NON-LOCAL THEORIES OF CONTINUUM MECHANICS
Abstract

The nonlocal theories of Continuum Mechanics provide an important field of application for fractional derivatives. Non-locality in time yields what we call materials with memory. In this presentation, we try to trace back the main achievements in the non-local theories of Continuum Mechanics, mainly in the field of Viscoelasticity.

It is the purpose of the presentation to enlighten the fact that fractional derivatives are not a matter of pure formalism, but they were motivated by real life requirements and they find genuine applications in practice.
Nonlocality is a manifestation of the atomic and molecular interactions in matter. In certain cases it is important to take into account the nonlocal intermolecular interactions, both in time and space. Stated otherwise, it is assumed that the state of a body at a certain location and at a certain time moment is influenced by its state at all the other locations, and at all the previous time moments. Phenomena like the fracture of solids, dislocations, stress concentration around holes and at the tips of cracks, ferroelectricity and viscoelastic behavior, and micropolar effects can be explained only if such a point of view is adopted. Such titles are now on the frontline of research because of their large domains of applicability.

Notions like hysteresis, relaxation and response functional are of primordial interest.

![Hysteresis loop in plasticity](image)

Domain structure in ferromagnetics. Cause of hysteresis.
At present, a popular approach in the literature is known under the general name of gradient theories, or sometimes higher gradient theories, which, according to Eringen's expression, describe models possessing limited nonlocality. Here, various order gradients and time rates of the thermodynamical determining parameters enter into response functions. Adapted from Maugin: "In most cases, however, the use of oversimplified kernels in nonlocal theories leads many practitioners to prefer an approach using the concept of a 'gradient theory' which yields 'nice' (although of an increased order) differential equations instead of integrodifferential equations with seemingly and physically equivalent solutions at the output."

Physically, the nonlocality is intimately linked to two intermolecular scales: The length scale over which the interaction between molecules is sensed, and the time scale of transmission of signal between neighboring molecules. In the former, "neighboring" may be taken 10 molecules away from the considered molecule, i.e. distances of the order of $10^{-5}$ cm and even more (as for granular media, for example).

Non-local kernels have to be built along clear strategies. They are calibrated based on the atomic scale structure of test materials obtained from atomistic simulations. The functional form of the kernels may involve more than one internal length scale. The success of one or another model is judged by its success in predicting results closer to reality.
MATERIALS WITH MEMORY

Assuming the relation between stress and strain in linear elasticity

\[ \sigma(t) = E \varepsilon(t). \]

It determines the value of stress at a certain moment in terms of the value of strain at the same moment. If now we assume that the stress depends on the whole history of strain, then one may write:

\[ \sigma(t) = E \left[ \varepsilon(t) - \chi \int_{-\infty}^{t} K(t-\tau) \varepsilon(\tau) d\tau \right]. \]

It was proved by Volterra that the kernel of relaxation \( K \) depends only on the combination \( t - \tau \) as a result of invariance of the value of stress with respect to the zero time reference. Later on, the lower bound of integration was changed to become a finite time value as a reference time.

Models based on nonlocality must comply with 8 axioms, 6 of which are common to all models of continuous media, viz.
Axioms of Causality, Determinism, Equipresence, Objectivity, Material Invariance and Admissibility,

in addition to two Axioms proper to nonlocality, which will be used specifically to suggest concrete mathematical forms for the relaxation kernels:

**Axiom of Neighborhood:** *The values of the independent constitutive variables at distant points do not appreciably affect the value of the constitutive-dependent variables at the considered point*

and

**Axiom of Memory:** *The values of the constitutive-independent variables at distant past do not appreciably affect the values of the constitutive functionals at the present time.*
Vito Volterra (1860-1940)
Italian
Mathematician

Tutors: Ulisse Dini, Enrico Betti

Visited Henri Poincaré in Paris in 1888

Met Hermann Schwartz and Leopold Kronecker in Berlin in 1891

Met Paul Painlevé, Emile Borel, Henri Lebesgue in Zurich in 1897

Succeeded Eugenio Beltrami in 1900 as Chair Mathematical Physics
at Rome University La Sapienza

THEORY OF INTEGRAL EQUATIONS
DISLOCATING SOLIDS
ELASTOSTATICS
Andrew Gemant (1895–1983)  
American, Born Hungarian  
Physicist

LINEAR VISCOELASTICITY  
FRACTIONAL CALCULUS

Andrew Gemant introduced the concept of complex viscosity which is often used for characterizing the viscous and elastic contributions to the rheological response measured in oscillatory shear experiments.
Blair wanted to describe something in-between elasticity and viscosity. He used fractional powers of time to model the characteristics of viscoelastic materials, but he could not achieve a rigorous definition of the fractional derivative that would satisfy the mathematicians. He abandoned his trials later on.
THEORY OF SHELLS
THEORY OF CREEP IN PLASTICITY
MECHANICS OF FRACTURE
MECHANICS OF COMPOSITE MATERIALS
FRACTIONAL CALCULUS

MOSCOW STATE UNIVERSITY

He used integro-differential operators with weakly singular kernels
Andrey Nicolaevich Gerasimov

The first to use proper fractional derivatives in viscoelasticity

In his paper

A generalization of linear laws of deformation and its applications to problems of internal friction. 
*Prikladnaia Matematika iMekhanica (J. of Appl. Mathematics and Mechanics)* 12, No 3 (1948), 251–260

A. Gerasimov introduced
History of Fractional derivatives

This concept appeared in 1695 and is most probably due to Gottfried Wilhelm Leibnitz (1646 - 1716), but it is also hidden in the work of Niels Henrick Abel (1802-1829) on the tautochrone problem.

List of fractional derivatives

Not like classical Newtonian derivatives, a fractional derivative is defined via a fractional integral.

Riemann–Liouville fractional derivative

The corresponding derivative is calculated using Lagrange's rule for differential operators. Computing $n$-th order derivative over the integral of order $(n - \alpha)$, the $\alpha$ order derivative is obtained. It is important to remark that $n$ is the nearest integer bigger than $\alpha$.

$$aD_t^\alpha f(t) = \frac{d^n}{dt^n}aD_t^{-(n-\alpha)} f(t) = \frac{d^n}{dt^n}aI_t^{n-\alpha} f(t)$$

Caputo fractional derivative

There is another option for computing fractional derivatives; the Caputo fractional derivative. It was introduced by M. Caputo in his 1967 paper.\[4\] In contrast to the Riemann Liouville fractional derivative, when solving differential equations using Caputo's definition, it is not necessary to define the fractional order initial conditions. Caputo's definition is illustrated as follows.

$$C_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}.$$

The following list summaries the fractional derivatives defined in the literature.\[5\]

Other types

Classical fractional derivatives include:

- Grünwald–Letnikov derivative
- Sonin–Letnikov derivative
- Liouville derivative
- Caputo derivative
- Hadamard derivative
- Marchaud derivative
- Riesz derivative
- Riesz-Miller derivative
• Miller–Ross derivative
• Weyl derivative
• Erdélyi–Kober derivative

New fractional derivatives include:

• Machado derivative
• Chen-Machado derivative
• Udita derivative
• Coimbra derivative
• Caputo-Katugampola derivative
• Hilfer derivative
• Davidson derivative
• Chen derivative


Generalizations

Erdélyi–Kober operator

The Erdélyi–Kober operator is an integral operator introduced by Arthur Erdélyi (1940) and Hermann Kober (1940) and is given by

\[ \frac{x^{-\nu-\alpha+1}}{\Gamma(\alpha)} \int_0^x (t-x)^{\alpha-1}t^{-\alpha-\nu} f(t) dt, \]

which generalizes the Riemann-Liouville fractional integral and the Weyl integral.

Further generalizations

A recent generalization introduced by Udita Katugampola (2011) is the following, which generalizes the Riemann-Liouville fractional integral and the Hadamard fractional integral. It is given by

\[ (\rho T_{a+}^\alpha f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho-1} f(\tau)}{(x^\rho - \tau^\rho)^{1-\alpha}} d\tau, \quad x > a. \]

Even though the integral operator in question is a close resemblance of the famous Erdélyi–Kober operator, it is not possible to obtain the Hadamard fractional integral as a direct consequence of the Erdélyi–Kober operator. Also, there is a Udita-type fractional derivative, which generalizes the Riemann-Liouville and the Hadamard fractional derivatives. As with the case of fractional integrals, the same is not true for the Erdélyi–Kober operator.
**Functional calculus**

In the context of functional analysis, functions $f(D)$ more general than powers are studied in the functional calculus of spectral theory. The theory of pseudo-differential operators also allows one to consider powers of $D$. The operators arising are examples of singular integral operators; and the generalisation of the classical theory to higher dimensions is called the theory of Riesz potentials. So there are a number of contemporary theories available, within which fractional calculus can be discussed. See also Erdélyi–Kober operator, important in special function theory (Kober 1940), (Erdélyi 1950–51).

**Applications**

**Fractional conservation of mass**

As described by Wheatcraft and Meerschaert (2008), a fractional conservation of mass equation is needed to model fluid flow when the control volume is not large enough compared to the scale of heterogeneity and when the flux within the control volume is non-linear. In the referenced paper, the fractional conservation of mass equation for fluid flow is:

$$-\rho \left( \nabla^\alpha \cdot \vec{u} \right) = \Gamma(\alpha + 1) \Delta x^{1-\alpha} \rho (\beta_s + \phi \beta_w) \frac{\partial p}{\partial t}$$

**Groundwater flow problem**

In 2013-2014 Atangana et al. described some groundwater flow problems using the concept of derivative with fractional order. In these works, The classical Darcy law is generalized by regarding the water flow as a function of a non-integer order derivative of the piezometric head. This generalized law and the law of conservation of mass are then used to derive a new equation for groundwater flow.

**Fractional advection dispersion equation**

This equation has been shown useful for modeling contaminant flow in heterogenous porous media. Atangana and Kilicman extended fractional advection dispersion equation to variable order fractional advection dispersion equation. In their work, the hydrodynamic dispersion equation was generalized using the concept of variational order derivative. The modified equation was numerically solved via the Crank-Nicholson scheme. The stability and convergence of the scheme in this case were presented. The numerical simulations showed that, the modified equation is more reliable in predicting the movement of pollution in the deformable aquifers, than the constant fractional and integer derivatives.

**Time-space fractional diffusion equation models**

Anomalous diffusion processes in complex media can be well characterized by using fractional-order diffusion equation models. The time derivative term is
corresponding to long-time heavy tail decay and the spatial derivative for diffusion nonlocality. The time-space fractional diffusion governing equation can be written as

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = -K(-\Delta)^\beta u. 
\]

A simple extension of fractional derivative is the variable-order fractional derivative, the \( \alpha, \beta \) are changed into \( \alpha(x, t), \beta(x, t) \). Its applications in anomalous diffusion modeling can be found in reference.\(^{[15]}\)\(^{[18]}\)

**Structural damping models**

Fractional derivatives are used to model viscoelastic damping in certain types of materials like polymers.\(^{[19]}\)

**Acoustical wave equations for complex media**

The propagation of acoustical waves in complex media, e.g. biological tissue, commonly implies attenuation obeying a frequency power-law. This kind of phenomenon may be described using a causal wave equation which incorporates fractional time derivatives:

\[
\nabla^2 u - \frac{1}{c_0^2} \frac{\partial^2 u}{\partial t^2} + \tau^\alpha_0 \frac{\partial^\alpha}{\partial t^\alpha} \nabla^2 u - \frac{\tau^\beta}{c_0^2} \frac{\partial^{\beta+2} u}{\partial t^{\beta+2}} = 0. 
\]

See also \(^{[20]}\) and the references therein. Such models are linked to the commonly recognized hypothesis that multiple relaxation phenomena give rise to the attenuation measured in complex media. This link is further described in \(^{[21]}\) and in the survey paper, \(^{[22]}\) as well as the acoustical attenuation article. See \(^{[23]}\) for a recent paper which compares fractional wave equations which model power-law attenuation.

**Fractional Schrödinger equation in quantum theory**

The fractional Schrödinger equation, a fundamental equation of fractional quantum mechanics, has the following form:\(^{[24]}\)

\[
i\hbar \frac{\partial \psi(r, t)}{\partial t} = D_\alpha (-\hbar^2 \Delta)^\frac{\alpha}{2} \psi(r, t) + V(r, t)\psi(r, t). 
\]

where the solution of the equation is the wavefunction \( \psi(r, t) \) - the quantum mechanical probability amplitude for the particle to have a given position vector \( r \) at any given time \( t \), and \( \hbar \) is the reduced Planck constant. The potential energy function \( V(r, t) \) depends on the system.

Further, \( \Delta = \partial^2 \partial r^2 \) is the Laplace operator, and \( D_\alpha \) is a scale constant with physical dimension \( [D_\alpha] = \text{erg}^{1-\alpha} \text{cm}^\alpha \text{sec}^{-\alpha} \), (at \( \alpha = 2, D_2 = 1/2m \) for a particle of mass \( m \)), and the operator \((-\hbar^2 \Delta)^\frac{\alpha}{2}\) is the 3-dimensional fractional quantum Riesz derivative defined by
\[ (-\hbar^2 \Delta)^{\alpha/2} \psi(r, t) = \frac{1}{(2\pi\hbar)^3} \int d^3p e^{i\mathbf{p} \cdot \mathbf{r}} |\mathbf{p}|^\alpha \varphi(p, t). \]

The index \( \alpha \) in the fractional Schrödinger equation is the Lévy index, \( 1 < \alpha \leq 2 \).

**References**


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