



ORIGINAL PAPER

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# Heat traveling waves in rigid thermal conductors with phase lag and stability analysis

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**Abstract** Recently, a model equation that describes nonlinear heat waves in a rigid thermal conductor has been derived. The system of the governing equations for temperature and heat flux is nonlinear. The objective of the present work is to find a variety of traveling wave solutions of this system of equations in the whole space. This is achieved by implementing the unified method. The obtained solutions are evaluated numerically and represented graphically. The behavior of these solutions is investigated, where it is shown that the temperature and the heat flux attain steady states in space, but increase with time. The effects of the characteristic length, time, heat flux, and reference temperature are studied via some material data. It is shown that the solutions may have the form of solitary wave, soliton, or soliton with double kinks. It is observed that the heat flux in the material is negative, this reflects the fact that heat flux is in the opposite direction of the normal vector to the material surface on which it is evaluated. The steady state solution of the considered model equation is studied. It is found that the stability of the solutions depends significantly on the wave number.

## 1 Introduction

Rigid heat conductors are classified as metal and nonmetal conductors. Examples of metal conductors are copper, aluminum, silver, and gold. Nonmetal conductors are metalloid, grease and graphite. The uses of thermal conductors in life manifests via a catenary, which is a system of overhead wires that supply electricity to a locomotive, streetcar or light rail vehicle. The study of heat wave propagation in continuous media has found growing interest in the past few studies. Such models have helped revealing interesting phenomena with practical applications in media of complex structure in which nonlinearity is tightly linked to stability in working conditions. Coleman and Newman [1] studied the implications of introducing a squared heat flux term in the free energy of the system, by which the heat flux and the temperature are treated as independent thermodynamical variables. Tarabek [2] investigated the existence of smooth solutions in one-dimensional nonlinear thermoelasticity with second sound, while Messaoudi et al. [3] considered the blow up of solutions in such systems. Ghaleb [4] and Gorgi and Montanaro [5] discussed models of nonlinear thermo-electroelasticity. Ghaleb et al. [6] proposed a model of nonlinear thermo-electroelasticity with many nonlinearities in extended thermodynamics, following Coleman. This model electroelasticity was further investigated by Abou-Dina and

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Ghaleb [7]. Rawy [8] discussed a restriction of this model to thermoelasticity. Shakeriaski and Ghodrat [9] studied the response of a thermoelastic material under a laser pulse in extended thermodynamics. Mahmoud et al. [10] studied nonlinear heat wave propagation in a rigid thermal conductor. In [10], solutions were obtained by different methods, and the effect of various material parameters was considered. Here is a variety of techniques which are used to find the exact solutions of nonlinear partial differential equations. Among them, the tanh and extended tanh methods [11, 12]. In [11], the extended tanh method is used to derive new soliton solutions for several forms of the fifth-order nonlinear KdV equation, Lax, Sawada–Kotera, Sawada–Kotera–Parker–Dye, Kaup–Kupershmidt, Kaup–Kupershmidt–Parker–Dye, and the Ito equations. Traveling wave solutions are obtained by using the modified extended tanh method for space-time fractional nonlinear partial differential equations [12]. In [13] the exact solutions of a compound KdV–Burgers equation are obtained, where in [14] the solitary wave solutions of the approximate equations for long water waves, the coupled KdV equations, and the dispersive long wave equations in  $2 + 1$  dimensions are constructed by using a homogeneous balance method. In [15] explicit formalisms for deep reductions of matrix differential equations and Darboux covariance properties are presented to explicit formulas of  $N$ -soliton solutions. In [16] Darboux transformation yields the variable separable solutions with two space-variable separated functions to find a new saddle-type ring soliton solution with completely elastic interaction and nonzero phase shifts. In [17] the  $\tilde{G}/G$ -expansion method is proposed and used to obtain the (TWS) involving parameters of the KdV equation, the mKdV equation, a variant of Boussinesq equations, and the Hirota–Satsuma equations. In [18] a generalized  $\tilde{G}/G$ -expansion method is proposed to seek exact solutions of the Benjamin–Bona–Mahony equation,  $(2+1)$ -dimensional generalized Zakharov–Kuznetsov equation, and a variant of Boussinesq equations. Triangular periodic wave solutions, hyperbolic function solutions, and Jacobian elliptic function solutions can be obtained as well. Moreover, it can also be used for many other nonlinear evolution equations in mathematical physics.

Here, the exact solutions are found by the unified method (UM) [19]. It has wide applications in investigating the behavior of the propagation of waves in shallow or in deep water. Solitary waves are also produced in compensated semiconductors [20] for determining the structure of pulse propagation in optical fibers. Also, solitary wave conduction appears in superionic conductors [21]. The (UM) covers most of all known methods in the literature such as the tanh, modified, and extended versions, the F-expansion, the exponential, and the  $\tilde{G}/G$ -expansion method [22–26]. The extended unified method [27] proposed by the first author may be sufficient to replace the analysis of inspecting the symmetries of partial differential equations that result when using Lie groups.

In the present work, we investigate a one-dimensional nonlinear system of two partial differential equations describing the propagation of heat waves in an infinite rigid thermal conductor. In these equations, the basic unknowns are the temperature and the heat flux. Dependence of the wave speed on the unknowns is taken in consideration. A multitude of wave solutions is obtained and illustrated graphically. This may be of interest in studying such materials in working conditions.

In view of the nonlinearity of the governing equations, we have restricted our considerations to the Cattaneo–Vernotte model, i.e., a model containing only one thermal relaxation time. However, inclusion of more than one thermal relaxation time is also possible, and will be dealt with in future work. Such complicated models provide better description of the physics, but involve more mathematical difficulties [28, 29]. The model used in this work finds application in the continuum description of media with complex structure [30].

## 2 The model equation

Recently, a one-dimensional system of equations has been presented in [10] for the propagation of heat waves in rigid thermal conductors. The main characteristic of the model is nonlinearity of the equations arising from two sources:

- (i) the presence of a quadratic dependence of the free energy on heat flux.
- (ii) the dependence of the thermal relaxation time and the coefficient of heat conduction on temperature and heat flux. It reads,

$$(1 + \theta)\theta_t + \eta_1(Q_x - Q\theta_x - \eta Q^2) = 0, \quad (1)$$

$$(1 + (1 + \mu_1)\theta + \mu_2 Q)Q_t + \frac{1}{\eta_1}(\eta Q + Q_x + \eta_2\theta\theta_x + \eta_3 Q\theta_x) = 0.$$

In Eq. (1), all symbols are dimensionless

**Table 1** List of symbols

Symbol	Name
$\theta$	Absolute temperature as measured from a reference temperature
$Q$	Heat flux component
$\mu_1, \mu_2$	Coupling constants
$C$	Specific heat capacity
$K$	Coefficient of heat conduction
$K_0$	Coefficient of heat conduction at reference temperature
$K_1, K_2$	Coupling constants for the coefficient of heat conduction
$L_0$	Characteristic length
$Q_0$	Characteristic heat flux
$T_0$	Characteristic time
$\Theta_0$	Reference temperature
$\rho$	Mass density
$\tau_0$	Thermal relaxation time at reference temperature
$\tau_1, \tau_2$	Coupling constants for the thermal relaxation time

There,  $\eta = \frac{L_0 Q_0}{\theta_0 K_0}$ ,  $\eta_1 = \frac{Q_0 \sqrt{\tau_0}}{\sqrt{\Theta_0 C K_0}}$ ,  $\eta_2 = \frac{K_1}{K_0}$ ,  $\eta_3 = \frac{K_2}{K_0}$ . Here we are interested in studying the traveling waves solution (TWS) of Eq. (1). To this issue, we introduce the transformations  $\theta(x, t) = \psi(z)$ ,  $Q(x, t) = \varphi(z)$ ,  $z = \alpha x + \beta t$ . Thus, Eq. (1) reduces to,

$$\beta(\psi + 1)\psi' - \eta_1(\eta\varphi^2 - \alpha\varphi' + \alpha\varphi\psi') = 0, \quad (2)$$

$$\varphi(\eta + \beta\eta_1\mu_2\varphi' + \alpha\eta_3\psi') + \varphi'(\alpha + \beta\eta_1((\mu_1 + 1)\psi + 1)) + \alpha\eta_2\psi\psi' = 0,$$

together with the boundary conditions  $\varphi(\infty) = A_1$ ,  $\psi(\infty) = B_1$ ,  $\varphi(-\infty) = A_2$ ,  $\psi(-\infty) = B_2$ , where  $A_i$  and  $B_i$  are given in the parameters  $\alpha, \beta, \mu_1, \eta_i, i = 1, 2, 3$ .

Here, the exact solutions of Eq. (2) are found using the unified method. Which asserts that, the solutions of a nonlinear partial differential equation are expressed in polynomial or rational forms in an auxiliary function that satisfies appropriate auxiliary equations.

### 3 Polynomial solutions of Eq. (2)

The solutions are represented in polynomial forms as,

$$\psi(z) = \sum_{i=0}^{n_1} a_i g^i(z), \quad \varphi(z) = \sum_{i=0}^{n_2} b_i g^i(z), \quad (3)$$

$$g'(z) = \sum_{i=0}^r c_i g^i(z).$$

We mention that, in Eq. (3),  $g(z)$  is the auxiliary function and the second equation is the auxiliary equation. Here,  $n_i$  and  $r$  are integers. The objective is to find  $n_i$  and  $r$ . To this end, balance of the nonlinear and higher order derivative terms is invoked. In Eq. (2), the balance is between  $\psi\psi'$  and  $\psi\varphi'$ . By writing  $\psi \sim g^{n_1}$ ,  $\varphi \sim g^{n_2}$  and  $g' \sim g^r$ , we get  $2n_1 + (r - 1) = n_1 + n_2 + (r - 1)$ . This holds when  $n_1 = n_2$  and when  $r$  is an arbitrary integer  $r = 1, 2, 3, \dots$

#### 3.1 When $r = 2$ and $n = 2$

In this case Eq. (3) becomes

$$\psi(z) = a_2 g(z)^2 + a_1 g(z) + a_0, \quad \varphi(z) = b_2 g(z)^2 + b_1 g(z) + b_0, \quad (4)$$

$$g'(z) = c_2 g(z)^2 + c_1 g(z) + c_0.$$

Inserting Eq. (4) into Eq. (2) and setting the coefficients of  $g(z)^j$ ,  $j = 0, 1, \dots$ , equal to zero, one gets

$$\begin{aligned}
 a_0 &= \frac{a_1 c_0 (\alpha b_0 \eta_1 - \beta) - \alpha b_1 c_0 \eta_1 + b_0^2 \eta_1}{a_1 \beta c_0}, \quad a_2 = \frac{a_1 b_1 (\alpha a_1 c_0 - b_0 \eta)}{4 \alpha a_1 b_0 c_0 + 2 \alpha b_1 c_0 - 2 b_0^2 \eta}, \\
 b_2 &= \frac{4 a_1 b_0^3 \eta^2 (\alpha a_1 c_0 - b_0 \eta)}{4 \alpha^3 a_1^2 c_0^3 - \alpha b_0^2 c_0 \eta^2}, \quad c_1 = \frac{b_1 c_0}{b_0}, \quad c_2 = \frac{4 a_1 b_0^2 \eta^2 (\alpha a_1 c_0 - b_0 \eta)}{\alpha (2 \alpha a_1 c_0 - b_0 \eta) (2 \alpha a_1 c_0 + b_0 \eta)}, \\
 \eta_2 &= -\frac{\beta b_2 \eta_1 (a_2 \mu_1 + a_2 - b_2 \mu_2)}{2 \alpha a_2^2}, \quad \eta_3 = -\frac{\beta \eta_1 (a_2 \mu_1 + a_2 - b_2 \mu_2)}{\alpha a_2} - \frac{a_2 \eta}{b_2}, \\
 \mu_1 &= \frac{(2 \alpha a_1 c_0 - b_0 \eta) (4 \alpha^2 a_1^2 c_0^2 - \alpha a_1 b_0 (b_0 + 6) c_0 \eta + b_0^2 \eta^2 (3 b_0^2 \eta^2 + b_0 + 2))}{b_0^4 \eta^2 \eta^2 (5 b_0 \eta - 8 \alpha a_1 c_0)}, \\
 \mu_2 &= -\left( (b \eta - 2 \alpha a_1 c_0 (-8 \alpha^3 a_1^3 c_0^3 - 6 \alpha^2 a_1^2 (b_0 - 2) b_0 c_0^2 \eta \right. \\
 &\quad \left. + \alpha + a_1 b_0^2 c_0 \eta^2 (-6 b_0^2 \eta_1^2 + 7 b_0 - 6) + b_0^3 \eta^3 (4 b_0^2 \eta_1^2 - 2 b_0 + 1)) \right) \\
 &\quad / (2 b_0^6 \eta^3 \eta^2 (5 b_0 \eta - 8 \alpha a_1 c_0)), \quad \beta = \frac{4 \alpha b_0^2 \eta \eta_1}{2 \alpha a_1 c_0 - b_0 \eta}.
 \end{aligned}
 \tag{5}$$

Inserting Eq. (5) into Eq. (4), we have

$$\begin{aligned}
 \psi(z) &= \frac{1}{P} (4 a_1^3 c_0^3 \alpha^3 - 12 a_1^2 b_0 c_0^2 \eta \alpha^2 - 5 a_1 b_0^2 c_0 \alpha \eta^2 + b_0^3 \eta^3 \\
 &\quad + 4 a_1^2 b_0 c_0 \alpha \eta (2 a_1 c_0 \alpha + b_0 \eta) g(z) + 4 a_1^2 b_0^2 \eta^2 (a_1 c_0 \alpha - b_0 \eta) g(z)^2), \\
 P &= 4 a_1 b_0 c_0 \alpha \eta (2 a_1 c_0 \alpha + b_0 \eta), \\
 \varphi(z) &= b_0 + \frac{2 b_0^2}{\alpha c_0} g(z) + \frac{4 a_1 b_0^3 \eta^2 (a_1 c_0 \alpha - b_0 \eta) g(z)^2}{4 a_1^3 c_0^3 \alpha^3 - b_0^2 c_0 \alpha \eta^2}.
 \end{aligned}
 \tag{6}$$

The solution of the auxiliary equation is

$$\begin{aligned}
 g(z) &= \frac{-M_2 M_1 \tanh\left(\frac{\eta M_2 (4 \alpha^2 A a_1^2 b_0^2 c_0^2 - A b_0^4 \eta^2 + c_0 z)}{\alpha c_0 \sqrt{2 \alpha a_1 c_0 - b_0 \eta} (2 \alpha a_1 c_0 b_0 \eta)}\right) - 4 \alpha^2 a_1^2 b_0 c_0^2 + b_0^3 \eta^2}{4 \alpha a_1^2 b_0^2 c_0 \eta - 4 a_1 b_0^3 \eta^2}, \\
 \eta &> 0, \quad \alpha > 0, \quad c_0 > 0, \quad b_0 > 0, \quad \frac{b_0 \eta}{4 c_0 \alpha} < a_1 < \frac{b_0 \eta}{c_0 \alpha}.
 \end{aligned}
 \tag{7}$$

Inserting Eq. (7) into Eq. (6), we get

$$\begin{aligned}
 \theta(x, t) &= \frac{1}{4 P} (b_0 (-40 a_1^2 c_0^2 \alpha^2 + 34 a_1 b_0 c_0 \eta \alpha - 3 b_0^2 \eta^2) \\
 &\quad - 2 M_1 M_2 \tanh\left[\frac{(A_0 + c_0 z) \eta M_2}{c_0 \alpha M_1 (2 a_1 c_0 \alpha + b_0 \eta)}\right] + b_0 (8 a_1^2 c_0^2 \alpha^2 \\
 &\quad - 6 a_1 b_0 c_0 \eta \alpha + b_0^2 \eta^2) \tanh\left[\frac{(A_0 + c_0 z) \eta M_2}{c_0 \alpha M_1 (2 a_1 c_0 \alpha + b_0 \eta)}\right]^2),
 \end{aligned}
 \tag{8}$$

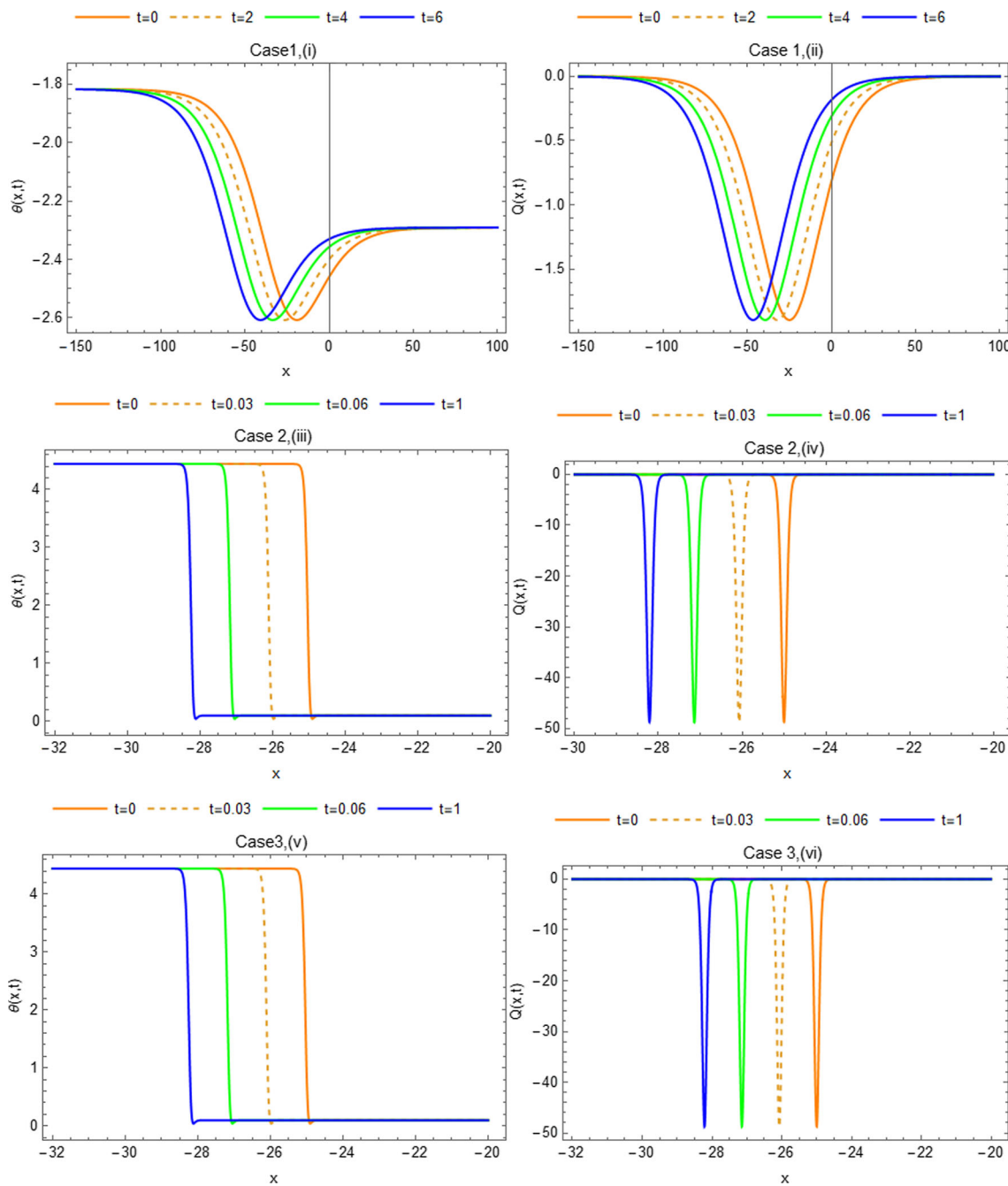
$$Q(x, t) = \frac{(b_0^2 \eta - 4 a_1 c_0 \alpha + b_0 \eta) \operatorname{sech}\left(\frac{(A_0 + c_0 z) \eta M_2}{c_0 \alpha M_1 (2 a_1 c_0 \alpha + b_0 \eta)}\right)^2}{4 a_1 c_0 (a_1 c_0 \alpha - b \eta)}, \quad M_1 = \sqrt{2 a_1 c_0 \alpha - b_0 \eta},$$

$$M_2 = \sqrt{b_0^3 \eta (8 a_1^2 c_0^2 \alpha^2 + 2 a_1 b_0 c_0 \eta \alpha - b_0^2 \eta^2)}, \quad z = \alpha x + \beta t,$$

and  $P$  is given by Eq. (6). Here, we focus our study on the behavior of temperature and heat flux of the material with properties given in Table 2.

**Table 2** Dimensions of the parameters

Quantity	Unit	Case-I	Case-II	Case-III
$L_0$	$m$	$0.69 \times 10^{-10}$	$0.69 \times 10^{-10}$	$0.69 \times 10^{-10}$
$\phi_0$	$W.m^{-2}$	$0.42 \times 10^{15}$	$3.422 \times 10^{17}$	$2.421 \times 10^{17}$
$\theta_0$	$K$	293	293	293
$\tau_0$	$s.m^{-3}.kg^{-1}.K^{-1}$	$0.5 \times 10^{-18}$	$0.1 \times 10^{-20}$	$0.8 \times 10^{-21}$
$K_0$	$W.m^{-1}.K^{-1}$	$1.0 \times 10^4$	$1.0 \times 10^4$	$1.0 \times 10^4$
$C$	$J.kg^{-1}.K^{-1}$	$0.3 \times 10^5$	$0.4 \times 10^8$	$1.6 \times 10^7$



**Fig. 1** (i)–(vi) When  $b_0 = 10, c_0 = 0.1, a_1 = 3, \alpha = 2, A_0 = 5$ . Case 1:  $\eta = 0.00989078, \eta_1 = 1.00171$ , Case 2:  $\eta = 0.999621, \eta_1 = 0.999577$ , Case 3:  $\eta = 0.999799, \eta_1 = 1.00011$  (see Table 2)

The numerical results of the solutions in Eq. (8) for  $\theta(x, t)$  and  $Q(x, t)$  are displayed against  $x$  for different values of  $t$  in Fig. 1(i)–(ii) and (iv)–(vi), respectively. The values of  $\eta = \frac{L_0\phi_0}{\theta_0 K_0}$ ,  $\eta_1 = \frac{Q_0\sqrt{\tau_0}}{\sqrt{\Theta_0 C K_0}}$  are taken as in [10].

- (i) Shows a soliton with double kinks for the temperature, which attains a steady state for large  $x$ , while it increases with time  $t$ , while (ii) shows a soliton for the heat flux.
- (iii) Shows a solitary wave for the temperature, while (iv) shows a soliton for the heat flux. In (v), the behavior of the TWS of the temperature is solitary, while in (vi) it is soliton for the heat flux.

### 3.2 When $r = 3$ and $n = 4$

In this case, we write,

$$\begin{aligned} \psi(z) &= a_4g(z)^4 + a_3g(z)^3 + a_2g(z)^2 + a_1g(z) + a_0, \\ \varphi(z) &= b_4g(z)^4 + b_3g(z)^3 + b_2g(z)^2 + b_1g(z) + b_0, \\ g'(z) &= (a^2 - b^2g(z)^2)g(z). \end{aligned} \tag{9}$$

Inserting Eq. (9) into Eq. (2), and by the same way as in the above, we get

$$\begin{aligned} b_0 &= b_1 = b_3 = 0, \quad a_1 = a_3 = 0, \\ b_4 &= \left( b^4 (M - a^2 (17a_0^3 + 43a_0^2 + 23a_0 - 3) \eta)^2 \right. \\ &\quad \left. a^2 (17a_0^3 + 43a_0^2 + 23a_0 - 3) \eta + M \right) / \\ &\quad 64a^{10}(a_0 + 2)^2 (17a_0^2 + 26a_0 - 3)^2 (37a_0^2 + 98a_0 + 61) \eta^3 \eta_1^4, \\ b_2 &= - \left( 3(5a_0 + 8)b^2 (a^2 (17a_0^3 + 43a_0^2 + 23a_0 - 3) \eta - M) / \right. \\ &\quad \left. 4a^4(37a_0 + 61) (17a_0^2 + 26a_0 - 3) \eta \eta_1^2 \right), \\ a_2 &= \frac{3(a_0+1)(3a_0+4)b^2}{2a^2(37a_0+61)}, \quad a_4 = -\frac{3(a_0+1)^2b^4}{2a^4(37a_0+61)}, \quad \beta = -\frac{\alpha\eta_1(\mu_1+1)}{\mu_2}, \\ \eta_3 &= -\frac{\beta\eta_1(a_4\mu_1+a_4+a_4\mu_2)}{\alpha a_4} - \frac{a_4\eta_2}{b_4}, \quad \eta_2 = -\frac{\beta b_4\eta_1(a_4\mu_1+a_4-3b_4\mu_2)}{4\alpha a_4^2}, \\ \mu_1 &= \frac{-8\alpha^2 a^4 (a_0^3 + 5a_0^2 + 7a_0 + 2)\eta_1^2 + \alpha a^2 (a_0 + 1)^2 \eta + (a_0 + 1)^2 \eta^2}{8a^4 \alpha^2 a_0 (a_0 + 2)^2 \eta_1^2}, \\ \alpha &= \frac{a^2 (17a_0^3 + 43a_0^2 + 23a_0 - 3) \eta - M}{16a^4 (17a_0^3 + 60a_0^2 + 49a_0 - 6) \eta_1^2}, \quad \beta = -\frac{\alpha\eta_1(\mu_1+1)}{\mu_2}. \end{aligned} \tag{10}$$

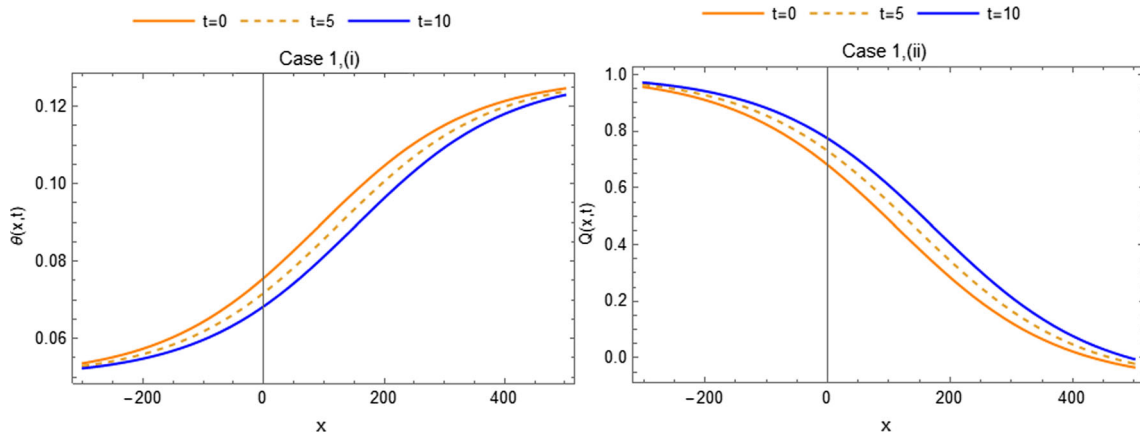
The solution of the auxiliary equation is

$$g(z) = \frac{aA_0e^{a^2z}}{\sqrt{1+b^2A_0^2e^{2a^2z}}}. \tag{11}$$

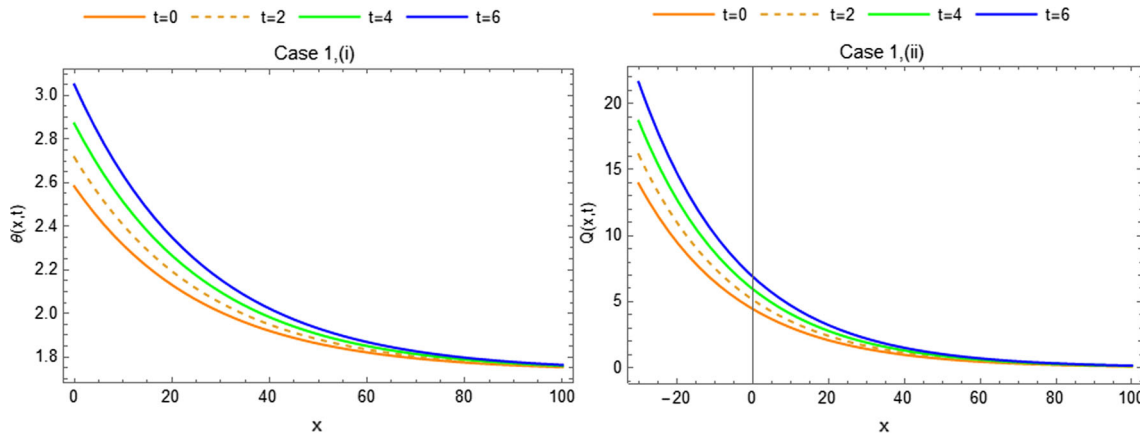
Finally, the solutions are

$$\begin{aligned} \theta(x, t) &= a_0 - \frac{3(a_0+1)^2 A_0^4 b^4 e^{4a^2z}}{2(37a_0+61)(A_0^2 b^2 e^{2a^2z} + 1)^2} + \frac{3(a_0+1)(3a_0+4)A_0^2 b^2 e^{2a^2z}}{2(37a_0+61)(A_0^2 b^2 e^{2a^2z} + 1)}, \\ Q(x, t) &= - \left( \left( 3A_0^2 b^2 e^{2a^2z} \left( a_0 \left( 4A_0^2 b^2 e^{2a^2z} + 5 \right) + 7A_0^2 b^2 e^{2a^2z} + 8 \right) \right. \right. \\ &\quad \left. \left. - \frac{M}{a^2\eta} + 17a_0^3 + 43a_0^2 + 23a_0 - 3 \right) / (4(37a_0 + 61) \right. \\ &\quad \left. \left. (17a_0^2 + 26a_0 - 3) \left( A_0^2 b^2 \eta_1 e^{2a^2z} + \eta_1 \right)^2 \right) \right), \quad z = \alpha x + \beta t, \end{aligned} \tag{12}$$

$$M = a^2\eta\sqrt{-(a_0 + 1)^2 (17a_0^2 + 26a_0 - 3) (a_0^2 (48\eta_1^2 - 17) + 2a_0 (96\eta_1^2 - 13) + 192\eta_1^2 + 3)}.$$



**Fig. 2** (i) and (ii) Numerical results of the solutions in Eq. (12) for  $\theta(x, t)$  and  $Q(x, t)$  are displayed against  $x$  for different values of  $t$  when  $b_0 = -10, c_0 = 0.1, a_1 = 3, \alpha = 2, A_0 = 5$ . In case-I,  $\eta = 0.00989078, \eta_1 = 1.00171$



**Fig. 3** (i) and (ii) When  $\alpha = 1.3, \beta = -2.5, A_0 = -5$  and in case-I, from the Table 2,  $\eta = 0.00989078, \eta_1 = 1.00171$  It is remarked that the temperature and heat flux increase with time

where  $b_i, i = 2, 4$  are given in Eq. (10). The results of the solutions in Eq. (12) for  $\theta(x, t)$  and  $Q(x, t)$  are displayed against  $x$  for different values of  $t$  in Fig. 2(i) and (ii). We focus on case-I in Table 2 and  $\eta = \frac{L_0 \phi_0}{\theta_0 K_0}, \eta_1 = \frac{Q_0 \sqrt{\tau_0}}{\sqrt{\Theta_0 C K_0}}$ .

(i) and (ii) show that the temperature and the heat flux decrease with time and attain steady states in space. The qualitative behavior is solitary wave.

**4 Rational solutions of Eq. (2)**

A rational solution (TWS) of Eq. (2) is written in the form

$$\psi(z) = \frac{a_1 g(z) + a_0}{s_1 g(z) + s_0}, \quad \varphi(z) = \frac{b_1 g(z) + b_0}{s_1 g(z) + s_0}, \tag{13}$$

$$g'(z) = \sum_{j=0}^{j=k} c_j g(z)^j.$$

Here, we consider two cases.

4.1 When  $k = 2$ 

The auxiliary equation reads

$$g'(z) = c_2 g(z)^2 + c_1 g(z) + c_0. \quad (14)$$

From Eqs. (13) and (14) and insertion into Eq. (2), we have

$$\begin{aligned} c_2 &= \frac{1}{2s_1}, c_1 = \frac{2\beta\eta(\sqrt{2\eta_1^2+1}\eta_1^2 - \sqrt{2\eta_1^2+1} - 1)}{\alpha^2\eta_1(2\eta_1^2-3)}, c_0 = 0, \\ a_1 &= \frac{1}{c_2} - s_1, a_0 = \frac{4\beta\eta\eta_1^3s_1^2 - \alpha^2s_0}{\alpha^2(\eta_1^2-1)}, b_1 = \frac{2\beta\eta_1s_1}{\alpha}, \\ b_0 &= \frac{a_1s_0(\beta - \alpha b_1c_2\eta_1) + a_0s_1(\alpha b_1c_2\eta_1 - \beta) + b_1\eta_1s_1(\alpha c_2s_0 - b_1\eta)}{\alpha c_2\eta_1s_1^2}, \\ s_0 &= -\frac{4(\alpha^2\beta\eta s_1^2 - \alpha^2\beta\eta(\eta_1^2-1)\sqrt{2\eta_1^2+1}s_1^2)}{\alpha^4(2\eta_1^3-3\eta_1)}, \\ \eta_2 &= \left(2\alpha\beta(2\eta_1^6-3\eta_1^4-2\sqrt{2\eta_1^2+1}\eta_1^2+2\sqrt{2\eta_1^2+1}+2)\right. \\ &\quad + \alpha^2\eta_1(2\eta_1^2-3)\left(\sqrt{2\eta_1^2+1}\eta_1^2-\sqrt{2\eta_1^2+1}-1\right) + 2\beta^2\eta_1\left(2\left(\sqrt{2\eta_1^2+1}+1\right)(\mu_1+2)\right. \\ &\quad \left. + \eta_1^2\left(-2\sqrt{2\eta_1^2+1}+2\mu_1+2\right) + 2\eta_1^6\left(\sqrt{2\eta_1^2+1}+1\right.\right. \\ &\quad \left.\left.\sqrt{2\eta_1^2+1}\mu_1\right) - \eta_1^4\left(3\sqrt{2\eta_1^2+1}\mu_1+3\sqrt{2\eta_1^2+1}+4\mu_1+7\right)\right)\Big)/ \\ &\quad \alpha\eta_1\left(-2\eta_1^2+\sqrt{2\eta_1^2+1}+1\right)\left(-2\eta_1^4+\eta_1^2+\sqrt{2\eta_1^2+1}+1\right), \\ \eta_3 &= \left(-\alpha^3\eta_1(2\eta_1^2-3)\left(-2\eta_1^2+\sqrt{2\eta_1^2+1}+1\right) + 2\alpha^2\beta\left(2\left(\sqrt{2\eta_1^2+1}-1\right)\eta_1^4+2\left(\sqrt{2\eta_1^2+1}+1\right)\right.\right. \\ &\quad \left.\left.-\eta_1^2\left(3\sqrt{2\eta_1^2+1}+1\right)\right) + 2\alpha\beta^2\eta_1\left(2\eta_1^4\left(\sqrt{2\eta_1^2+1}-1\right) + 2\left(\sqrt{2\eta_1^2+1}+1\right) - \eta_1^2\left(3\sqrt{2\eta_1^2+1}+1\right)\right)\right) \\ &\quad + 4\beta^3\left(2\sqrt{2\eta_1^2+1}\eta_1^6+2\eta_1^2+2\left(\sqrt{2\eta_1^2+1}+1\right)\right. \\ &\quad \left.-\eta_1^4\left(3\sqrt{2\eta_1^2+1}+4\right)\right)\mu_2\Big)/\left(2\alpha^2\beta\left(-2\eta_1^2+\sqrt{2\eta_1^2+1}+1\right) - 2\eta_1^4+\eta_1^2+\sqrt{2\eta_1^2+1}+1\right). \end{aligned} \quad (15)$$

The solution of the auxiliary equation gives rise to

$$\begin{aligned} g(z) &= -\frac{1}{\alpha^2\eta_1(-3+2\eta_1^2)}(2s_1\beta\eta\left(-1+\sqrt{1+2\eta_1^2}(-1+\eta_1^2)\right) \\ &\quad (1+\tanh((A_0+z)\beta\eta\left(-1+\sqrt{1+2\eta_1^2}(-1+\eta_1^2)\right))). \end{aligned} \quad (16)$$

Finally the solutions are

$$\begin{aligned} \theta(x,t) &= \frac{P_1}{R_1}, \quad P_1 = 1 - 4\eta_1^4 + \sqrt{1+2\eta_1^2}(1+\eta_1^2) + 2\eta_1^2 + (-1 + \sqrt{1+2\eta_1^2} \\ &\quad (-1 + \eta_1^2)) \left(1 + \tanh((A_0 + \alpha x + \beta t)\beta\eta\left(-1 + \sqrt{1+2\eta_1^2}(-1 + \eta_1^2)\right))\right), \\ R_1 &= \left(-1 + \sqrt{1+2\eta_1^2}(-1 + \eta_1^2)\right) (-1 + \tanh((A_0 + \alpha x + \beta t) \end{aligned}$$



$$\begin{aligned}
& \beta\eta \left( -1 + \sqrt{1 + 2\eta_1^2(-1 + \eta_1^2)} \right), \\
Q(x, t) = & \frac{2\beta\eta_1 \left( \tanh \left( \frac{\beta\eta(\sqrt{2\eta_1^2+1}\eta_1^2 - \sqrt{2\eta_1^2+1}) (A_0+z)}{\alpha^2\eta_1(2\eta_1^2-3)} \right) + 1 \right)}{\alpha \left( \tanh \left( \frac{\beta\eta(\sqrt{2\eta_1^2+1}\eta_1^2 - \sqrt{2\eta_1^2+1}) (A_0+z)}{\alpha^2\eta_1(2\eta_1^2-3)} \right) - 1 \right)}, \quad z = \alpha x + \beta t.
\end{aligned} \tag{17}$$

The numerical results of the solutions in Eq. (17) for  $\theta(x, t)$  and  $Q(x, t)$  are displayed against  $x$  for different values of  $t$  in Fig. 3(i) and (ii), respectively.

#### 4.2 When $k = 2$

Here, we take the auxiliary equation

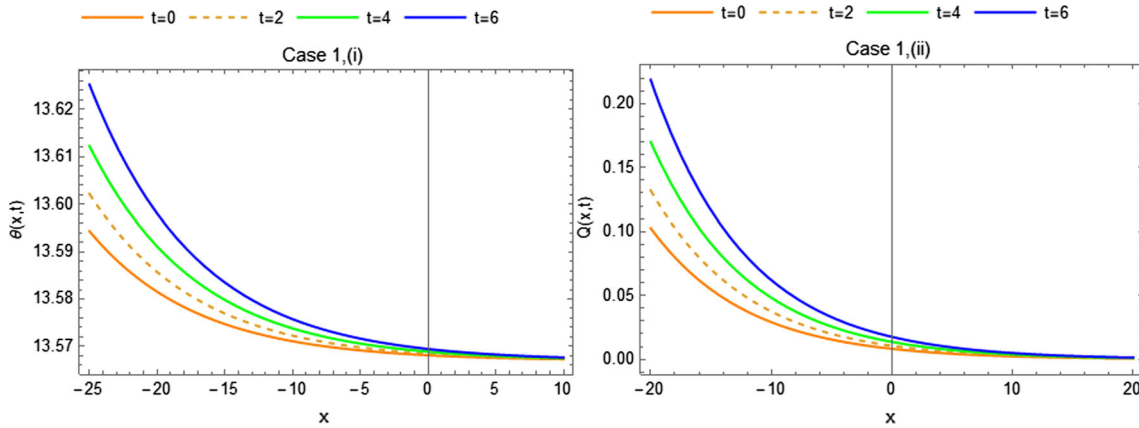
$$g'(z) = a - bg(z)^2. \tag{18}$$

By the same way, we have

$$\begin{aligned}
a &= \frac{bs_1^2}{s_0^2} \quad b_1 = -\frac{b_0s_1}{s_0}, \\
a_1 &= \frac{s_1(a_0^2b\beta s_0 + a_0bs_0(\beta s_0 - \alpha b_0\eta_1) + b_0\eta_1(2\alpha bs_0^2 + b_0\eta s_1))}{bs_0^2(a_0\beta - \alpha b_0\eta_1 + \beta s_0)}, \\
a_0 &= \frac{1}{2bs_1s_0\beta^2\eta} (-2b^2s_0^3\alpha^2\beta\eta_1 + bs_1s_0\beta\eta(-2s_0\beta + b_0\alpha\eta_1) + H), \\
\eta_2 &= (2bs_0a_0\beta - \alpha b_0\eta_1 + \beta s_0(2a_0^2b\beta^2\eta_1(\mu_1 + 1)s_0 \\
& \eta + s_1(\beta b_0^2\eta_1^2(\mu_1 + 1) + \alpha b_0\eta_1s_0 - \beta s_0^2) + 2bs_0^2(\beta s_0 \\
& (\alpha + \beta\eta_1) + \alpha b_0\eta_1(\beta\eta_1\mu_1 - \alpha) + a_0\beta s_0 \\
& - 2\alpha bb_0\eta_1^2(\mu_1 + 1) + 2bs_0(\alpha + \beta\eta_1(\mu_1 + 2)) - \eta s_1)) / \\
& (\alpha 2\alpha bs_0^2 + b_0\eta s_1\eta_1(2a_0^2b\beta s_0 \\
& 2a_0bs_0(\beta s_0 - \alpha b_0\eta_1) + b_0\eta_1(2\alpha bs_0^2 + b_0\eta s_1))),
\end{aligned} \tag{19}$$

$$\begin{aligned}
\eta_3 &= (s_0a_0\beta - \alpha b_0\eta_1 + \beta s_0(2b\eta s_1 - b_0\eta^2s_1^2 \\
& \beta b_0^2\eta_1\mu_2 + b_0s_0(\alpha + \beta\eta_1) - \alpha s_0^2 + 4b^2s_0(\alpha s_0^2(\alpha + \beta\eta_1) \\
& + a_0\beta\mu_2(a_0\beta - \alpha b_0\eta_1) + \beta\mu_2s_0(a_0\beta + \alpha b_0\eta_1)))) / \\
& (\alpha^2\alpha bs_0^2 + b_0\eta s_1(2a_0^2b\beta s_0 + b_0 \\
& 2a_0bs_0(\beta s_0 - \alpha b_0\eta_1) + \eta_1(2\alpha bs_0^2 + b_0\eta s_1))),
\end{aligned}$$

$$H = (2bs_0^2\alpha + b_0s_1\eta)\sqrt{bs_1s_0\beta^2\eta_1}\sqrt{-2s_1\beta\eta + bs_0\alpha^2\eta_1}.$$



**Fig. 4** (i) and (ii). When  $A_0 = -5, b = 0.07, s_1 = 2.5, s_0 = 1.5, b_0 = 5, \alpha = 1.5, \beta = -1.5,$  and  $\eta = 0.00989078, \eta_1 = 1.00171$

Finally, the solutions are,

$$\begin{aligned} \theta(x, t) = & (2\alpha^2 b^2 \beta \eta \eta_1 s_1 s_0^3 (\alpha b_0 \eta_1 - \beta s_0) + 4\alpha^4 b^3 \beta \eta_1^2 s_0^5 \\ & \alpha(-b) \eta_1 s_0^2 (3\beta^2 b_0 \eta^2 s_1^2 + 2\alpha H) + \beta \eta(-s_1) (\beta b_0^2 \eta^2 \eta_1 s_1^2 + H s_0) \\ & + (2\alpha^2 b^2 \beta \eta \eta_1 s_1 s_0^3 (\beta s_0 - \alpha b_0 \eta_1) - 4\alpha^4 b^3 \beta \eta_1^2 s_0^5 \\ & \alpha b \eta_1 s_0^2 (2\alpha H - \beta^2 b_0 \eta^2 s_1^2) + \beta \eta(+s_1) (H s_0 - \beta b_0^2 \eta^2 \eta_1 s_1^2)) \\ & \tanh\left(\frac{s_0(A_0 - bz)}{s_1}\right) / (s_1 s_0 \beta \eta (2\alpha^2 b^2 \beta \eta_1 s_0^3 \\ & \alpha + b \beta b_0 \eta \eta_1 s_1 s_0 - H) \tanh\left(\frac{s_0(A_0 - bz)}{s_1}\right) - 1), \end{aligned} \tag{20}$$

$$Q(x, t) = \frac{b_0(1 + \tanh(\frac{s_0(A_0 - bz)}{s_1}))}{s_0(-1 + \tanh(\frac{s_0(A_0 - bz)}{s_1}))}, \quad z = \beta t + \alpha x.$$

The numerical results of the solutions in Eq. (20) for  $\theta(x, t)$  and  $Q(x, t)$  are displayed against  $x$  for different values of  $t$  in Fig. 4(i) and (ii), respectively. (i) and (ii) show that the temperature and heat flux increase with time.

It is worthy to mention that the role of varying the parameters  $\eta_2, \eta_3,$  and  $\mu_2$  is considered, but we have observed that there is no significant contribution. So, the figures were omitted.

### 5 Stability analysis

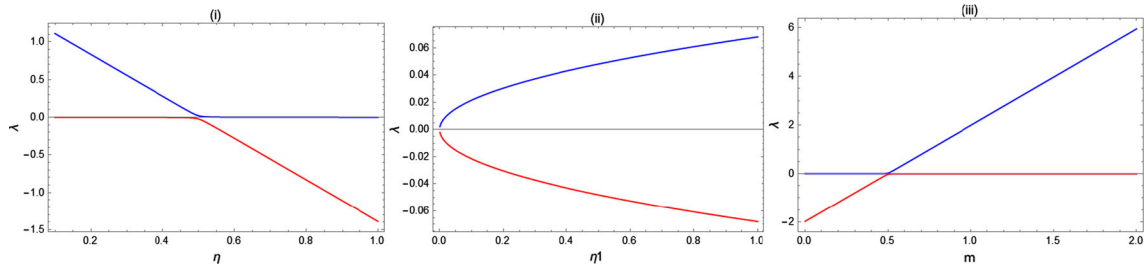
Here, we analyze the stability of the steady state solutions of Eq. (1). The steady state solutions hold by setting  $\theta_t = 0$  and  $Q_t = 0$ , where  $\theta(x, t) = h(x)$  and  $Q(x, t) = p(x)$ , which satisfy the equations

$$\begin{aligned} -\eta p(x)^2 - p(x)h'(x) + p'(x) &= 0, \\ \eta_2 h(x)h'(x) + p(x)\eta + \eta_3 h'(x) + p'(x) &= 0. \end{aligned} \tag{21}$$

The solutions are  $p(x) = 0, h(x) = h_0.$

We write

$$\theta(x, t) = h_0 + \varepsilon_1 e^{\lambda t} H(x), \quad Q(x, t) = \varepsilon_2 e^{\lambda t} K(x). \tag{22}$$



**Fig. 5** (i), (ii), and (ii), Eigenvalue  $\lambda$  against  $\eta$ ,  $\eta_1$ , and  $m$ , respectively. When (i)  $h_0 = 2$ ,  $m = 0.5$ ,  $\mu_1 = 0.3$ ,  $\eta_1 = 0.1$ ,  $\eta_2 = 0.1$ , (ii)  $h_0 = 2$ ,  $m = 0.5$ ,  $\mu_1 = 0.3$ ,  $\eta = 0.5$ ,  $\eta_2 = 1$ , and (iii)  $h_0 = 0.2$ ,  $\eta = 0.5$ ,  $\eta_1 = 0.2$ ,  $\eta_2 = 0.1$ ,  $\mu_1 = 0.3$ .

Substituting Eq. (22) into Eq. (1), we get

$$M \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = 0, \quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},$$

$$m_{11} = \lambda h(x)H(x) - \eta_1 p(x)H'(x) + \lambda H(x),$$

$$m_{12} = -\eta_1 h'(x)K(x) - 2\eta\eta_1 p(x)K(x) + \eta_1 K'(x), \tag{23}$$

$$m_{21} = \eta_2 H(x) h'(x) + \eta_2 h(x) H'(x) + \eta_3 p(x) H'(x),$$

$$m_{22} = \frac{\eta_3 h'(x)K(x)}{\eta_1} + \lambda\mu_1 h(x)K(x) + \lambda h(x)K(x) + \lambda\mu_2 p(x)K(x) + \frac{\eta K(x)}{\eta_1} + \lambda K(x) + \frac{K'(x)}{\eta_1},$$

which gives rise

$$\eta_1 (\eta_2 H(x)h'(x) + H'(x)(\eta_2 h(x) + \eta_3 p(x))) (K(x) (h'(x) + 2\eta p(x)) - K'(x)) + \frac{1}{\eta_1} [\lambda(h(x) + 1) H(x) - \eta_1 p(x)H'(x)(K(x)(\eta + \eta_1\lambda + \eta_1\lambda) + \eta_3 h'(x) + (\mu_1 + 1)h(x) + \eta_1\lambda\mu_2 p(x)) + K'(x))] = 0. \tag{24}$$

The eigenvalue problem in Eq. (24) is subjected to the boundary conditions (BCs)  $H(\pm\infty) = 0$  and  $K(\pm\infty) = 0$ . To this issue, we assume that

$$H(x) = H_0 \begin{cases} e^{-mx} & m > 0, x > 0 \\ e^{mx} & m > 0, x < 0, \end{cases} \tag{25}$$

$$K(x) = K_0 \begin{cases} e^{-mx} & m > 0, x > 0 \\ e^{mx} & m > 0, x < 0. \end{cases}$$

Substituting Eq. (25) into Eq. (24), we have,

$$\lambda = \frac{-\eta - \eta h_0 \pm \sqrt{(h_0 + 1) (4\eta_1^3 \eta_2 h_0 m^2 (h_0 \mu_1 + h_0 + 1) + (h_0 + 1)(m - \eta)^2) + h_0 m + m}}{2\eta_1 (h_0 + 1)(h_0 \mu_1 + h_0 + 1)}. \tag{26}$$

We mention that the steady state solution is saddle node. The results in (26) are shown in Fig. 5 (i) and (ii).

- (i) the solutions are stable or unstable, and the point at  $m = 0.5$  is a saddle point.
- (ii) the solutions are stable or unstable with critical point  $\eta_1 = 0.01$ .
- (iii) when  $m > 0.5$  the solutions are stable, when  $m < 0.5$  the solutions are stable or unstable and the point at  $m = 0.5$  is a saddle point.

## 6 Conclusions

The temperature–heat flux model equation for nonlinear heat waves in a rigid conductor is considered. Exact traveling waves solutions are found using the (UM). The results are illustrated in a variety of graphs. It is found that nonlinear heat waves are solitary waves. The attained states in space and the temperature and heat flux increase with time. The effects of the characteristic parameters, length, time, and heat flux are investigated and shown in a graph. It is found that the solutions are solitary, soliton, or soliton with double kinks. It is remarked that the stability of the solutions depends critically on the wave number of the perturbed solutions with a critical value, above it the solutions are stable. Otherwise they are unstable.

In view of the nonlinearity of the governing equations, considerations were confined to a single thermal phase lag. Cases with more than one thermal relaxation time will be considered in future work.

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## Declarations

**Conflict of interest** The authors declare that there is no conflict of interest.

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