Thermodynamic properties of a rotating Bose–Einstein condensation in a deep optical lattice

Hend A. ABDEL-GANY[1,*], **Ali Y. ELLITHI**[1], **Abdelhamid A. GALAL**[1], **Ahmed S. HASSAN**[2]
[1]Department of Physics, Faculty of Science, Cairo University, Giza, Egypt
[2]Department of Physics, Faculty of Science, El-Minia University, El-Minia, Egypt

Abstract: In this paper, we suggest a conventional semiclassical approximation to calculate several thermodynamic quantities of a rotating Bose– Einstein condensation in a deep optical lattice. Expressions for the condensation fraction, critical temperature, and heat capacity are derived analytically. These expressions are obtained by considering the standard harmonic approximation of the deep optical lattice potential. The suggested approach is able to include the the finite size and the positive chemical potential effects simultaneously. The advantage of our suggested approach is in its simplicity, in comparison to the quantum-mechanical calculations (Bose–Hubbard model). Moreover, its generality allows the treatment of a finite temperature regime

Key words: Bose–Einstein condensates in periodic potentials, semiclassical approximation, Boson degeneracy in quantum fluids

1. Introduction

Ultra-cold Bose gas in a state of a rotating Bose–Einstein condensation (BEC) trapped in a deep optical lattice is an important system for studying several fundamental problems in condensed matter physics [1, 2, 3], including the lattice system of charged particles subject to a uniform magnetic field [4], superfluid to Mott insulator phase transition [5], type II superconductors [6], quantum Hall effect [7], and fully frustrated Josephson junction arrays [8].

In a deep optical lattice, previous studies have confirmed that, for an appropriate parameters regime, the atoms are localized in lattice sites; moreover, a band-like structure appears [9, 10]. When the hopping between lattice sites is negligible, the optical potential can be approximated by an equivalent local harmonic potential at these sites. This approximation considerably simplifies the problem and provides a starting point for calculating the finite temperature regime for the thermodynamic properties for which the tight-binding approximation fails.

While the preceding papers elaborate the thermodynamic properties of a BEC in an optical lattice [11, 12, 13, 14] and a rotating BEC in harmonic potential [15, 16, 17] , the present paper investigates the thermodynamic properties of a rotating BEC in a deep optical lattice [18]. Separately, fast rotation or the deepness of the optical potential is affected significantly by the thermodynamic properties of the system [19, 20] compared to the pure harmonically BEC. Consequently, study of the thermodynamic properties of a rotating Bose gas in an optical lattice under a realistic experimental condition is quite interesting.

---

*Correspondence: hendahmedsayed@gmail.com

Motivated by the experiments by Tung et al. [21] and Williams et al. [22, 23], the condensate fraction, critical temperature, and heat capacity are calculated. We suggest a modified semiclassical approximation, which is the density of state (DOS) approach. In this approach, the sums over the energy levels for the thermodynamic quantities are approximated directly by ordinary integrals weighted by the appropriate DOS. The parametrized DOS provides a consistent way for treating the effects of finite size, the fast rotation regime, deepness of the optical potential, and the positive chemical potential, all of them simultaneously. The latter effect is coincidental with the effect of repulsive interaction provided by the mean-field theory approach.

The outcome results show that the ultra-cold rotating BEC in an optical lattice displays distinct thermodynamic behavior compared to a rotating BEC or a confined BEC in lattice potential. The effects of the optical lattice and the rotation are to act as a stirring mechanism.

This paper is planned as follows: the next section includes a simple model for calculating the energy and momentum operators for the rotating harmonic oscillator in an optical lattice. In section 3, we present our results for the condensation fraction, critical temperature, and heat capacity. The last section outlines the conclusion.

2. Energy and momentum operators

Properties of the single particle energy eigenvalues have been discussed for the combined harmonic optical trap [19], as well as rotating harmonic potential [20, 24]. Here we use the results of these studies to suggest an approximated single particle spectrum for the rotating harmonic oscillator in an optical lattice.

Consider a single atom of mass m trapped in a combined 3D cylindrically symmetric harmonic potential with a 2D optical lattice that rotates uniformly in the xy-plane with an angular velocity $\boldsymbol{\Omega}(\equiv \Omega \hat{e}_z)$ around the z-axis. In the rotating frame, the noninteracting single particle Hamiltonian has the form

$$H = \frac{p^2}{2m} + V_{har} + V_{lat} - \Omega\, L_z \tag{1}$$

where

$$V_{har} = \frac{m}{2}\omega_\perp^2 r_\perp^2 + \frac{m}{2}\omega_z^2 z^2, \quad V_{lat} = V_0[\sin^2\left(\frac{\pi x}{d_x}\right) + \sin^2\left(\frac{\pi y}{d_y}\right)], \tag{2}$$

With $p^2 = p_x^2 + p_y^2 + p_z^2$, $r_\perp^2 = x^2 + y^2$, $\{\omega_\perp(\equiv \omega_{x,y}), \text{and } \omega_z\}$ are the effective trapping frequencies of the harmonic potential, V_0 is the lattice potential depth, d_j is the lattice spacing in a direction j, and $L_z = xp_y - yp_x$ is the $z-$component for the angular momentum.

For the Hamiltonian (1) it is impossible to find an exact analytical expression for the energy eigenvalues. An approximated expression can be readily obtained by extending the standard harmonic approximation for the optical lattice potential V_{lat}. In a deep optical lattice where the atoms are localized at the potential minima of the optical lattice and hopping between different lattice sites is negligible, the atoms will trapped in a harmonic potential [25],

$$V_{lat} = \frac{m}{2}\omega_{lat}^2(x^2 + y^2) \tag{3}$$

with on-site trapping frequency, $\omega_{lat}^2 = \frac{4\omega_R^2 V_0}{E_R}$, where $E_R \equiv \hbar\omega_R = \pi^2\hbar^2/2md_{x,y}^2$ is an energy scale for specifying the lattice depth. It is defined as the recoil energy that 1 atom requires when it absorbs 1 lattice photon.

Gathering Eqs.(1), (2), and (3) [26], the noninteracting single particle Hamiltonian can be approximated by

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{m}{2}\left[\left(\omega_\perp^2 + \omega_{lat}^2\right) r_\perp^2 + \omega_z^2 z^2\right] - \Omega(xp_y - yp_x) \tag{4}$$

In terms of the well-known dimensionless creation, annihilation, and number operators [27], Eq.(4) becomes

$$H = \hbar\omega_+(a_+^\dagger a_+ + \frac{1}{2}) + \hbar\omega_-(a_-^\dagger a_- + \frac{1}{2}) + \hbar\omega_z(a_z^\dagger a_z + \frac{1}{2}), \tag{5}$$

where $\omega_\pm = \sqrt{\omega_\perp^2 + \omega_{lat}^2} \mp \Omega$, and $a_{x,y} = \frac{1}{\sqrt{2}}\left(\frac{x,y}{\sqrt{\frac{\hbar}{m\sqrt{\omega_\perp^2 + \omega_{lat}^2}}}} + i\frac{p_{x,y}\sqrt{\frac{\hbar}{m\sqrt{\omega_\perp^2 + \omega_{lat}^2}}}}{\hbar}\right), a_z = \frac{1}{\sqrt{2}}\left(\frac{z}{\sqrt{\frac{\hbar}{m\omega_z}}} + i\frac{p_z\sqrt{\frac{\hbar}{m\omega_z}}}{\hbar}\right),$

$a_\pm = \frac{1}{\sqrt{2}}(a_x \mp ia_y)$. Hamiltonian in (5) is characterized by a discrete single particle energy levels given by

$$E(n_+, n_-, n) \quad = \quad n_+\hbar\omega_\perp\gamma_+ + n_-\hbar\omega_\perp\gamma_- + n_z\hbar\omega_z + E_0 \tag{6}$$

With $\gamma_\pm = \left[\sqrt{1 + \frac{\omega_{lat}^2}{\omega_\perp^2}} \mp \frac{\Omega}{\omega_\perp}\right]$, and $E_0 = \frac{1}{2}\hbar\{\gamma_+\omega_\perp + \gamma_-\omega_\perp + \omega_z\}$ is the ground state energy.

3. General behavior for the thermodynamic properties

First of all, we are coming to treat the problem as in ordinary statistical mechanics, that is, via the familiar relation between the number of atoms, temperature, and the chemical potential. The number of atoms in any specified energy state is defined by the familiar Bose–Einstein distribution:

$$n_{n_+, n_-, n_z} = \sum_{n_{+,-,z}=0}^{\infty} \frac{ze^{-\beta E(n_+, n_-, n_z)}}{1 - ze^{-\beta E(n_+, n_-, n_z)}} = \sum_{j=0}^{\infty} z^j \sum_{n_{+,-,z}=0}^{\infty} e^{-j\beta E(n_+, n_-, n_z)}, \tag{7}$$

where $\beta = (1/K_B T)$ and $z = e^{\beta(\mu(\Omega, V_0) - E_0)}$ is the fugacity. Degeneracy factors in Eq. (8) are avoided by accounting for quantum number individually. The chemical potential is defined as the ground state energy of the rotating boson in the optical lattice and determined by the constraint that the total number of particles in the system is given by

$$N = \sum_{n_{+,-,z}=0}^{\infty} n_{n_+, n_-, n_z} = \sum_{j=0}^{\infty} z^j \sum_{n_{+,-,z}=0}^{\infty} e^{-j\beta E(n_+, n_-, n_z)}, \tag{8}$$

where $\beta = (1/K_B T)$ and $z = e^{\beta(\mu(\Omega, V_0) - E_0)}$ is the fugacity. Degeneracy factors in Eq. (8) are avoided by accounting for quantum number individually.

It is impossible to evaluate the sum in Eq. (8) analytically in a closed form. Another possible way to do this analysis is to separate out the lowest energy state population from the sum and approximate the sum over the excited energy state directly by an ordinary integral weighted by the accurate DOS.

3.1. Approximated DOS of a rotating BEC in an optical lattice

An accurate approximated DOS formula of a rotating BEC in an optical lattice can be parametrized by using the calculation techniques given in reference [28] for the magnetically trapped fermions. Two slight modifications

for the DOS in [29, 30, 31, 32] are considered by one of the authors for rotating condensate [15] and trapped boson gas in a combined magnetic-optical potential [11]. However, for the spectrum in Eq. (6), the DOS for the discrete particle energy eigenvalue is given by

$$\rho(\epsilon) = \frac{1}{(\gamma_+\gamma_-)}\left\{\frac{1}{2}\frac{\epsilon^2}{(\hbar\omega_g)^3} + \frac{3}{2}\left(\frac{\bar{\omega}}{\omega_g}\right)\frac{\epsilon}{(\hbar\omega_g)^2} + \frac{(9\bar{\omega}^2 - \bar{\omega}_r^2)}{8\hbar\omega_g^3}\right\},$$
(9)

where

$$\omega_g = [\omega_\perp^2\omega_z]^{1/3}, \quad \bar{\omega} = \frac{\omega_\perp}{3}[\gamma_+ + \gamma_- + \frac{\omega_z}{\omega_\perp}], \text{ and } \quad \bar{\omega}_r = \frac{\omega_\perp}{\sqrt{3}}\sqrt{\gamma_+^2 + \gamma_-^2 + \omega_z^2/\omega_\perp^2}.$$
(10)

In Eq. (9) the variable ϵ refers to the continuous spectrum. For the nonrotating trap and absence of the optical potential, the parameter $\gamma_+\gamma_- = 1$. In this limit, Eq. (9) reduces to the calculated DOS in [29, 33], $\rho(\epsilon) = \frac{1}{2}\frac{\epsilon^2}{(\hbar\omega)^3} + \frac{3}{2}\frac{\epsilon}{(\hbar\omega)^2} + \frac{1}{(\hbar\omega)}$, where ω is the trap frequency.

Generalization to a many particle system is straightforward. Following the method outlined in our previous paper [11, 15], the DOS for the many particle system in a fast rotation regime, i.e. $\Omega \sim \omega_\perp$, is given by

$$\rho(\epsilon) = \frac{1}{(\gamma_+\gamma_-)}\left\{\frac{1}{2}\frac{\epsilon^2}{(\hbar\omega_g)^3} + \frac{\epsilon}{(\hbar\omega_g)^2}\left[\frac{3}{2}\frac{\bar{\omega}}{\omega_g} + \frac{(9\bar{\omega}^2 - \bar{\omega}_r^2)}{4\hbar\omega_g} \times \frac{\mu(\Omega, V_0)}{[\gamma_+\omega_\perp + \gamma_-\omega_\perp + \omega_z]^2}\right]\right\}$$
(11)

In Eq. (11), $\mu(\Omega, V_0)$ is the chemical potential of a rotating boson in an optical lattice. Once $\mu(\Omega, V_0)$ is known, the thermodynamic properties of the rotating boson in an optical lattice can be calculated. The nontrivial aspect here is the determination of $\mu(\Omega, V_0)$ as a function of Ω and V_0.

3.1.1. Chemical potential of a rotating BEC in an optical lattice

Several authors have discussed the rotation dependence of the chemical potential $\mu(\Omega)$ and the optical potential depth dependence of the chemical potential $\mu(V_0)$ for a finite number of particles N, but did not discuss the rotation- lattice dependence of the chemical potential $\mu(\Omega, V_0)$ presented in this work.

A simple method to calculate the rotation-lattice dependence of the chemical potential $\mu(\Omega, V_0)$ is to follow Hadzibabic and co-workers [34]. In their recent work, they identified a relevant interaction energy scale to explore the relationship between the nonsaturation of the *ideal* Bose gases and the interatomic interactions for the pure harmonically trapped gas. The identified energy scale is given by

$$\mu_0(\omega_g) = \frac{\hbar\omega_g}{2}\left(15N_0\frac{a}{a_{har}}\right)^{2/5},$$
(12)

where a is the s-wave scattering length and $a_{har} = \sqrt{\frac{\hbar}{m\omega_g}}$ is the ground state spatial extension for the harmonic oscillator. It is clear that the energy in Eq. (12) is equivalent to the mean-field prediction for the nonrotating chemical potential $\mu(T = 0)$ of a harmonically trapped gas with N_0 condensate atoms in the Thomas–Fermi limit. Since in our approximation both the rotation and the optical potential lead to a shift in the harmonic oscillator frequency, then we have to generalize Hadzibabic's results to calculate an accurate expression for $\mu(\Omega, V_0)$.

Generalization of Hadzibabic's results to include the rotation effect can be obtained by using Fetter's [16] results, i.e.

$$\mu_0(\Omega) = \mu_0(\omega_g)\left(1 - \Omega^2/\omega_\perp^2\right)^{2/5} = \mu_0(\omega_g)(1 - \alpha^2)^{2/5}, \tag{13}$$

where α is the rotation rate. Eq. (13) gives the relevant interaction energy scale for the rotating trapped gas.

When the rotating bosons are adiabatically loading in optical potential, the generalization for the result given in Eq. (13) can be obtained by using Pedri and co-workers' [35] results for the local chemical potential at the lattice sites. Thus Hadzibabic's scaling energy for the rotating bose gas in an optical lattice becomes

$$\mu(\Omega, V_0) = \mu_0(\omega_g)(1 - \alpha^2)^{2/5}\left(\frac{\pi^2 V_0}{4E_R}\right)^{1/10} = \frac{1}{2}k_B T_0\left(\frac{\zeta(3)}{N}\right)^{1/3}\left[15N_0\frac{a}{a'_{har}}\right]^{2/5}\left[\frac{\pi^2 V_0}{E_R}\right]^{1/10}, \tag{14}$$

where $a'_{har} = \sqrt{\frac{\hbar}{m\omega_g(\gamma_+\gamma_-)^{1/3}}}$, and

$$T_0 = \frac{\hbar\omega_g}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3}(\gamma_+\gamma_-)^{1/3} \tag{15}$$

is the BEC transition temperature for the noninteracting gas. Now we can use Dalfovo's interaction scaling parameter η [36] which fixed the chemical potential of the harmonically trapped gas in units of the transition temperature for the non-interacting gas in the same trap,

$$\eta \equiv \frac{\mu_0}{k_B T_0} = \frac{1}{2}\left(\frac{\zeta(3)}{N}\right)^{1/3}\left[15N_0\frac{a}{a'_{har}}\right]^{2/5}. \tag{16}$$

The scaling parameter η describes the strength of the atomic interactions within the condensate. It is independent of the system size when the thermodynamic limit is taken in the usual way ($N \to \infty$). In terms of η, Eq. (14) becomes

$$\mu(\Omega, V_0) = \eta k_B T_0\left[1 - \frac{\alpha^2}{\kappa^2}\right]^{2/5}\left[\frac{\pi^2 V_0}{E_R}\right]^{1/10} \tag{17}$$

where $\kappa = \sqrt{1 + \frac{\omega_{lat}^2}{\omega_\perp^2}} \equiv \sqrt{1 + 4S_\perp^2\frac{V_0}{E_R}}$ gives the shift in the harmonic oscillator frequencies due to the combined optical potential-rotation effects, and $S_\perp = \omega_R/\omega_\perp$ is the ratio between recoil frequency of the optical potential and the radial frequency of the harmonic potential (we shall refer to this ratio as the reduced recoil frequency). The parametrized $\mu(\Omega, V_0)$ in Eq. (17) looks like a generalization for the well-known Thomas–Fermi approximation of a harmonically trapped boson [36]. Furthermore, it embodies the interatomic interaction effect in our semiclassical approach. Finally, in the thermodynamic limit the contribution from η term is vanished.

3.2. Condensate fraction

Now it is straightforward to calculate the total number of particles in Eq. (8) by using the DOS,

$$\begin{aligned}
N &= \sum_{j=0}^{\infty} z^j \sum_{n_{+,-,z}=0}^{\infty} e^{-j\beta E(n_+,n_-,n_z)} \equiv N_0 + \sum_{j=1}^{\infty} z^j \int_0^{\infty} \rho(\epsilon)e^{-j\beta\epsilon}d\epsilon \\
&= N_0 + \frac{g_3(z)}{[\kappa^2 - \alpha^2]}\left\{\left(\frac{k_B T}{\hbar\omega_g}\right)^3 + \left(\frac{k_B T}{\hbar\omega_g}\right)^2\frac{g_2(z)}{g_3(z)}R(\Omega, V_0)\right\}
\end{aligned}$$

$$\tag{18}$$

where $R(\Omega, V_0) = \left[\frac{3}{2}\frac{\bar{\omega}}{\omega_g} + \frac{(9\bar{\omega}^2 - \bar{\omega}_r^2)}{36\bar{\omega}^2}\frac{\mu(\Omega, V_0)}{\hbar\omega_g}\right]$. In Eq. (18) N_0 is the number of particles in the lowest energy state (this number can be macroscopic, i.e. of the order of N, when $\mu(\Omega, V_0) = E_0$, and $T = 0$). This approximation requires that the condition $k_B T >> \hbar\gamma_{\pm}, \hbar\omega_z$ is satisfied. It is not easy to recover this validity condition directly. Naively one could have expected that $k_B T >> \hbar\gamma_-, \hbar\omega_z$ especially for a fast rotating trap, i.e. $\alpha = 1$. However, the condition $k_B T >> \hbar\gamma_+$ can be satisfied for deep lattice potential.

In terms of the reduced temperature $t = \frac{T}{T_0}$, the condensate fraction is given by

$$\frac{N_0}{N} = 1 - t^3 - R_1(\Omega, V_0)\ t^2 \tag{19}$$

with

$$R_1(\Omega, V_0) = \frac{\zeta(2)}{\zeta(3)[\kappa^2 - \alpha^2]^{1/3}}\left[\frac{3}{2}\frac{\bar{\omega}}{\omega_g}\left(\frac{\zeta(3)}{N}\right)^{\frac{1}{3}} + \eta\left[\kappa^2 - \alpha^2\right]^{\frac{1}{3}}\left[\frac{\pi^2 V_0}{E_R}\right]^{1/10}\left[1 - \frac{\alpha^2}{\kappa^2}\right]^{\frac{2}{5}}\frac{(9\bar{\omega}^2 - \bar{\omega}_r^2)}{36\bar{\omega}^2}\right] \tag{20}$$

where Eq. (15), and $(\gamma_+\gamma_-) = \left[\kappa^2 - \alpha^2\right]$ are used here. The last term in Eq. (19) provides the perturbation correction to the ideal gas result, which is $\frac{N_0}{N} = 1 - t^3$. This correction is included in the parameter $R_1(\Omega, V_0)$: in the brackets the first term gives the finite size effect, while the second term accounts for the effect of the chemical potential when it becomes equal to the lowest energy state. The latter is similar to the effect of repulsive interaction provided by the mean-field theory approach.

In our approach the changeable parameters are: the potential depth, V_0, in units of the recoil energy, E_R, the rotation frequency, Ω, in units of $\omega_{\perp}$, α, and the reduced recoil frequency, $S_{\perp}$. The usual parameters for finite size, N, and the interatomic interaction effect, η, are used. The following parameters values, to reproduce Williams' experiment setup [23], are also used: the harmonic oscillator frequencies are $\{\omega_{\perp}, \omega_z\} = 2\pi \times \{20.1,\ 53.0\}\ Hz$, while the optical potential depth, V_0, and the recoil frequency are set in units of $E_R = 143\ Hz$, and $\omega_{\perp} = 20.1\ Hz$, respectively.

In Figure 1, a direct comparison between the measured condensed fraction by Williams et al. [23] and the calculated theoretical results from Eq. (19) is given. This figure shows that the calculated results are in considerable agreement with the measured data.

In Figure 2 the dependence of the condensate fraction on the reduced temperature t and the depth of the lattice potential $V_0(E_R)$ is presented for rotation rate $\alpha = 0.0$. It has a monotonically decreasing nature everywhere. As the lattice depth increases the condensate fraction dependence on temperature changes from $N_0/N = (1 - t^3)$ to $N_0/N = (1 - t^{3/2})$. This temperature dependence was previously predicted by Blakie and Wang [19] for optically trapped nonrotating Bose gas.

In Figure 3 the dependence of the condensate fraction on the depth of the lattice potential $V_0(E_R)$ and the rotation rate α is presented. It has a monotonically decreasing nature for all α range $\{0, 1\}$, rather rapid in small potential depth range and minor in intermediate depth. In the range of greater depths the rate of decreasing slows down.

The effects of finite size and interatomic interaction on the condensate fraction are summarized in Figure 4 for rotation rate $\alpha = 0.9$, potential depth $V(E_R) = 4$. The reduced temperature $t = 0.7$, and the reduced recoil energy $S_{\perp} = 1.0$. This figure shows that the simultaneous effects of the finite size and interatomic interaction lead to a reduction in the condensed fraction by about $\sim 15\%$.
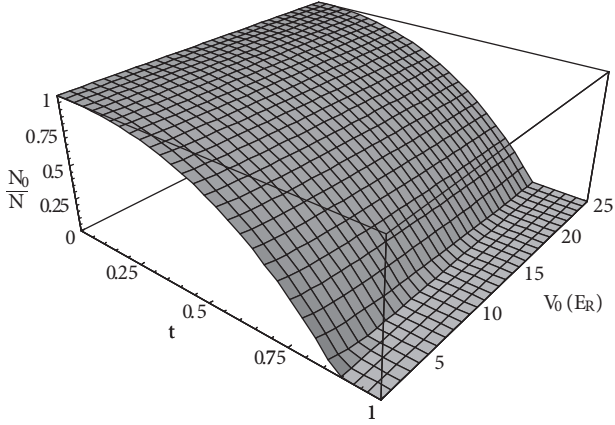
Figure 1. Condensate fraction versus the lattice depth $V_0(E_R)$ for different values of the reduced temperature t. The rotation rates $\alpha = \Omega/\omega_\perp = 0.9$. Lines represent the theoretical results calculated from Eq. (18), while solid circles are the measured data from Williams et al. [23]. The parameter η is taken to be $\eta = 0.4$, and the total number of particles is $N = 1.0 \times 10^4$.
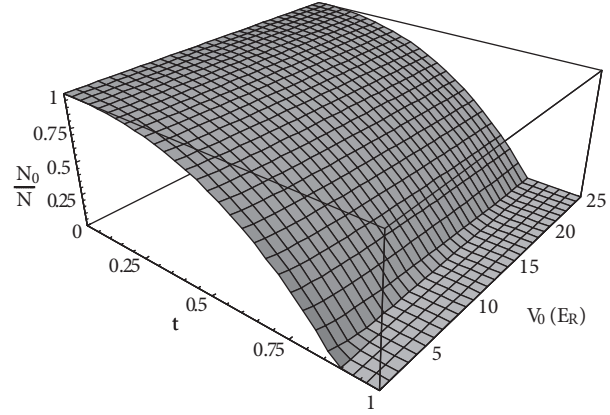

Figure 2. Condensate fraction as a function of the lattice depth $V_0(E_R)$ and the reduced temperature t. The trap parameters are $\{\omega_\perp, \omega_z\} = 2\pi \times \{20.1, 53.0\}\ Hz$, reduced recoil energy $S_\perp = 10$, $\eta = 0.3$ and $N = 10^4$, and the rotation rate is $\alpha = 0.0$.
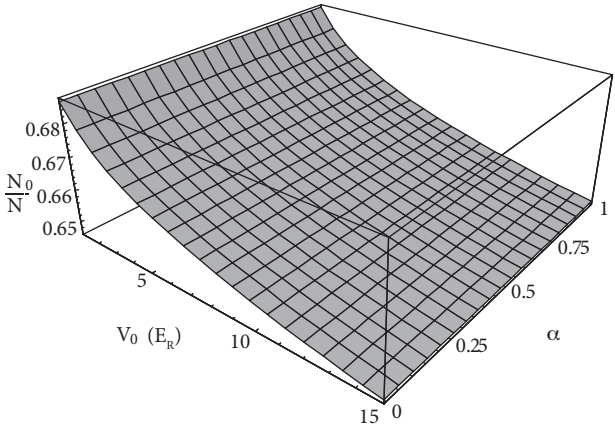

Figure 3. Condensate fraction as a function of the lattice depth $V_0(E_R)$ and the rotation rate α. The trap parameters are $\{\omega_\perp, \omega_z\} = 2\pi \times \{20.1, 53.0\}\ Hz$, reduced recoil energy $S_\perp = 1$, $\eta = 0.4$ and $N = 10^4$, and the reduced temperature is $t = 0.6$.
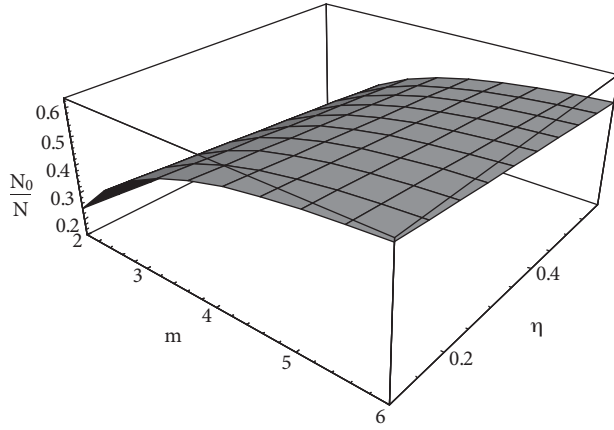

Figure 4. Condensate fraction versus the atoms number $N = 10^m$ and the interatomic interaction parameter η.

3.3. Critical temperature

One of the main proposals of the present work is to study the effects of the finite size and interatomic interaction on the transition temperature of the ideal Bose gas trapped in rotating optical lattice, T_0, given in Eq. (15). These effects can be seen more clearly by calculating the critical temperature. The latter is obtained as usual [11, 30, 32] by setting N_0/N in Eq. (19) equal to zero, thus

$$T_c = \frac{\hbar \omega_g}{k_B} \Big(\frac{N}{\zeta(3)}\Big)^{1/3} [\kappa^2 - \alpha^2]^{1/3} \Big[1 - \frac{1}{3} R_1(\Omega, V_0)\Big] \tag{21}$$

45

The calculated results for T_0 and T_c from Eqs. (15) and (21), respectively, provide a consistent way for treating the effects of the finite size and interatomic interaction. Figure 5 illustrates the fundamental change in the transition temperature due to finite size and interaction effect in the fast rotation regime, i.e. $\alpha = 0.9$, for $V_0(E_R) = 4$, $S_\perp = 2.0$. However, the simultaneous effects of the finite size and interatomic interaction on the transition temperature are the same as the above-mentioned behavior for the condensed fraction, both of them decrease T_0.
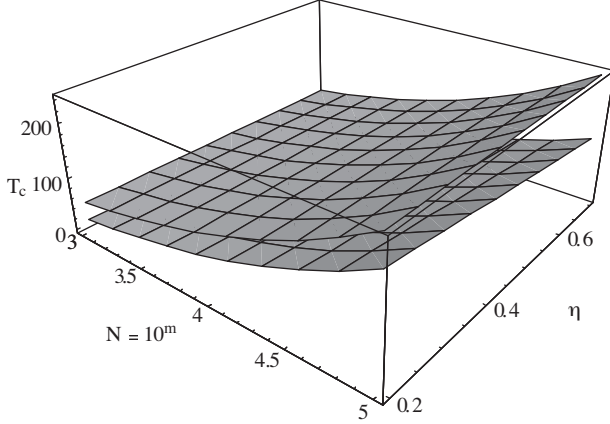

Figure 5. Finite size and interatomic interaction effects on the transition temperature of the ideal Bose gas trapped in a rotating optical lattice.
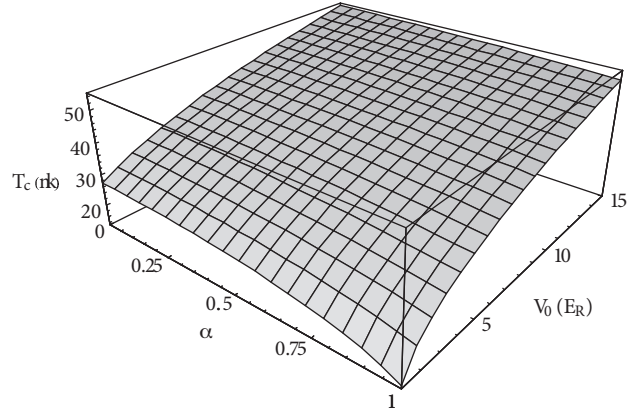
Figure 6. Critical temperature versus the lattice depth $V_0(E_R)$ and the rotation rate α.

In Figure 6 the dependence of the critical temperature T_c on the rotation rate α and optical lattice depth $V_0(E_R)$ is illustrated. It is clear that the critical temperature decreases by increasing the rotation rate α, while it increases by increasing the lattice depth $V_0(E_R)$. Since the rate of decreasing is rapid in small depth due to rotation and it slows down for high depth, we have to conclude that for the rotating boson in optical lattice, the compensate of the critical temperature can be balanced by the localization of the atoms at the optical trap lattice sites due to the centrifugal force. The increase or decrease in the critical temperature as a function of V_0 is considered by Blakie and Wang [19] for optically trapped nonrotating Bose gas.

3.4. Heat capacity

In the DOS approach, the heat capacity can be calculated by differentiating the total energy $E(T)$ with respect to the temperature, i.e.

$$C_V(T) = \left(\frac{\partial E}{\partial T}\right)_{N,V} = \frac{\partial}{\partial T} \sum_{j=1}^{\infty} z^j \int_0^\infty \epsilon\, \rho(\epsilon) e^{-j\beta\epsilon} d\epsilon \tag{22}$$

However, the treatment of the heat capacity is slightly more complicated because we have to take into account 2 different temperature regimes, which are T less or greater than T_0.

For $T < T_0$ the chemical potential $\mu(\Omega, V_0)$ is fixed (independent on T) and the rotating condensate atoms in the ground state N_0 depend on the temperature as well as the rotation rate α and the optical potential depth V_0. From Eqs. (22), the heat capacity is given by

$$\frac{C_{V,T<T_0}}{Nk_B} = 12\frac{\zeta(4)}{\zeta(3)}t^3 + 6\ R_1(\Omega, V_0)t^2 \tag{23}$$

On the other hand, for $T > T_0$, the rotating condensate N_0 vanishes (note that $N_0 \to 0$, for $\alpha > 1$ in the absence of the optical potential). However, for $T > T_0$ the chemical potential $\mu_{(\Omega, V_0)}$ depends on the temperature for any rotation rate and optical potential depth. Following Grossmann and Holthaus [29], we have

$$\frac{C_{V,T>T_0}}{Nk_B} = 12\frac{g_4(z)}{\zeta(3)}t^3 + 6\frac{g_3(z)}{\zeta(3)}R_1(\Omega, V_0)t^2 - \left[\frac{3g_3(z)}{\zeta(3)}t^3 + 2\frac{g_2(z)}{\zeta(3)}R_1(\Omega, V_0)t^2\right] \times \frac{3g_3(z) + 2R_1(\Omega, V_0)g_2(z)/t}{g_2(z) + R_1(\Omega, V_0)g_1(z)/t} \tag{24}$$

In the absence of the optical potential for the nonrotating condensate, i.e. $\gamma_+\gamma_- \to$ unity, the results previously obtained by Crossmann and Holthaus [29] can be obtained by setting $\eta = 0$. When $\frac{C_{V,T>T_0}}{Nk_B}$ is calculated, the convenient integral representation for the Bose function, $g_k(z)$, is used [37]. For $|z| < 1$ and k is real positive ≥ 1 this integral representation is given by

$$g_k(z) = \frac{z}{\Gamma(k)}\int_0^1 \frac{[\log(1/s)]^{k-1}}{1 - zs}ds, \quad \text{Im } s = 0 \tag{25}$$

Note that this representation includes $|z| = 1$ if $k \geq 2$. Another useful approximated formula for the Bose function is given by Klünder and Pelster [38] when $\hbar\omega_g/k_BT < 1$.

In Figures 7 and 8 the calculated results from Eqs. (23) and (24) for the heat capacity as a function of the reduced temperature t with $V_0(E_R)$ and α play as a parameter are represented graphically. The reduced recoil frequency and the interaction parameter are taken to $S_\perp = 0.4$ and $\eta = 0.5$, respectively. These 2 figures show that the specific heat increases with the reduced temperature and is discontinuous at $t = 1$. It gets larger for $t < 1$ and smaller for $t \geq 1$. For a finite system, the jump is quite significant and its magnitude is given by

$$\frac{\Delta C_{V,t=1}^{(\infty)}}{Nk_B} = \left[3 + 2\frac{\zeta(2)}{\zeta(3)}R_1(\Omega, V_0)\right] \times \frac{3\zeta(3) + 2R_1(\Omega, V_0)\zeta(2)}{\zeta(2) + R_1(\Omega, V_0)\zeta(1)} \tag{26}$$

For an infinite system, i.e. the thermodynamic limit, the result first obtained by Crossmann and Holthaus [29] for the magnitude of the jump is recovered, $\frac{\Delta C_{V,t=1}^{(\infty)}}{Nk_B} = 9\frac{\zeta(3)}{\zeta(2)}$.
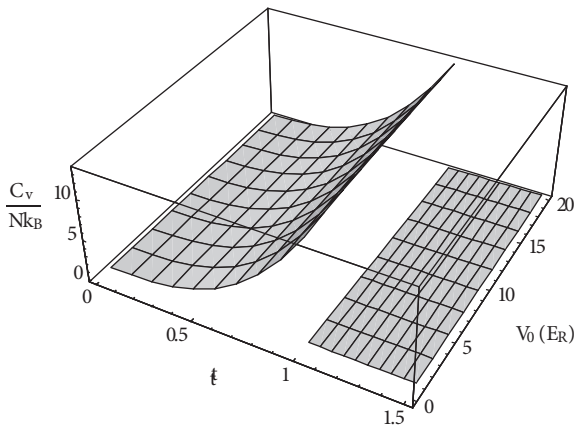

Figure 7. The heat capacity C_V/Nk_B as a function of the reduced temperature and lattice depth $V_0(E_R)$.
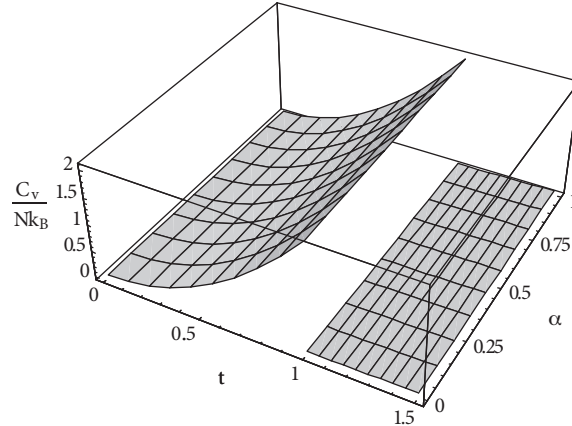
Figure 8. The heat capacity C_V/Nk_B as a function of the reduced temperature and the rotation rate α.

Figure 9 is devoted to investigate the heat capacity as a function of α and $V_0(E_R)$ for 2 different values of t: $t < 1$ and $t > 1$. This figure shows that the dependence of the heat capacity on temperature does not change, which means that the effect of both α and $V_0(E_R)$ is to act as a stirring mechanism.

Finally, using C_V as indicator, the discontinuity characterizes the phase transition to be of second order, according to the Ehrenfest definition. Remarkably in the thermodynamic limit, Eq. (23) obeys the third law of thermodynamics, which demands the vanishing of the heat capacity at zero temperature. Eq. (24) corresponds to the Dulong–Petit law in the very high temperature limit, $(\frac{C_{V,T>T_0}}{Nk_B})_{T\to\infty} = 3$.
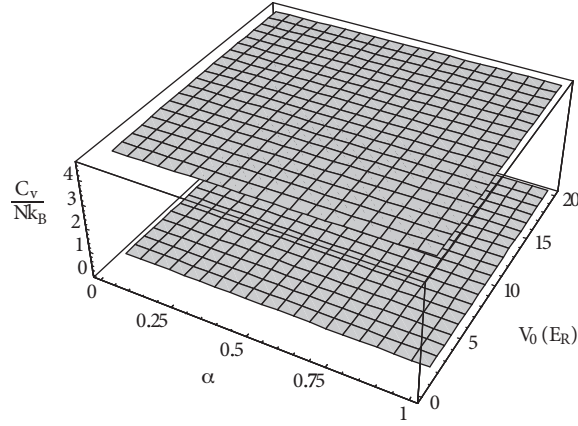

Figure 9. The heat capacity C_V/Nk_B as a function of the rotation rate α and lattice depth $V_0(E_R)$.

4. Conclusion

In this paper, an accurate semiclassical approach is used to investigate the thermodynamic properties of the rotating boson in an optical lattice. An expression for the condensed fraction, critical temperature, and the heat capacity are given. Our approach includes the important physical effects, such as finite size effect, interatomic interaction, and the effect of a positive chemical potential at condensation. The calculated results show that, in the presence of the optical lattice, the effect of the fast rotation (centrifugal suppression) is extinguished. For small lattice depth, the effect of optical potential is to act as a stirring mechanism. As well as for the deeper lattice depth, atoms may be only localized in lattice sites. Consequently, the effect of centrifugal force is quenched.

References

[1] N. R. Cooper, *Adv. Phys.*, **57**, 539, (2008), 616.

[2] A. L. Fetter, *Rev. Mod. Phys.*, **81**, (2009), 647.

[3] I. Bloch, J. Dalibard, and W. Zwerger, *Rev. Mod. Phys.*, **80**, (2008), 885.

[4] K. Kasamatsu, *Phys. Rev. A*, **79**, (2009), 021604(R).

[5] I. B. Spielman, P. R. Johnson, J. H. Huckans, C. D. Fertig, S. L. Rolston, W. D. Phillips, and J. V. Porto, *Phys. Rev. A*, **73**, (2006), 020702(R).

[6] W. Hofstetter, J. Cirac, P. Zoller, E. Demler, and M. Lukin, *Phys. Rev. Lett.*, **89**, (2002), 220407.

[7] N. K. Wilkin and J. M. F. Gunn, *Phys. Rev. Lett.*, **84**, (2000), 000006.

[8] Y. Azizi, A. Valizadeh, Physica B, **406**, (2011), 1017.

[9] L. Viverit, C. Menotti, T. Calarco, and A. Smerzi, Phys. Rev. Lett. **93**, 110401 (2004).

[10] H. Ott, E. de Mirandes, F. Ferlaino, G. Roati, V. Turck, G. Modugno, and M. Inguscio, *Phys. Rev. Lett*, **92**, (2004), 160601.

[11] A. S. Hassan, *Phys. Lett. A*, **374**, (2010), 2106.

[12] Azza M. El-Badry, *Turk J. Phys.*, **36**, (2012), 361.

[13] Shemi S. M. Soliman, *Turk J. Phys.*, **35**, (2011), 65.

[14] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, *Nature*, **415**, (2002), 39; M. Greiner, O. Mandel, T. Rom, A. Altmeyer, A. Widera, T. W. Han, and I. Bloch, *Physica B*, **329**, (2003), 11.

[15] A. S. Hassan, A. M. El-Badry, and S. M. Soliman, *Phys. Lett. A*, **376**, (2012), 1781.

[16] A. L. Fetter, *Phys. Rev. A*, **75**, (2007), 013620 ; *Phys. Rev. A*, **64**, (2001), 063608; *Physica C*, **404**, (2004), 158.

[17] I. Coddington, P. C. Haljan, P. Engels, V. Schweikhard, S. Tung and E. A. Cornell, *Phys. Rev. A*, **70**, (2004), 063607.

[18] A. Kato, Y. Nakano, K. Kasamatsu, and T. Matsui, *Phys. Rev. A*, **84**, (2011), 053623.

[19] P. B. Blakie, A. Bezett, P. F. Buonsante, *Phys. Rev. A*, **75**, (2007), 063609; D. Baillie and P. B. Blakie, *Phys. Rev. A*, **80**, (2009), 033620; P. B. Blakie and Wen-Xin Wang, *Phys. Rev. A*, **76**, (2007), 053620.

[20] S. Kling and A. Pelster, *Phys. Rev. A*, **76**, (2007), 023609.

[21] S. Tung, V. Schweikhard, E. A. Cornell, *Phys. Rev. Lett.*, **97**, (2006), 240402.

[22] R. A. Williams, J. D. Pillet, S. Al-Assam, B. Fletcher, M. Shotter, and C. J. Foot, *Opt. Express*, **16**, (2008), 16977.

[23] R. A. Williams, S. Al-Assam, and C. J. Foot, *Phys. Rev. Lett.*, **104**, (2010), 050404.

[24] S. Stock, B. Battelier, V. Bretin, Z. Hadzibabic, J. Dalibard , *Laser Physics Lett*, **2**, (2005), 275.

[25] D. Jaksch, P. Zoller, *New J. Phys.*, **5**, (2003), 1; D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, *Phys. Rev. Lett.*, **81**, (1998), 3108.

[26] F. Gerbier, S. Trotzky, S. Fölling, U. Schnorrberger, J. D. Thompson, A. Widera, I. Bloch, L. Pollet, M. Troyer, B. Capogrosso-Sansone, N. V. Prokofev, and B. V. Svistunov, *Phys. Rev. Lett.*, **101**, (2008) 155303; *Phys. Rev. Lett.* **99**, (2007), 120405; F. Gerbier and J. Dalibard, *New J. of Phys.*, **12**, (2010), 033007.

[27] L. D. Landau, E. M. Lifschitz, Quantum Mechanics: Non-Relativistic Theory, (Pergamon Press, Oxford. 1977).

[28] G. Su, L. Chen, J. Chen, *Phys. Lett. A*, **326**, (2004), 252.

[29] S. Grossmann, S. and M. Holthaus, *Phys. Lett. A*, **208**, (1995), 188.

[30] K. Kirsten, and D. J. Toms, *Phys. Lett. A*, **222**, (1996) 148; *Phys. Rev. A*, **54**, (1996), 4188.

[31] N. J. Van Druten, and W. Ketterle, *Phys. Rev. Lett.*, **79**, (1997), 549.

[32] A. S. Hassan, and A. M. El-Badry, *Physica B*, **404**, (2009), 1947.

[33] K. Kirsten, and D. J. Toms, *Phys. Rev. E*, **59**, (1999), 158; K. Kirsten, and D. J. Toms, *Phys. Lett. A*, **243**, (1998), 137.

[34] R. L. D. Campbell, R. P. Smith, N. Tammuz, S. Beattie, S. Moulder, and Z. Hadzibabic, *Phys. Rev. A*, **82**, (2010), 063611.

[35] P. Pedri, L. Pitaevskii, S. Stringari, *Phys. Rev. Lett.*, **87**, (2001), 220401.

[36] F. Dalfovo, S. Giorgini, L.P. Pitaevskii and S. Stringari, *Rev. of Mod. Phys.*, **71**, (1999), 463.

[37] M. H. Lee, *Phys. Rev. E*, **56**, (1997), 3909.

[38] B. Klünder, and A. Pelster, *Eur. Phys. J. B*, **68**, (2009), 457.