Journal of
Computational and Theoretical Nanoscience
Vol. 12, 1–7, 2015

Deformation for a Rectangle by a Finite Fourier Transform

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In this paper, we introduce a simple method to solve a static, plane boundary value problem in elasticity for an isotropic rectangular region. The method depends on finite Fourier transform to transfer the biharmonic equation to a nonhomogeneous ordinary differential equation of the fourth order. Also, by transfering the boundary conditions, one can find the general solution for the nonhomogeneous ordinary differential equation. Finally, the inverse Fourier transfer allows to get the analytical solution for the biharmonic equation. Using expressions for displacements proved by two of the authors [MSA and AFG], one can obtain the displacements for the rectangular domain.

Keywords: Finite Fourier Transform, Statics, Plane Boundary Value Problem, Elasticity, Rectangle.

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1. INTRODUCTION

The boundary-value problems in elasticity for rectangular domains have received a special treatment due to the existence corners on the boundary of the domain. Meleshko in Ref. [2] used the superposition method, proposed earlier by Mathieu (1890), Inglis (1921) and Pickett (1944). The author studied the stress field near the boundaries, including specific cases of discontinuous and concentrated normal and shear loading.

A solution by Vihak et al. in Ref. [3], consists of two parts and satisfies both the conditions imposed on the boundary of the rectangle and the original relationships.

Golovchan in Ref. [4] proposed an algorithm based on a complex-valued representation of the general solution to the differential equations of the plane problem and on the use of Lagrange polynomials to satisfy the boundary conditions.

Meleshko in Ref. [5] discussed the biharmonic equation for a rectangle using the method of superposition for creeping Stokes flows.

Pravina et al. in [6] found the solution of the inverse transient problem of quasi-static thermal stresses in a rectangular plate by the finite Fourier and Laplace transforms for the governing equations.

In Ref. [1], two of the authors (MSA and AFG) used the boundary integral representation of harmonic functions to solve the biharmonic equation for a long cylinder with a circular cross-section. Their solution is valid for any simply connected, smooth boundary subjected to a uniform pressure.

In this work, we use the finite Fourier transform for the governing equations and the boundary conditions to get the analytical solution for the first fundamental problem of elasticity. The solution coincides with that in Ref. [1]. More problems for a rectangular domains can be found in Refs. [7, 8]. An extensive literature on the topic is now available and we can only mention a few recent interesting investigations in Refs. [11–17].

2. DESCRIBING THE PROBLEM

Let us consider a rectangular domain with sides a and b subjected to a constant, pressures P_1 and P_2 on its boundary. In the absence of body forces, the equations of equilibrium are

$$\sigma_{ij,j} = 0$$

They are automatically satisfied if the stress components are defined through a stress function $U(x, y)$ as

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \tag{1}$$

Let

$$\mathbf{f} = f_x \mathbf{i} + f_y \mathbf{j}$$

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denote the external force per unit length of the boundary. Then, at a general boundary point Q the stress vector satisfies

$$\mathbf{f} = \boldsymbol{\sigma}^T \mathbf{n}$$

The normal pressure at the boundary of the rectangle can be defined by the second derivatives for the stress function.

For the first fundamental problem of elasticity for a rectangular domain, we solve the homogeneous biharmonic equation[1]

$$\nabla^4 U = 0 \tag{2}$$

with the boundary conditions (from relation (1))

$$\begin{cases} \left.\dfrac{\partial^2 U}{\partial x^2}\right|_1 = -P_1, & \text{at } y = 0 \\ \left.\dfrac{\partial^2 U}{\partial y^2}\right|_2 = -P_2, & \text{at } x = a \\ \left.\dfrac{\partial^2 U}{\partial x^2}\right|_3 = -P_1, & \text{at } y = b \\ \left.\dfrac{\partial^2 U}{\partial y^2}\right|_4 = -P_2, & \text{at } x = 0 \end{cases} \tag{3}$$

as shown in the Figure 1.

2.1. The Boundary Conditions at the Sides

Integrate the boundary conditions (3) to get

$$\begin{cases} U|_1 = D_0 + D_1 x - \dfrac{P_1}{2}x^2 \\ U|_2 = A_0 + A_1 y - \dfrac{P_2}{2}y^2 \\ U|_3 = C_0 + C_1 x - \dfrac{P_1}{2}x^2 \\ U|_4 = B_0 + B_1 y - \dfrac{P_2}{2}y^2 \end{cases} \tag{4}$$

where D_0, D_1, A_0, A_1, C_0, C_1, B_0 and B_1 are constants to be determined.

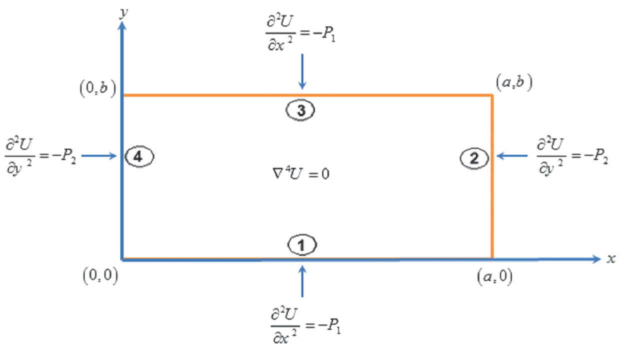

Fig. 1. Rectangular domain with sides a and b.

At the sides of the rectangle, one can write truncated Taylor's expansion for the two functions $\partial^2 U/\partial y^2$ and $\partial^2 U/\partial x^2$ as

$$\begin{cases} \left.\dfrac{\partial^2 U}{\partial y^2}\right|_1 = d_0 + d_1 x + d_2 x^2 \\ \left.\dfrac{\partial^2 U}{\partial x^2}\right|_2 = a_0 + a_1 y + a_2 y^2 \\ \left.\dfrac{\partial^2 U}{\partial y^2}\right|_3 = c_0 + c_1 x + c_2 x^2 \\ \left.\dfrac{\partial^2 U}{\partial x^2}\right|_4 = b_0 + b_1 y + b_2 y^2 \end{cases} \tag{5}$$

where d_0, d_1, d_2, a_0, a_1, a_2, c_0, c_1, c_2, b_0, b_1 and b_2 are constants to be determined.

2.2. The Boundary Condition at the Corners

Applying the continuity of the function U at the corners

1. At the point $(0,0)$, we have

(a) The vanishing of the function U and its first derivatives at the point $(0,0)$

$$U|_1 = U|_4 = 0, \Rightarrow D_0 = 0, \quad B_0 = 0$$

$$\left.\frac{\partial U}{\partial x}\right|_1 = \left.\frac{\partial U}{\partial y}\right|_4 = 0, \Rightarrow D_1 = 0, \quad B_1 = 0$$

(b) The continuity of the second derivatives of the function U

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_1 = \left.\frac{\partial^2 U}{\partial x^2}\right|_4, \Rightarrow b_0 = -P_1$$

$$\left.\frac{\partial^2 U}{\partial y^2}\right|_1 = \left.\frac{\partial^2 U}{\partial y^2}\right|_4, \Rightarrow d_0 = -P_2$$

(c) The continuity of the mixed second derivatives of the function U

$$\left.\frac{\partial^4 U}{\partial x^2 \partial y^2}\right|_1 = \left.\frac{\partial^4 U}{\partial x^2 \partial y^2}\right|_4, \Rightarrow d_2 = b_2 = 0$$

2. At the point $(a,0)$:

$$U|_1 = U|_2, \Rightarrow A_0 = -\frac{P_1}{2}a^2$$

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_1 = \left.\frac{\partial^2 U}{\partial x^2}\right|_2, \Rightarrow a_0 = -P_1$$

$$\left.\frac{\partial^2 U}{\partial y^2}\right|_1 = \left.\frac{\partial^2 U}{\partial y^2}\right|_2, \Rightarrow d_1 = 0$$

$$\left.\frac{\partial^4 U}{\partial x^2 \partial y^2}\right|_1 = \left.\frac{\partial^4 U}{\partial x^2 \partial y^2}\right|_2 \Rightarrow d_2 = a_2 = 0$$

3. At the point $(0, b)$:

$$U|_3 = U|_4, \quad \Rightarrow \quad C_0 = -\frac{P_2}{2}b^2$$
$$\left.\frac{\partial^2 U}{\partial x^2}\right|_3 = \left.\frac{\partial^2 U}{\partial x^2}\right|_4, \quad \Rightarrow \quad b_1 = 0$$
$$\left.\frac{\partial^2 U}{\partial y^2}\right|_3 = \left.\frac{\partial^2 U}{\partial y^2}\right|_4, \quad \Rightarrow \quad c_0 = -P_2$$
$$\left.\frac{\partial^4 U}{\partial x^2 \partial y^2}\right|_3 = \left.\frac{\partial^4 U}{\partial x^2 \partial y^2}\right|_4 \quad \Rightarrow \quad b_2 = c_2 = 0$$

4. At the point (a, b):

$$U|_2 = U|_3, \quad \Rightarrow \quad A_1 = \frac{a}{b}C_1 = C$$
$$\left.\frac{\partial^2 U}{\partial x^2}\right|_2 = \left.\frac{\partial^2 U}{\partial x^2}\right|_3, \quad \Rightarrow \quad a_1 = 0$$
$$\left.\frac{\partial^2 U}{\partial y^2}\right|_2 = \left.\frac{\partial^2 U}{\partial y^2}\right|_3, \quad \Rightarrow \quad c_1 = 0$$
$$\left.\frac{\partial^4 U}{\partial x^2 \partial y^2}\right|_2 = \left.\frac{\partial^4 U}{\partial x^2 \partial y^2}\right|_3 \quad \Rightarrow \quad a_2 = c_2 = 0$$

3. THE METHOD OF SOLUTION

Equation (2) may be written as

$$\nabla^4 U = \frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0 \tag{6}$$

To solve this biharmonic equations in a rectangular domain, multiply Eq. (6) by $\sin((n\pi)/a)x$ and integrate in the interval $[0, a]$, then

$$\int_0^a \frac{\partial^4 U}{\partial x^4}\sin\frac{n\pi}{a}x\,dx + 2\int_0^a \frac{\partial^4 U}{\partial x^2 \partial y^2}\sin\frac{n\pi}{a}x\,dx + \int_0^a \frac{\partial^4 U}{\partial y^4}\sin\frac{n\pi}{a}x\,dx = 0 \tag{7}$$

Take

$$U_n(y) = \int_0^a U(x, y)\sin\frac{n\pi}{a}x\,dx \tag{8}$$

Integrate by parts for the following

$$\begin{aligned}\int_0^a \frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi}{a}x\,dx &= \frac{\partial U}{\partial x}\sin\frac{n\pi}{a}x\Big|_0^a - \frac{n\pi}{a}\int_0^a \frac{\partial U}{\partial x}\cos\frac{n\pi}{a}x\,dx \\ &= -\frac{n\pi}{a}\int_0^a \frac{\partial U}{\partial x}\cos\frac{n\pi}{a}x\,dx \\ &= -\frac{n\pi}{a}\left(U\cos\frac{n\pi}{a}x\Big|_0^a + \frac{n\pi}{a}\int_0^a U\sin\frac{n\pi}{a}x\,dx\right)\end{aligned}$$

but, from the second and fourth of Eq. (4) one has

$$\begin{aligned}U\cos\frac{n\pi}{a}x\Big|_0^a &= (-1)^n\, U(a, y)|_4 - U(0, y)|_2 \\ &= -((-1)^n - 1)\frac{P_2}{2}y^2 - Cy + \frac{P_1}{2}a^2\end{aligned}$$

Use Eq. (8) to get

$$\int_0^a \frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi}{a}x dx = -\left(\frac{n\pi}{a}\right)^2 U_n(y) + n\pi(-1)^n\frac{aP_1}{2} - \frac{n\pi}{a}(-1)^n Cy + \frac{n\pi}{a}P_2\frac{(-1)^n - 1}{2}y^2 \tag{9}$$

The second term in Eq. (7) can be written as

$$\int_0^a \frac{\partial^4 U}{\partial x^2 \partial y^2}\sin\frac{n\pi}{a}x\,dx = \frac{d^2}{dy^2}\int_0^a \frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi}{a}x\,dx$$

use Eq. (9), then

$$\int_0^a \frac{\partial^4 U}{\partial x^2 \partial y^2}\sin\frac{n\pi}{a}x\,dx = -\left(\frac{n\pi}{a}\right)^2\frac{d^2 U_n(y)}{dy^2} + \frac{n\pi}{a}((-1)^n - 1)P_2 \tag{10}$$

The third term of Eq. (7) may be written as

$$\frac{d^4 U_n(y)}{dy^4} = \int_0^a \frac{\partial^4 U}{\partial y^4}\sin\frac{n\pi}{a}x\,dx \tag{11}$$

The first term of Eq. (7), integrate by parts, yields

$$\begin{aligned}&\int_0^a \frac{\partial^4 U}{\partial x^4}\sin\frac{n\pi}{a}x\,dx \\ &= \frac{\partial^3 U}{\partial x^3}\sin\frac{n\pi}{a}x\Big|_0^a - \frac{n\pi}{a}\int_0^a \frac{\partial^3 U}{\partial x^3}\cos\frac{n\pi}{a}x\,dx \\ &= -\frac{n\pi}{a}\int_0^a \frac{\partial^3 U}{\partial x^3}\cos\frac{n\pi}{a}x\,dx \\ &= -\frac{n\pi}{a}\left[\frac{\partial^2 U}{\partial x^2}\cos\frac{n\pi}{a}x\Big|_0^a + \frac{n\pi}{a}\int_0^a \frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi}{a}x\,dx\right]\end{aligned}$$

but, from the second and fourth of Eq. (5) one gets

$$\begin{aligned}\frac{\partial^2 U}{\partial x^2}\cos\frac{n\pi}{a}x\Big|_0^a &= (-1)^n\left.\frac{\partial^2 U}{\partial x^2}(a, y)\right|_4 - \left.\frac{\partial^2 U}{\partial x^2}(0, y)\right|_2 \\ &= -((-1)^n - 1)P_1\end{aligned}$$

then

$$\int_0^a \frac{\partial^4 U}{\partial x^4}\sin\frac{n\pi}{a}x\,dx = \frac{n\pi}{a}((-1)^n - 1)P_1 - \left(\frac{n\pi}{a}\right)^2\int_0^a \frac{\partial^2 U}{\partial x^2}\sin\frac{n\pi}{a}x\,dx$$

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and using Eqs. (8) and (9):

$$\int_0^a \frac{\partial^4 U}{\partial x^4}\sin\frac{n\pi}{a}x\,dx = \left(\frac{n\pi}{a}\right)^4 U_n(y)+\frac{n\pi}{a}\left[2((-1)^n-1)-n^2\pi^2(-1)^n\right]\frac{P_1}{2} + \left(\frac{n\pi}{a}\right)^3(-1)^n Cy-\left(\frac{n\pi}{a}\right)^3 P_2\frac{(-1)^n-1}{2}y^2 \quad (12)$$

4. THE BIHARMONIC EQUATION

Substitution from Eqs. (10)–(12) into Eq. (7) yields

$$U_n^{(iv)}(y)-2\left(\frac{n\pi}{a}\right)^2 U_n^{(ii)}(y)+\left(\frac{n\pi}{a}\right)^4 U_n(y)=F_n(y) \quad (13)$$

with

$$F_n(y) = -\frac{n\pi}{a}\left[2((-1)^n-1)-n^2\pi^2(-1)^n\right]\frac{P_1}{2} - 2\frac{n\pi}{a}((-1)^n P_2-P_2) - \frac{n\pi}{a}\left[\left(\frac{n\pi}{a}\right)^2(-1)^n C-((-1)^n-1)\frac{P_1-P_1}{b}\right]y + \left(\frac{n\pi}{a}\right)^3\frac{(-1)^n P_2-P_2}{2}y^2 \quad (14)$$

Now, one gets a nonhomogeneous ordinary differential equation of the fourth order , the solution of which is

$$U_n(y)=U_n^H(y)+U_n^p(y)$$

where $U_n^H(y)$ is the homogeneous solution and $U_n^p(y)$ any particular solution.

4.1. The Homogeneous Solution

The homogeneous solution of Eq. (13)

$$\frac{d^4U_n(y)}{dy^4}-2\left(\frac{n\pi}{a}\right)^2\frac{d^2U_n(y)}{dy^2}+\left(\frac{n\pi}{a}\right)^4 U_n(y)=0$$

where, the auxiliary equation is

$$\lambda^4-2\left(\frac{n\pi}{a}\right)^2\lambda^2+\left(\frac{n\pi}{a}\right)^4=0$$

or

$$\left(\lambda^2-\left(\frac{n\pi}{a}\right)^2\right)^2=0$$

then

$$\lambda=\pm\left(\frac{n\pi}{a}\right),\quad \pm\left(\frac{n\pi}{a}\right)$$

Finally, the homogeneous solution is

$$U_n^H(y) = (\alpha_n+(y-b)\beta_n)\frac{\sinh((n\pi)/a)y}{\sinh((n\pi)/a)b} + (\gamma_n+y\delta_n)\frac{\sinh((n\pi)/a)(y-b)}{\sinh((n\pi)/a)b} \quad (15)$$

4.2. The Particular Solution

The particular solution must satisfy Eq. (13), then

$$\left(D^2-\left(\frac{n\pi}{a}\right)^2\right)^2 U_n^p(y)=F_n(y)$$

or

$$U_n^p(y) = \frac{1}{(D^2-((n\pi)/a)^2)^2}F_n(y) = \frac{1}{((n\pi)/a)^4}\left(1-\frac{D^2}{((n\pi)/a)^2}\right)^{-2}F_n(y) = \frac{1}{((n\pi)/a)^4}\left(1+2\left(\frac{aD}{n\pi}\right)^2+\cdots\right)F_n(y) = \frac{1}{((n\pi)/a)^4}F_n(y)+2\frac{D^2}{((n\pi)/a)^6}F_n(y)+\cdots$$

Use Eq. (14) to get

$$U_n^p(y) = -\frac{a^3}{\pi^3}\frac{(-1)^n-1}{n^3}P_1+\frac{a^3}{\pi}\frac{(-1)^n}{n}\frac{P_1}{2} - \frac{a}{\pi}\frac{(-1)^n}{n}Cy+\frac{1}{2}\frac{a}{\pi}P_2\frac{(-1)^n-1}{n}y^2 \quad (16)$$

From Eqs. (15) and (16), the general solution for Eq. (13) is

$$U_n(y) = (\alpha_n+(y-b)\beta_n)\frac{\sinh((n\pi)/a)y}{\sinh((n\pi)/a)b} + (\gamma_n+y\delta_n)\frac{\sinh((n\pi)/a)(y-b)}{\sinh((n\pi)/a)b} - \frac{a^3}{\pi^3}\frac{(-1)^n-1}{n^3}P_1+\frac{a^3}{\pi}\frac{(-1)^n}{n}\frac{P_1}{2}-\frac{a}{\pi}\frac{(-1)^n}{n}Cy + \frac{1}{2}\frac{a}{\pi}P_2\frac{(-1)^n-1}{n}y^2 \quad (17)$$

5. TRANSFORMATION OF BOUNDARY CONDITIONS

Using the same technique, we transfer the boundary conditions (4 and 5)

1. At $y=0$:

$$\int_0^a U(x,0)\sin\frac{n\pi}{a}x\,dx=-\frac{P_1}{2}\int_0^a x^2\sin\frac{n\pi}{a}x\,dx$$

$$\int_0^a\frac{\partial^2 U(x,0)}{\partial y^2}\sin\frac{n\pi}{a}x\,dx=-P_2\int_0^a\sin\frac{n\pi}{a}x\,dx$$

and by use of Eq. (8), for the LHS and integrating by parts for the RHS, then

$$U_n(0)=\left[\frac{a^3}{\pi}\frac{(-1)^n}{n}-\frac{2a^3}{\pi^3}\frac{(-1)^n-1}{n^3}\right]\frac{P_1}{2} \quad (18)$$

$$\frac{d^2}{dy^2}U_n(0)=\frac{a}{\pi}P_2\frac{(-1)^n-1}{n} \quad (19)$$

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From Eq. (17) and the second derivative with respect to y:

$$U_n(0) = -\gamma_n - \frac{a^3}{\pi^3}\frac{(-1)^n-1}{n^3}P_1 + \frac{a^3}{\pi}\frac{(-1)^n}{n}\frac{P_1}{2} \tag{20}$$

$$\frac{d^2}{dy^2}U_n(0) = 2\frac{\pi}{a}\frac{n}{\sinh((n\pi)/a)b}\beta_n + 2\delta_n\frac{n\pi}{a}\frac{\cosh((n\pi)/a)b}{\sinh((n\pi)/a)b} - \left(\frac{n\pi}{a}\right)^2\gamma_n + \frac{a}{\pi}P_2\frac{(-1)^n-1}{n} \tag{21}$$

Equating Eqs. (18) and (19) with Eqs. (20) and (21) gives

$$\gamma_n = 0, \quad \beta_n = -\delta_n\cosh\frac{n\pi}{a}b \tag{22}$$

2. At $y = b$:

$$\int_0^a U(x,b)\sin\frac{n\pi}{a}x\,dx = -\frac{P_2}{2}b^2\int_0^a \sin\frac{n\pi}{a}x\,dx + \frac{b}{a}C\int_0^a x\sin\frac{n\pi}{a}x\,dx - \frac{P_1}{2}\int_0^a x^2\sin\frac{n\pi}{a}x\,dx$$

$$\int_0^a \frac{\partial^2 U(x,b)}{\partial y^2}\sin\frac{n\pi}{a}x\,dx = -P_2\int_0^a \sin\frac{n\pi}{a}x\,dx$$

Use of Eq. (8) for the LHS and integrating by parts for the RHS yields

$$U_n(b) = \frac{ab^2}{\pi}\frac{(-1)^n-1}{n}\frac{P_2}{2} - \frac{ab}{\pi}\frac{(-1)^n}{n}C + \left(\frac{(-1)^n}{n}\frac{a^3}{\pi} - 2\frac{a^3}{\pi^3}\frac{(-1)^n-1}{n^3}\right)\frac{P_1}{2} \tag{23}$$

$$\frac{d^2}{dy^2}U_n(b) = \frac{a}{\pi}P_2\frac{(-1)^n-1}{n} \tag{24}$$

From Eq. (17) and the second derivative with respect to y:

$$U_n(b) = \alpha_n - \frac{a^3}{\pi^3}\frac{(-1)^n-1}{n^3}P_1 + \frac{a^3}{\pi}\frac{(-1)^n}{n}\frac{P_1}{2} - \frac{ab}{\pi}\frac{(-1)^n}{n}C + \frac{1}{2}\frac{ab^2}{\pi}P_2\frac{(-1)^n-1}{n} \tag{25}$$

$$\frac{d^2}{dy^2}U_n(b) = 2\frac{n\pi}{a}\beta_n\frac{\cosh((n\pi)/a)b}{\sinh((n\pi)/a)b} + 2\frac{\pi}{a}\frac{n}{\sinh((n\pi)/a)b}\delta_n + \left(\frac{n\pi}{a}\right)^2\alpha_n + \frac{a}{\pi}P_2\frac{(-1)^n-1}{n} \tag{26}$$

Equating Eqs. (23) and (24) with Eqs. (25) and (26) gives

$$\alpha_n = 0, \quad \beta_n\cosh\frac{n\pi}{a}b = -\delta_n \tag{27}$$

Solving the second of Eqs. (22) and (27):

$$\beta_n = \delta_n = 0 \tag{28}$$

Making use of Eqs. (22), (27) and (28), Eq. (17) may be written as

$$U_n(y) = -\frac{a^3}{\pi^3}\frac{(-1)^n-1}{n^3}P_1 + \frac{a^3}{\pi}\frac{(-1)^n}{n}\frac{P_1}{2} - \frac{a}{\pi}\frac{(-1)^n}{n}Cy + \frac{1}{2}\frac{a}{\pi}P_2\frac{(-1)^n-1}{n}y^2 \tag{29}$$

Then from Eq. (8) the inverse function is obtained as

$$U(x,y) = \frac{2}{a}\sum_{n=1}^{\infty}U_n(y)\sin\frac{n\pi}{a}x \tag{30}$$

Substituting from Eq. (29) into Eq. (30), and using (Ref. [10], p. 46):

$$\begin{cases} \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi}{a}x = -\frac{1}{2}\frac{\pi}{a}x, & 0 < x < a \\ \sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{a}x = \frac{\pi}{2} - \frac{x}{2}\frac{\pi}{a}, & 0 < x < a \\ \sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^3}\sin n\frac{\pi}{a}x = \frac{\pi^3}{4a^3}(-a^2x + ax^2), & 0 \le x \le a \end{cases} \tag{31}$$

then

$$U(x,y) = -\frac{P_1}{2}x^2 - \frac{P_2}{2}y^2 + \frac{C}{a}xy \tag{32}$$

Applying the boundary condition

$$U_{xy} = 0$$

at the sides of the rectangle gives the value

$$C = 0$$

hence

$$U(x,y) = -\frac{P_1}{2}x^2 - \frac{P_2}{2}y^2 \tag{33}$$

6. THE HARMONIC FUNCTIONS

It is well-known that the solution of the homogeneous biharmonic equation can be written as[1]

$$U(x,y) = x\Phi(x,y) + y\Phi^c(x,y) + \Psi(x,y) \tag{34}$$

where the functions $\Phi(x,y)$, $\Psi(x,y)$ and its conjugate $\Phi^c(x,y)$, $\Psi^c(x,y)$ are harmonic functions.

Differentiate Eq. (34) with respect to x and y respectively

$$\frac{\partial^2 U}{\partial x^2} = x\frac{\partial^2\Phi}{\partial x^2} + 2\frac{\partial\Phi}{\partial x} + y\frac{\partial^2\Phi^c}{\partial x^2} + \frac{\partial^2\Psi}{\partial x^2} \tag{35}$$

$$\frac{\partial^2 U}{\partial y^2} = x\frac{\partial^2\Phi}{\partial y^2} + 2\frac{\partial\Phi^c}{\partial y} + y\frac{\partial^2\Phi^c}{\partial y^2} + \frac{\partial^2\Psi}{\partial y^2} \tag{36}$$

by addition of the two Eqs. (35) and (36) and use of Cauchy-Riemann relations

$$\frac{\partial\Phi}{\partial x}=\frac{\partial\Phi^c}{\partial y},\quad \frac{\partial\Phi}{\partial y}=-\frac{\partial\Phi^c}{\partial x}$$

then

$$\nabla^2 U = 2\frac{\partial\Phi}{\partial x}+2\frac{\partial\Phi^c}{\partial y}$$

Using Cauchy-Riemann relations, one obtains

$$\frac{\partial\Phi}{\partial x}=\frac{1}{4}\nabla^2 U \tag{37}$$

$$\frac{\partial\Phi^c}{\partial y}=\frac{1}{4}\nabla^2 U \tag{38}$$

Integrate Eqs. (37) and (38) with respect to x and y respectively

$$\Phi(x,y)=\frac{1}{4}\int_0^x \nabla^2 U dx + f(y) \tag{39}$$

$$\Phi^c(x,y)=\frac{1}{4}\int_0^y \nabla^2 U dy + g(x) \tag{40}$$

Differentiation of Eq. (37) with respect to x and Eq. (39) with respect to y twice, addition and use Eq. (33), one obtains

$$\frac{d^2}{dy^2}f(y)=0 \tag{41}$$

Similarly, differentiate Eq. (38) with respect to y and differentiate Eq. (40) with respect to x twice, add and use Eq. (33) to get

$$\frac{d^2}{dx^2}g(x)=0 \tag{42}$$

Integrate Eqs. (41) and (42):

$$f(y)=\kappa_0+\kappa_1 y \tag{43}$$

$$g(x)=\varkappa_0+\varkappa_1 x \tag{44}$$

Substitution from Eqs. (43) and (44) into Eqs. (39) and (40) and using Eq. (33) gives

$$\Phi(x,y)=-\frac{P_1+P_2}{4}x+\kappa_0+\kappa_1 y \tag{45}$$

$$\Phi^c(x,y)=-\frac{P_1+P_2}{4}y+\varkappa_0+\varkappa_1 x \tag{46}$$

From Eqs. (33), (34), (45) and (46), then

$$\Psi(x,y)=-\frac{P_1-P_2}{4}(x^2-y^2)-\kappa_0 x-\varkappa_0 y - (\kappa_1+\varkappa_1)xy \tag{47}$$

7. THE DISPLACEMENT VECTOR COMPONENTS

From:[1]

$$\frac{E}{1+\nu}u=-\frac{\partial U}{\partial x}+4(1-\nu)\Phi \tag{48}$$

and

$$\frac{E}{1+\nu}v=-\frac{\partial U}{\partial y}+4(1-\nu)\Phi^c \tag{49}$$

where ν, E are Poisson's ratio and Young's modulus respectively.

Subsitution from Eqs. (33), (45) and (46) into Eqs. (48) and (49) yields

$$\frac{E}{1+\nu}u=P_1 x+4(1-\nu)\left(-\frac{P_1+P_2}{4}x+\kappa_0+\kappa_1 y\right) \tag{50}$$

$$\frac{E}{1+\nu}v=P_2 y+4(1-\nu)\left(-\frac{P_1+P_2}{4}y+\varkappa_0+\varkappa_1 x\right) \tag{51}$$

7.1. Conditions for Eliminating the Rigid Body Translation

These are two conditions, to be applied only for the first fundamental problem: Following,[9] we require that the displacement at point O vanish, i.e.,

$$u(0,0)=0,\quad v(0,0)=0$$

then

$$\kappa_0=0,\quad \varkappa_0=0 \tag{52}$$

7.2. Condition for Eliminating the Rigid Body Rotation

This condition is applied only for the first fundamental problem. Following,[9] we shall require that

$$\frac{\partial u}{\partial y}(0,0)-\frac{\partial v}{\partial x}(0,0)=0$$

then

$$\frac{E}{1+\nu}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)=-4(1-\nu)(\varkappa_1-\kappa_1)$$

This condition is satisfied only for

$$\varkappa_1=\kappa_1=0$$

Finally the displacements are

$$\frac{E}{1+\nu}u=(\nu P_1-(1-\nu)P_2)x \tag{53}$$

$$\frac{E}{1+\nu}v=-((1-\nu)P_1-\nu P_2)y \tag{54}$$

8. CONCLUSIONS

One arrives at the following conclusions:

(1) We proposed a new method based on the well-known definition of Finite Fourier Transform.

(2) The method has been applied for finding the solution to a static, plane boundary value problem for long elastic, isotropic cylinder of rectangular normal cross-section subjected to different pressures on the lateral sides.
(3) The method is valid only for rectangular domains with no shear stress at the boundary.
(4) The solution in Ref. [1] was recovered.
(5) The method is simple and easy.
(6) The series involved in the solution is summable. This, however, may not be true for more complicated situations.
(7) The proposed method will be applied for a rectangular domain with variable heat supply in the bulk and variable pressure at the boundary.

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Received: 25 October 2013. Accepted: 24 November 2013.