under accelerated condition.

The observed values of the total lifetime under PSPALTs are given by:

$$t_{(1)}\le\ldots\le t_{(n_u)}\le\tau\le t_{(n_u+1)}\le\ldots\le t_{(n_u+n_a)}\le\eta$$

where:

n_u and n_a = the number of items failed at use and accelerated conditions, respectively, and

$t_{(i)}$ = the order statistic realization of t_i based on *i.i.d* random variables. Let δ_{1i} and δ_{2i} be two indicator functions such that $\delta_{1i}\equiv I(t_i\le\tau)$ and $\delta_{2i}\equiv I(\tau<t_i\le\eta)$, $i=1,\ldots,n$.

Since the total lifetimes $Y_1,\ldots,Y_n$ of n units are *i.i.d* r.v.'s, then the total likelihood function for them is given by:

$$\begin{aligned}L(a,b,\alpha,\beta)\propto\prod_{i=1}^{n}&[\alpha a\,k_1^b\,t_i^b\left(a\,k_1^b\,t_i^{b+1}\right)^{\alpha-1}\exp\{-[a\,k_1^b\,t_i^{b+1}]^\alpha\}]^{\delta_{1i}}\\ &\cdot[\alpha\beta a\,k_2^b(\beta(t_i-\tau)+\tau)^b\left(a\,k_2^b(\beta(t_i-\tau)+\tau)^{b+1}\right)^{\alpha-1}\\ &\times\exp\{-[a\,k_2^b(\beta(t_i-\tau)+\tau)^{b+1}]^\alpha\}]^{\delta_{2i}}\\ &\cdot[\exp\{-[a\,k_2^b(\beta(\eta-\tau)+\tau)^{b+1}]^\alpha\}]^{\bar{\delta}_{1i}\bar{\delta}_{2i}}\end{aligned} \quad (9)$$

where $\bar{\delta}_{1i}=1-\delta_{1i}$ and $\bar{\delta}_{2i}=1-\delta_{2i}$.

The natural logarithm of the above likelihood function is given by

$$\begin{aligned}\ln L\propto\,&(n_u+n_a)\ln\alpha+(n_u+n_a)\alpha\ln a+n_a\ln\beta\\ &+(n_u\ln k_1+n_a\ln k_2)\alpha b+((b+1)\alpha-1)\\ &\times\left[\sum_{i=1}^{n_u}\ln t_i+\sum_{i=1}^{n_a}\ln(\beta(t_i-\tau)+\tau)\right]\\ &-a^\alpha\left[k_1^{b\alpha}\sum_{i=1}^{n_u}t_i^{(b+1)\alpha}+k_2^{b\alpha}\sum_{i=1}^{n_a}(\beta(t_i-\tau)+\tau)^{(b+1)\alpha}\right]\\ &-a^\alpha k_2^{b\alpha}n_c(\beta(\eta-\tau)+\tau)^{(b+1)\alpha}\end{aligned} \quad (10)$$

To obtain the MLEs of model parameters, the required derivations are given in Appendix.

Properties of Estimators When $\alpha=1$

The ML estimates of the model parameters for $\alpha=1$ (the exponential distribution case) have the following properties.

$\hat{b}$ AND K_I, I = 1, 2 ARE RELATED BY TEST DATA

From the maximum likelihood equations presented in Appendix, we have

$$(n_u+n_a)+\left(\sum_{i=1}^{n_u}\ln t_i+\sum_{i=1}^{n_a}\ln\psi_i\right)-\frac{(n_u+n_a)\left[\sum_{i=1}^{n_u}t_i^{(\hat{b}+1)}\ln t_i+\sum_{i=1}^{n_a}\psi_i^{(\hat{b}+1)}\ln\psi_i\right]}{\left(\sum_{i=1}^{n_u}t_i^{(\hat{b}+1)}+\sum_{i=1}^{n_a}\psi_i^{(\hat{b}+1)}\right)}=0 \quad (11)$$

where $\psi_i=\beta(t_i-\tau)+\tau$.

$\hat{b}$ can be found without K_i, $i=1,2$. That is, $\hat{b}$ is indirectly related to K_i, $i=1,2$ by test data.

UNIQUENESS OF $\hat{b}$

Let

$$f\equiv\sum_{i=1}^{n_u}\ln t_i+\sum_{i=1}^{n_a}\ln\psi_i \quad (12)$$

and

$$g(x)\equiv\frac{(n_u+n_a)\left[\sum_{i=1}^{n_u}t_i^x\ln t_i+\sum_{i=1}^{n_a}\psi_i^x\ln\psi_i\right]}{\left(\sum_{i=1}^{n_u}t_i^x+\sum_{i=1}^{n_a}\psi_i^x\right)}-\frac{(n_u+n_a)}{x} \quad (13)$$

then,

$$\frac{dg(x)}{dx} = \frac{(n_u+n_a)\left[\sum_{i=1}^{n_u} t_i^x (\ln t_i)^2 + \sum_{i=1}^{n_a} \psi_i^x (\ln \psi_i)^2\right]}{\left(\sum_{i=1}^{n_u} t_i^x + \sum_{i=1}^{n_a} \psi_i^x\right)} - \frac{(n_u+n_a)\left[\left(\sum_{i=1}^{n_u} t_i^x \ln t_i\right)^2 + \left(\sum_{i=1}^{n_a} \psi_i^x \ln \psi_i\right)^2\right]}{\left(\sum_{i=1}^{n_u} t_i^x + \sum_{i=1}^{n_a} \psi_i^x\right)^2} + \frac{(n_u+n_a)}{x^2}$$

$$= \frac{(n_u+n_a)\left[\sum_{i=1}^{n_u}\sum_{j=1}^{n_u} t_i^x t_j^x (\ln t_i - \ln t_j)^2 + \sum_{i=1}^{n_a}\sum_{j=1}^{n_a} \psi_i^x \psi_j^x (\ln \psi_i - \ln \psi_j)^2\right]}{\left(\sum_{i=1}^{n_u} t_i^x + \sum_{i=1}^{n_a} \psi_i^x\right)^2} + \frac{(n_u+n_a)}{x^2}, \quad i<j$$

Since $t_i > 0\ (i = 1, 2, \ldots, n_u + n_a)$, $(dg(x)/dx) > 0$.

That is, $g(x)$ is a strictly monotone increasing function. In addition, it is noted that $\hat{b} > 0$ and if $\hat{b}$ is the solution of Eq 11, then $g(\hat{b}+1) = f$. Therefore, with the strict monotonicity and continuity of $g(x)$ in $[1,\infty)$, we conclude that $\hat{b}$ determined by Eq 11 is unique. Similarly, we can conclude that $\hat{\beta}$ is also unique.

INVARIANCE OF $\hat{b}$

Let $\omega_i = (t_i/\zeta)$, where ζ is an arbitrary constant. Then, substituting $\zeta\omega_i$ for t_i into Eq 11, we can derive

$$(n_u+n_a) + \left(\sum_{i=1}^{n_u} \ln \omega_i + \sum_{i=1}^{n_a} \ln \varphi_i\right) + (n_u+n_a)\ln\zeta - \frac{(n_u+n_a)\left[\sum_{i=1}^{n_u} \omega_i^{(\hat{b}+1)} \zeta^{(\hat{b}+1)} (\ln \omega_i + \ln \zeta) + \sum_{i=1}^{n_a} \varphi_i^{(\hat{b}+1)} \zeta^{(\hat{b}+1)} (\ln \varphi_i + \ln \zeta)\right]}{\left(\sum_{i=1}^{n_u} \omega_i^{(\hat{b}+1)} \zeta^{(\hat{b}+1)} + \sum_{i=1}^{n_a} \varphi_i^{(\hat{b}+1)} \zeta^{(\hat{b}+1)}\right)} = 0$$

where $\varphi_i = \beta(\omega_i - \tau) + \tau$.

That is,

$$(n_u+n_a) + \left(\sum_{i=1}^{n_u} \ln \omega_i + \sum_{i=1}^{n_a} \ln \varphi_i\right) - \frac{(n_u+n_a)\left[\sum_{i=1}^{n_u} \omega_i^{(\hat{b}+1)} \ln \omega_i + \sum_{i=1}^{n_a} \varphi_i^{(\hat{b}+1)} \ln \varphi_i\right]}{\left(\sum_{i=1}^{n_u} \omega_i^{(\hat{b}+1)} + \sum_{i=1}^{n_a} \varphi_i^{(\hat{b}+1)}\right)} = 0 \quad (14)$$

It is noted that Eqs 11 and 14 have the same form. Equation 14 does not include ζ. This means that if the test data are divided (or multiplied) by an arbitrary constant,$\hat{b}$ does not change.

EXISTENCE $\hat{b}$

By definition of $g(x)$

$$g(1) - f = \frac{(n_u+n_a)\left[\sum_{i=1}^{n_u} t_i \ln t_i + \sum_{i=1}^{n_a} \psi_i \ln \psi_i\right]}{\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a} \psi_i} - (n_u+n_a) - \sum_{j=1}^{n_u} \ln t_j - \sum_{j=1}^{n_a} \ln \psi_j$$

$$= \left\{\left[\sum_{i=1}^{n_u}\sum_{j=1}^{n_u} (\ln t_i - \ln t_j)\cdot t_i + \sum_{i=1}^{n_a}\sum_{j=1}^{n_a} (\ln \psi_i - \ln \psi_j)\cdot \psi_i\right] - (n_u+n_a)\left[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a} \psi_i\right]\right\} \Big/ \left[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a} \psi_i\right]$$

$$= \left\{\left[\sum_{i=1}^{n_u}\sum_{j=1}^{n_u} (\ln t_i - \ln t_j)\cdot [t_i - t_j] + \sum_{i=1}^{n_a}\sum_{j=1}^{n_a} (\ln \psi_i - \ln \psi_j)\cdot[\psi_i - \psi_j]\right] - (n_u+n_a)\right.$$

$$\left. \times \left[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a} \psi_i\right]\right\} \Big/ \left[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a} \psi_i\right], \quad i<j$$

If

$$\left[\sum_{i=1}^{n_u}\sum_{j=1}^{n_u}(\ln t_i - \ln t_j).[t_i - t_j] + \sum_{i=1}^{n_a}\sum_{j=1}^{n_a}(\ln \psi_i - \ln \psi_j) \cdot [\psi_i - \psi_j]\right] \geq (n_u + n_a)\left[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a}\psi_i\right], \quad i < j \tag{15}$$

then

$$g(1) \geq f$$

When $g(1) \geq f$, it is certain that $g(x) > f$, $(x > 1)$ according to the strict monotonically and continuity of $g(x)$ in [1, $+\infty$), i.e., Eq 11 cannot provide solution $\hat{b} < 0$ when Eq 15 is true. Inversely, if

$$\left[\sum_{i=1}^{n_u}\sum_{j=1}^{n_u}(\ln t_i - \ln t_j) \cdot [t_i - t_j] + \sum_{i=1}^{n_a}\sum_{j=1}^{n_a}(\ln \psi_i - \ln \psi_j) \cdot [\psi_i - \psi_j]\right] < (n_u + n_a)\left[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a}\psi_i\right], \ i < j$$

then

$$g(1) < f$$

But

$$t_{(n_u+n_a)} \geq t_i, \quad i = 1, 2, 3, \ldots, n_u + n_a - 1$$

and there is at least one strict inequality,

$$\lim_{x \longrightarrow +\infty} g(x) = (n_u + n_a)\ln t_{(n_u+n_a)} > f$$

Thus, according to the continuity of $g(x)$ on $[1+\xi, \gamma)$, where ξ is a small constant and γ is a large constant such that $g(1+\xi) < f$ and $g(\gamma) > f$, there exists an $x^* \in [1+\xi, \gamma]$, which satisfies $g(x^*) = f$ and $\hat{b} = x^* - 1$ is the solution of Eq 11.

Moreover, the necessary and sufficient condition for Eq 14 providing solution $\hat{b} > 0$ is:

$$\left[\sum_{i=1}^{n_u}\sum_{j=1}^{n_u}(\ln t_i - \ln t_j)\cdot[t_i - t_j] + \sum_{i=1}^{n_a}\sum_{j=1}^{n_a}(\ln \psi_i - \ln \psi_j) \cdot [\psi_i - \psi_j]\right] < (n_u + n_a)\left[\sum_{i=1}^{n_u} t_i + \sum_{i=1}^{n_a}\psi_i\right], \quad i < j$$

Data Analysis

In this section, we provide a numerical example and simulation studies are also made to demonstrate the use of the proposed method in this article as shown from the following two subsections.

AN ILLUSTRATIVE EXAMPLE

An illustrative example has been presented to clarify the practical usage of PSPALTs:

To illustrate the use of the proposed methodology in this article a simulated data set from PSPALTs model assuming Weibull distribution with Type-I censoring is investigated as follows.

PSPALTs model assuming Weibull distribution with Type-I censoring is run to estimate both the life distribution parameters and the acceleration factor. We choose $n = 40$, $a = 1.2$, $b = 1.8$, $\alpha = 1.5$, $\beta = 3.9$ given $k_1 = 3$, $k_2 = 9$, with a combination of (τ, η) set to be (5,8). The number of failures observed at use and accelerated conditions are $n_u = 8$ and $n_a = 23$, respectively with censored items $n_c = 9$. The MLEs of the model parameters are $\hat{a} = 0.9231$, $\hat{b} = 1.2189$, $\hat{\alpha} = 1.4352$, and $\hat{\beta} = 3.5659$. Moreover, the biases and MSEs of the MLEs of the model parameters a, b, α, and β are (0.0063, 0.0082, 0.0028, 0.0053) and (0.3020, 0.2885, 0.2356, 0.2933), respectively.

In practice, PSPALTs are easier to implement and have many advantages include:

(1) *Time saving*: PSPALTs can substantially shorten the duration of the test without affecting the accuracy of lifetime distribution estimates.
(2) *Economical*: Testing units under PSPALTs can reduce the costs of experiments because not all test units are run at higher stresses.
(3) *Adaptable*: PSPALTs are flexible test strategy, especially for new products when one presumably has little information regarding appropriate test stresses. In such situations, it may not be easy for the experimenter to determine suitable test stress levels. In simple PALTs, the second stress level, as well as the transition time, could be dynamically adjusted as failure information is being gathered under the first stress level.

SOME SIMULATION RESULTS

In this section simulation studies are conducted to discuss the performance of the MLEs in terms of their biases and mean square errors (MSEs) for different choices of α, β, a, b, τ, K_1,K_2, and η values.

The simulation study is carried out according to the following algorithm:

1. Specify the values of n,τ K_1, K_2, and η
2. Specify the values of the parameters α, β, a, and b.
3. Generate a random sample of size n from the random variable Y given by Eq 7 and sort it. The Weibull random variable can be easily generated. For example, if U represents a uniform random variable from [0,1], then $Y = \left[-[\ln(1-U)]^{1/\alpha}/a\,k_1^b\right]^{1/(b+1)}$ has Weibull distribution with pdf $f_1(t)$ as given by Eq 8 if $t \leq \tau$. However, if $t > \tau$ then $Y = \tau + \beta^{-1}\{[-[\ln(1-U)]^{1/\alpha}/a\,k_2^b]^{1/(b+1)} - \tau\}$ has Weibull distribution with pdf $f_2(t)$ as given by Eq 8.
4. Use the model given by Eq 8 to generate Type-I censored data for a given n,τ, K_1,K_2, η, α, β, a, and b.
5. Use the Type-I censored data to compute the MLEs of the model parameters. Newton–Raphson method is applied for solving the nonlinear equations given by Eqs A2–A4

TABLE 1 Average values of the MLEs, biases (between brackets) and MSEs, when a, b, α, and β set at 1.2, 1.8, 1.5, and 3.9, respectively, with $K_1 = 3$ and $K_2 = 9$.

n	*Parameters* (τ, η)	(5, 8)	(7, 8)	(5, 11)
30	*a*	0.8229 (0.0073) 0.3076	0.8915 (0.0100) 0.4091	0.8311 (0.0063) 0.2676
	b	1.0965 (0.0045) 0.2982	1.1879 (0.0062) 0.3966	1.1075 (0.0037) 0.2594
	α	1.4076 (0.0032) 0.2504	1.5249 (0.0044) 0.3331	1.4217 (0.0026) 0.2179
	β	3.4860 (0.0058) 0.3035	3.7765 (0.0080) 0.4036	3.5209 (0.0045) 0.2640
50	*a*	1.0233 (0.0053) 0.2964	1.1086 (0.0073) 0.3942	1.0335 (0.0042) 0.2425
	b	1.3413 (0.0037) 0.2787	1.4531 (0.0051) 0.3706	1.3547 (0.0031) 0.1746
	α	1.4628 (0.0023) 0.2207	1.5847 (0.0031) 0.2669	1.4774 (0.0018) 0.1563
	β	3.6459 (0.0047) 0.2831	3.9497 (0.0065) 0.3765	3.6824 (0.0038) 0.2025
75	*a*	1.1556 (0.0041) 0.1753	1.2519 (0.0056) 0.2331	1.1672 (0.0035) 0.1355
	b	1.6173 (0.0034) 0.1557	1.7521 (0.0047) 0.2071	1.6335 (0.0029) 0.1178
	α	1.4811 (0.0016) 0.1355	1.6045 (0.0022) 0.1801	1.4959 (0.0012) 0.0866
	β	3.7974 (0.0027) 0.1682	4.1139 (0.0038) 0.2237	3.8354 (0.0023) 0.1163
100	*a*	1.2081 (0.0020) 0.0996	1.3088 (0.0028) 0.1325	1.2001 (0.0017) 0.0787
	b	1.7892 (0.0018) 0.0905	1.9383 (0.0025) 0.1204	1.8006 (0.0015) 0.0679
	α	1.5066 (0.0010) 0.0781	1.6322 (0.0015) 0.1039	1.5010 (0.0008) 0.0535
	β	3.8928 (0.0024) 0.0959	4.2172 (0.0033) 0.1275	3.9003 (0.0019) 0.0782

to obtain the MLEs of the parameters *b*, α, and β. Then one can easily obtain the value of *a* using Eq A5. Accordingly, λ is determined from the relation $\lambda = aS^b$ to obtain the estimated value of the parameter θ, where $\theta = 1/\lambda$.

6. Replicate the steps 3–5 10 000 times.
7. Compute the average values of biases and MSEs associated with the MLEs of the parameters.
8. Steps 1–7 are done with different values of n,τ, K_1,K_2, η, α, β, a, and b.
9. Conducting the above algorithm, the average values of biases and MSEs are obtained using 10 000 replications to avoid randomness. The results are reported in **Tables 1** and **2** based on different values of n, τ, K_1, K_2, η, α, β, a, and b to investigate the performance of the MLEs of the model parameters.

From **Tables 1** and **2**, the following observations can be made:

1. For fixed τ and η, the MSEs decrease as n increases.
2. For fixed τ and n, the MSEs decrease as η increases.
3. For fixed η and n, the MSEs are to be larger when τ is getting to be large and $\alpha > 1$.

TABLE 2 Average values of the MLEs, biases (between brackets) and MSEs, when a, b, α, and β set at 1.2, 1.8, 0.6, and 3.9, respectively, with $K_1 = 3$ and $K_2 = 9$.

n	*Parameters* (τ, η)	(5, 8)	(7, 8)	(5, 11)
30	*a*	0.8558 (0.0071) 0.2981	0.8887 (0.0079) 0.3812	0.8403 (0.0061) 0.2272
	b	1.1404 (0.0042) 0.2774	1.1842 (0.0053) 0.3655	1.1196 (0.0034) 0.2134
	α	0.4639 (0.0030) 0.2406	0.5202 (0.0039) 0.3108	0.5273 (0.0023) 0.1863
	β	3.6254 (0.0055) 0.2910	3.7649 (0.0073) 0.3865	3.5596 (0.0041) 0.2308
50	*a*	1.0642 (0.0051) 0.2732	1.1052 (0.0064) 0.3750	1.0449 (0.0039) 0.2124
	b	1.3949 (0.0034) 0.2644	1.4486 (0.0047) 0.3599	1.3696 (0.0028) 0.1489
	α	0.5213 (0.0020) 0.1987	0.5798 (0.0027) 0.2432	0.5437 (0.0014) 0.1107
	β	3.7917 (0.0045) 0.2635	3.9376 (0.0052) 0.3604	3.7228 (0.0033) 0.1828
75	*a*	1.2018 (0.0038) 0.1654	1.2480 (0.0046) 0.2217	1.1810 (0.0032) 0.1141
	b	1.6820 (0.0031) 0.1361	1.7467 (0.0041) 0.1946	1.6514 (0.0025) 0.0875
	α	0.5403 (0.0012) 0.1104	0.5996 (0.0020) 0.1487	0.5624 (0.0009) 0.0551
	β	3.9493 (0.0024) 0.1472	4.1012 (0.0035) 0.2106	3.8775 (0.0021) 0.0867
100	*a*	1.2564 (0.0017) 0.0811	1.3047 (0.0026) 0.1165	1.2336 (0.0013) 0.0598
	b	1.8608 (0.0015) 0.0786	1.9323 (0.0024) 0.1129	1.8270 (0.0011) 0.0494
	α	0.5869 (0.0007) 0.0517	0.6271 (0.0014) 0.0688	0.5984 (0.0005) 0.0354
	β	4.0485 (0.0021) 0.0782	4.2042 (0.0031) 0.1061	3.9749 (0.0013) 0.0527

4. For fixed η and n, the MSEs are to be slightly larger when τ is getting to be large and $\alpha < 1$.
5. It is observed that the MLEs of the model parameters are very close to the true values as n increases.

The same pattern is observed for the biases. That is, under the proposed model developed in this paper, we have obtained good estimates for the parameters of interest.

Conclusion

In this paper, we have studied PALTs with PS when the PS is directly proportional to time. That is, the stress is a linearly increasing function with time. The PS test pattern is more effective in time and money compared with constant- or step-stress. For highly reliable products, PSPALTs have been proposed to obtain timely information of the product's lifetime distribution. We have discussed some statistical properties of the MLEs such as uniqueness, invariance, and existence under PSPALTs assuming the Weibull distribution. As a future work, the progressive stress testing wherein the stress on every item is increased continuously in a non-linear pattern with time will be considered.

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Appendix

The MLEs of a, b, α, and β can be obtained by solving the following likelihood equations:

$$\frac{\partial \ln L}{\partial a} = \frac{(n_u + n_a)\alpha}{a} - a^{\alpha-1}\alpha\left[k_1^{b\alpha}\sum_{i=1}^{n_u} t_i^{(b+1)\alpha} + k_2^{b\alpha}\sum_{i=1}^{n_a}\psi_i^{(b+1)\alpha} + n_c k_2^{b\alpha}\psi_\eta^{(b+1)\alpha}\right] = 0 \tag{A1}$$

where $\psi_i = \beta(t_i - \tau) + \tau$ and $\psi_\eta = \beta(\eta - \tau) + \tau$

$$\frac{\partial \ln L}{\partial b} = (n_u \ln k_1 + n_a \ln k_2)\alpha + \alpha\left[\sum_{i=1}^{n_u}\ln t_i + \sum_{i=1}^{n_a}\ln\psi_i\right] - \alpha a^{\alpha}\left\{\left(k_1^{b\alpha}(\ln k_1)\sum_{i=1}^{n_u} t_i^{(b+1)\alpha} + k_2^{b\alpha}(\ln k_2)\sum_{i=1}^{n_a}\psi_i^{(b+1)\alpha}\right) + \sum_{i=1}^{n_u} t_i^{(b+1)\alpha}\ln t_i + \sum_{i=1}^{n_a}\psi_i^{(b+1)\alpha}\ln\psi_i + \psi_\eta^{(b+1)\alpha} n_c(\ln k_2 + \ln\psi_\eta)\right\} = 0 \tag{A2}$$

$$\frac{\partial \ln L}{\partial \alpha} = (n_u \ln k_1 + n_a \ln k_2)(1/\alpha + \ln a + b) + (b+1)\left[\sum_{i=1}^{n_u}\ln t_i + \sum_{i=1}^{n_a}\ln\psi_i\right] - a^{\alpha}\left\{(\ln a + b)\left[k_1^{b\alpha}(\ln k_1) \times \sum_{i=1}^{n_u} t_i^{(b+1)\alpha} + k_2^{b\alpha}(\ln k_2)\sum_{i=1}^{n_a}\psi_i^{(b+1)\alpha}\right] + (b+1)\left[\sum_{i=1}^{n_u} t_i^{(b+1)\alpha}\ln t_i + \sum_{i=1}^{n_a}\psi_i^{(b+1)\alpha}\ln\psi_i\right] + n_c\psi_\eta^{(b+1)\alpha}[\ln a + b\ln k_2 + (b+1)\ln\psi_\eta]\right\} = 0 \tag{A3}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n_a}{\beta} + ((b+1)\alpha - 1)\sum_{i=1}^{n_a}\frac{(t_i - \tau)}{\psi_i} - a^{\alpha}(b+1)\alpha\left[k_2^{b\alpha}\sum_{i=1}^{n_a}(t_i - \tau)\psi_i^{(b+1)\alpha-1} + n_c k_2^{b\alpha}(\eta - \tau)\psi_\eta^{(b+1)\alpha-1}\right] = 0 \tag{A4}$$

From Eq A1, the MLE of a can be obtained as

$$\hat{a} = \left(\frac{n_u + n_a}{\left[k_1^{b\alpha}\sum_{i=1}^{n_u} t_i^{(\hat{b}+1)\hat{\alpha}} + k_2^{b\alpha}\sum_{i=1}^{n_a}\psi_i^{(\hat{b}+1)\hat{\alpha}} + n_c k_2^{b\alpha}\psi_\eta^{(\hat{b}+1)\hat{\alpha}}\right]}\right)^{1/\hat{\alpha}} \tag{A5}$$

When α is known, one can eliminate Eq A3 and can put the known values of a and α in Eqs A2 and A4. Then, solve for b and β.

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