The Use of Conditional Probability Integral Transformation Method for Testing Accelerated Failure Time Models

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Abstract

This paper suggests the use of the conditional probability integral transformation (CPIT) method as a goodness of fit (GOF) technique in the field of accelerated life testing (ALT), specifically for validating the underlying distributional assumption in accelerated failure time (AFT) models. The CPIT method is based on transforming the data into independent and identically distributed (i.i.d) *Uniform (0, 1)* random variables and then applying a certain GOF technique to test the uniformity of the transformed random variables. In this paper, the CPIT method is used to validate each of the exponential and lognormal distributions' assumptions in an AFT model under constant stress and complete sampling. The performance of this method is investigated via a simulation study. Moreover, a real life example is presented to illustrate the application of it. Concluding comments about the good performance of the CPIT method are made.

Keywords: Accelerated life testing, Accelerated failure time model, Constant stress, Goodness of fit techniques, Conditional probability integral transformation method.

1. Introduction

Accelerated life testing (ALT) is the key tool to assess the reliability and durability of high reliable manufactured products. Under ALT, test units are exposed to high stress conditions that are more severe than those encountered in reality. The goal is to accelerate failures of these units so that failure times can be obtained sooner and the results are then used in companion with extrapolation procedures to draw inference about the units at the normal stress conditions. The extrapolation procedures are based on physical models called accelerated life models (ALM) which relate the lifetime distribution to the stress. The difference between the ALM proposed in the literature is in the influence of the applied stress on the reliability (for more details, see Bagadonavičius and Nikulin (2002)).

The majority of inference in ALT is built on an accelerated failure time (AFT) model. This model consists of two components; a lifetime distribution and a relationship between life and the stress called life-stress relationship. Examples of this relationship are the Arrhenius and the inverse power law relationships (for more details, see Nelson (1990)).

Although the importance of verifying the suitability of the models used in ALT, there is a lack in the studies presented in this area. Generally, these studies can be classified into two categories; the first one is concerning with goodness of fit (GOF) techniques proposed to assess the effect of the stress on the lifetime distribution, that is which ALM has the best fit for the data (see Bagadonavičius and Nikulin (2002); Bagadonavičius *et al.* (2004); Bagadonavičius *et al.* (2011); Balakrishnan *et al.* (2013)).

The second category of studies handles the problem of validating the assumptions of the AFT models. These studies assumed that a certain AFT model holds and proposed a GOF technique to verify the underlying assumptions concerning the life-stress relationship and the lifetime distribution at each stress level.

With respect to the life-stress relationship, Nelson (1990) used the F statistic to test the linear life-stress relationship for log-failure time variable $(\log T)$, where T was assumed to have exponential and lognormal distributions. Lawless (2003) dealt with the same case but using the likelihood ratio test (LRT). Eguchi (1992) investigated the validity of the inverse power law relationship assuming a bivariate exponential distribution using a test statistic based on a projection method. Teng and Yeon (2002) proposed D-statistic based on the transformed Least Square (LS) estimation method to assess the validity of the log-linear life-stress relationship against the log-quadratic one in case of step-stress ALT experiment under exponential type II censored data.

Regarding testing for the underlying distribution at each stress level, Sethurman and Singpurwalla (1982) used Kolmogrov-Smirnov statistic to test whether the unknown distribution at different stress levels belong to a common parametric location-scale family. Nelson (1990) used the LRT for the same case and the same distributions. Wang (2009) proposed a procedure based on sample spacing to test the exponentiality of the lifetime distribution for each stress level. This procedure was based on type II censored k-stages step-stress ALT in the existence of the log linear life-stress relationship. Galanova *et al.* (2012) used modified nonparametric GOF tests to validate parametric (exponential, Weibull, Gamma, Generalized Gamma, and lognormal) AFT models based on an analysis of a sample of residuals. Bagadonavičius *et al.* (2013) investigated the appropriateness of exponential, Weibull, log-logistic and lognormal AFT model using modified chi-square statistic under right censoring.

The novelty of this paper is to apply the conditional probability integral transformation (CPIT) method to examine the GOF of the log-location-scale family of distributions; specifically, the exponential and lognormal distributions, under the inverse power law AFT model. The case of constant stress and complete sampling is considered.

The paper is organized as follows. Section 2 presents the theoretical basis of the CPIT method. In Section 3, some applications of the CPIT method are explained. Section 4 clarifies how the CPIT method is used to test for the underlying distributions in AFT model with application on the exponential and lognormal distributions. A simulation study is carried out in Section 5. A real life example is given in Section 6 to illustrate the applicability of the method. Finally, the paper is concluded in Section 7.

2. Conditional Probability Integral Transformation Method

The CPIT method was introduced by O'Reilly and Quesenberry (1973). The idea of it is based on transforming the original set of n random variables into a smaller set of (n - p) - where p is the number of estimated parameters - i.i.d *Uniform (0, 1)* random variables by using certain conditional distributions obtained by conditioning on sufficient statistics. After transforming into *Uniform (0, 1)* random variables, tests of uniformity can be applied to the transformed set to assess whether to be i.i.d *Uniform (0, 1)* random variables. The theoretical basis of the method is explained as follows.

Let $T_1, T_2, ..., T_n$ be a set of i.i.d random variables with probability density function (pdf) $f(t;\beta)$ and corresponding absolutely continuous cumulative distribution function (CDF) $F(t;\beta)$. Let S_n be a p-component vector, that is the minimal sufficient statistic for $\beta' = (\beta_1, \beta_2, ..., \beta_p)$. Denote by $\tilde{F}(t_1, t_2, ..., t_n)$, the CDF of $(T_1, T_2, ..., T_n)$ given the statistic S_n. O'Reilly and Quesenberry (1973) proved that the (n - p) random variables

$$U_1 = \tilde{F}_n(T_1),\, U_2 = \tilde{F}_n(T_2 \mid T_1),\ \ ...\ ,\text{ and } U_{n-p} = \tilde{F}_n(T_{n-p} \mid T_1, T_2, ..., T_{n-p-1}), \qquad (2.1)$$

are i.i.d *Uniform (0, 1)* random variables.

This result does not require that $T_1, T_2, ..., T_n$ are i.i.d random variables. If $T_1, T_2, ..., T_n$ are i.i.d random variables and $(S_n)_{n \geq 1}$ is doubly transitive, then the *(n – p)* random variables

$$U_1 = \tilde{F}_{p+1}(T_{p+1}),\, U_2 = \tilde{F}_{p+2}(T_{p+2}),\ ...\ ,\quad \text{and } U_{n-p} = \tilde{F}_n(T_n), \qquad (2.2)$$

are i.i.d *Uniform (0, 1)*.

3. Applying the CPIT Method

The CPIT method has wide applications. O'Reilly and Quesenberry (1973) applied the CPIT for linear regression model as explained in sub-section 3.4. O'Reilly and Stephens (1982) used this method to transform from exponential distribution to uniform one as clarified in sub-section 3.1. While, Quesenberry *et al.* (1983) introduced the use of the CPIT method in testing the assumptions of analysis of variance (ANOVA) model. This will be explained in brief in sub-section 3.3. There were no applications of the CPIT method in case of lognormal distribution. Thus, applying the CPIT method for it, is explained in sub-section 3.2.

3.1 In case of exponential distribution

Let $T_1, T_2, ..., T_n$ be a set of i.i.d random variables having exponential distribution with pdf given by

$$f(t;\lambda) = \frac{1}{\lambda}\exp\left\{-\frac{t}{\lambda}\right\}, t > 0, \lambda > 0, \qquad (3.1)$$

where λ is the scale parameter. O'Reilly and Stephens (1982) transformed the sample order statistics using (2.1) into uniform random variables in the form

$$U_i = 1 - \left\{ \frac{1-(n-i+1)T_{(i)} / \left(T_{(i)} + ... + T_{(n)}\right)}{1-(n-i+1)T_{(i-1)} / \left(T_{(i)} + ... + T_{(n)}\right)} \right\}, i = 1,2,...,n-1. \quad (3.2)$$

where $T_{(1)}, T_{(2)}, ..., T_{(n)}$ are the sample order statistics and $T_{(0)} = 0$. Then, they stated that by testing the uniformity of these *(n - 1)* variables using a suitable GOF technique, the assumption of exponentiality can be validated. In this paper, we will use the modified Watson statistic as a GOF technique.

3.2 In case of normal and lognormal distributions

Let $T_1, T_2, ..., T_n$ be a set of i.i.d random variables having normal distribution with pdf given by

$$f(t;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ -\frac{(t-\mu)^2}{2\sigma^2} \right\}, -\infty < t < \infty, -\infty < \mu < \infty, \sigma > 0, \quad (3.3)$$

where μ and σ are the location and scale parameters, respectively.

The transformation from normal distribution to uniform one based on (2.2) was proposed by O'Reilly and Quesenberry (1973) and modified by D'Agostino and Stephens (1986) as follows

$$U_{i-2} = G_{i-2}(A_i), \ i = 3,4,...,n, \quad (3.4)$$

where $A_i = \left[(i-1)/i \right]^{1/2} \left(T_i - \bar{T}_{i-1} \right) / S_{i-1}$, $\bar{T}_i = \sum_{r=1}^{i} T_r / i$, $S_i^2 = \sum_{r=1}^{i} \left(T_r - \bar{T}_i \right)^2 / (i-1)$, and $G_c(A)$ denotes a Student-t CDF with *c* degrees of freedom evaluated at *A*.

In this paper, we try to apply the same technique on the case of lognormal distribution. To transform from lognormal distribution to uniform one, let $T_1, T_2, ..., T_n$ be a set of i.i.d random variables having lognormal distribution with pdf given by

$$f(t;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}t} \exp\left\{ -\frac{(\ln t-\mu)^2}{2\sigma^2} \right\}, -\infty < t < \infty, -\infty < \mu < \infty, \sigma > 0, \quad (3.5)$$

where μ and σ are unknown parameters.

The transformation $Y = \ln T$ results in a new set of random variables $Y_1, Y_2, ..., Y_n$ that have normal distribution with pdf given by (3.3). Then, the transformation to uniform distribution can be obtained by replacing $T_1, T_2, ..., T_n$ by $Y_1, Y_2, ..., Y_n$ in (3.4). By testing the uniformity of the *(n - 2) U* variables computed from (3.4), the assumption of normality can be validated.

3.3 In case of ANOVA model

In this sub-section, the work of Quesenberry *et al.* (1983) to test the assumptions of ANOVA model using the CPIT method, is summarized as follows.

Suppose that there are k mutually exclusive samples, $T_{ij},\ i=1,2,...,n_j,\ j=1,2,...,k,$ and the problem is to test

$$H_0: T_{ij} \sim N\left(\mu_j, \sigma^2\right),\ i=1,2,...,n_j,\ j=1,2,...,k.$$

Let $n=n_1+n_2+...+n_k,\ v_{ij}=n_1+...+n_{j-1}+i-j-1,\ \bar{T}_{ij}=\sum_{r=1}^{i} T_{rj}/i,$

$SS_{ij}=\sum_{r=1}^{i}\left(T_{rj}-\bar{T}_{ij}\right)^2,$ and $A_{ij}=\left[(i-1)v_{ij}/i\right]^{1/2}\left[T_{ij}-\bar{T}_{(i-1)j}\right]/\left[\sum_{r=1}^{j-1} SS_{n+r}+SS_{(i-1)j}\right]^{1/2}.$

The modification of the CPIT method under the assumptions of the ANOVA model was given by D'Agostino and Stephens (1986) as

$$U_{ij}=G_{v_{ij}}(A_{ij}), \tag{3.6}$$

for $j=1,\ i=3,4,...,n_1,$ and for $j=2,3,...,k,\ i=2,3,...,n_j,$ where $G_c(A)$ is the same as in (3.4). By testing the uniformity of the *(n - k - 1) U* values computed from (3.6), the assumptions of ANOVA model can be verified.

3.4 In case of linear regression model

O'Reilly and Quesenberry (1973) introduced the transformation to *Uniform (0, 1)* random variables in the case of linear regression model. They used (2.1) to get the transformation. The null hypothesis

$$H_0: Y_n \sim N\left(X_n\beta, \sigma^2 I\right),$$

is considered to test for the linear regression model in the form

$$Y_n = X_n\beta, \tag{3.7}$$

where Y_n is a vector of n observations, X_n is an $n\times q$ matrix, β is a vector of q parameters, and σ^2 is an another parameter to be estimated. Denoting the i^{th} observation, $i=1,2,...,n$ in Y_n by y_i and the i^{th} row of X_n by x_i', we can assume that Y_i is an $i\times 1$ vector which consists of the first *i* observations in Y_n and X_i be an $i\times q$ matrix consisting of the first *i* rows of X_n.

The transformation to the *Uniform (0, 1)* distribution under H_0 was proposed by O'Reilly and Quesenberry (1973) and was given by

$$U_{i-p}=G_{i-p}(A_i),\ i=p+1,p+2,...,n, \tag{3.8}$$

where $A_i = (i-p)^{1/2}(y_i - x_i'b_i) / \left\{ \left[1 - x_i'(X_i'X_i)^{-1} x_i \right] S_i^2 - (y_i - x_i'b_i)^2 \right\}^{1/2}$, $b_i = (X_i'X_i)^{-1} X_i'Y_i$ is the LS estimator of β computed using the first i observations, and $S_i^2 = Y_i'\left[I - X_i(X_i'X_i)^{-1} X_i' \right] Y_i$ is the LS sum of squares of the residuals computed using the first i observations.

By testing the uniformity of the $(n - p)$ U variables, given by (3.8), the assumptions of the linear regression model can be checked.

D'Agostino and Stephens (1986) recommended the use of the modified Watson statistic to test the uniformity of the U values. This statistic is referred to as U_{MOD}^2 and has the following form

$$U_{MOD}^2 = \left\{ U^2 - \frac{0.1}{(n-p)} + \frac{0.1}{(n-p)^2} \right\} \left\{ 1 + \frac{0.8}{(n-p)} \right\}, \tag{3.9}$$

where U^2 is defined as

$$U^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left[\frac{(2i-1)}{2n} - F_0(t_{(i)}) \right]^2 - n\left[\bar{F}_0(t_{(i)}) - 0.5 \right]^2, \tag{3.10}$$

where $\bar{F}_0(t_{(i)}) = \sum_{i=1}^{n} F_0(t_{(i)}) / n$ and $F_0(t)$ is the hypothesized distribution. The critical points for U_{MOD}^2 were given in D'Agostino and Stephens (1986).

4. Applying the CPIT Method in AFT Model

D'Agostino and Stephens (1986) applied the CPIT method in the case of multi-sample problems, that is the case in which there exist k, $k > 1$ samples. The target was to validate the normality assumption. They transformed each sample separately using equation (3.4), then, the transformed values obtained from all samples were pooled together as one sample from *Uniform (0, 1)* distribution. Finally, this sample was used to verify the normality assumption of all samples.

Under constant stress ALT, the units are tested at several high stress levels. We can assume that at each stress level, there is a different sample. To examine the underlying distributional assumption in this case, we will try to use the same technique of D'Agostino and Stephens (1986). It will be applied as follows. First, transform the failure times at each stress level separately into i.i.d. *Uniform (0, 1)* random variables. Then, the transformed random variables obtained at each stress level using the CPIT method are pooled together as one sample hypothesized to be drawn from *Uniform (0, 1)* distribution. Second, apply the modified Watson GOF technique on the pooled sample, the uniformity of the transformed variables and accordingly the adequacy of the hypothesized family of distributions can be judged.

In the following sub-sections, we try to apply the CPIT method to test for both the exponential and lognormal distributions under the AFT model.

4.1 Exponential AFT model

Under the exponential inverse power law AFT model, the experiment is conducted as follows.

1. A random sample of size *n* units is put on test, and all run to failure.
2. There are k test stress levels and n_j units are tested at stress level V_j, $j = 1, 2, ..., k,$
3. The total number of test units is $n = n_1 + n_2 + ... + n_k$.
4. The stress V_j affects the scale parameter λ_j, through the inverse power law relationship. This can be expressed as

   $$\lambda_j = C / V_j^P, C > 0. \quad (4.1)$$

5. T_{ij} denotes the failure time of the test unit *i* at stress level *j*. These failure times are assumed to have exponential distribution with pdf in the form

   $$f(t_{ij}; \lambda_j) = \frac{1}{\lambda_j} \exp\left\{-\frac{t_{ij}}{\lambda_j}\right\}, t_{ij} > 0, \lambda_j > 0. \quad (4.2)$$

By substituting (4.1) in (4.2), the pdf of T_{ij}, $i = 1,2,..., n_j$, $j = 1,2,..., k,$ takes the following form

$$f(t_{ij}; C, P) = \frac{v_j^P}{C} \exp\left\{-\frac{v_j^P t_{ij}}{C}\right\}, t_{ij} > 0, C > 0. \quad (4.3)$$

The CPIT method can be used to validate the assumption of the exponential distribution in AFT model regardless the inverse power law AFT relationship, as follows

- The CPIT method is used to get the transformed *U* values from each sub- sample at each stress level separately regardless the inverse power law relationship as follows

  $$U_{ij} = 1 - \left\{ \frac{1 - (n_j - i + 1) Z_{ij} / \left(Z_{ij} + ... + Z_{n_j j}\right)}{1 - (n_j - i + 1) Z_{(i-1)j} / \left(Z_{ij} + ... + Z_{n_j j}\right)} \right\}^{n_j - i}, \quad \begin{matrix} i = 1, 2, ..., n_j - 1 \\ j = 1, 2, ..., k, \end{matrix} \quad (4.4)$$

  where Z_{ij} is the i^{th} order statistic from the j^{th} stress level, and $Z_{0j} = 0$.

- The transformed (*n* - *k*) *U* values, obtained from all stress levels using (4.4), are pooled together to constitute one sample hypothesized to be drawn from *Uniform (0, 1)* distribution.
- The U_{MOD}^2 statistic is applied on the pooled sample to verify the simple uniformity of the transformed *U* values which is equivalent to testing for the assumption of the exponential distribution at each stress level.

4.2 Lognormal AFT model

Under the lognormal inverse power law AFT model, the experiment is conducted with the same first three assumptions of the exponential case defined above in addition to the following

1. The stress V_j does not affect the shape parameter $1/\sigma$ of the lognormal distribution at each stress level.

2. The scale parameter, $\exp(\mu_j)$ is related to the stress through the inverse power law relationship as

$$\exp(\mu_j)=C/V_j^P, C>0, or$$
$$\mu_j=\beta_0+\beta_1 x_j, \tag{4.5}$$

where $\beta_0=\ln C$, $\beta_1=-P$, and $x_j=\ln V_j$.

3. The failure times at stress level V_j, T_{ij}, $i=1,2,...,n_j$, $j=1,2,...,k,$ are assumed to have lognormal distribution with pdf given by

$$f(t_{ij};\beta_0,\beta_1,\sigma)=\frac{1}{\sigma\sqrt{2\pi}t_{ij}}\exp\left\{-\frac{(\ln t_{ij}-\beta_0-\beta_1 x_j)^2}{2\sigma^2}\right\}, t>0, \sigma>0. \tag{4.6}$$

When dealing with the log-failure times, $Y_{ij}=\ln T_{ij}$, $i=1,2,...,n_j$, $j=1,2,...,k,$ the lognormal inverse power law AFT model is reduced to the ordinary linear regression model since the following assumptions are satisfied.

- The distribution of the log-lifetime variable (Y), at each stress level V_j, $j=1,2,...,k,$ belongs to the normal family of distributions.
- The scale parameter of the log-lifetime distribution, σ, is constant at each stress level.
- The location parameter of the log-lifetime distribution, μ_j, $j=1,2,...,k,$ at each stress level is related to the stress through the linear specification defined in (4.6).

Thus, the transformed U values needed to validate the assumption of the normal (lognormal) distribution at each stress level can be obtained using 3 different CPIT methods as follows

i. CPIT 1 method

This method does not take into account neither the constancy of the scale parameter of the normal distribution at each stress level nor the linearity of the relationship between the location parameter and the transformed stress level. Using CPIT 1, the transformed U values are obtained as follows

$$U_{(i-2)j}=G_{i-2}(A_{ij}),\ i=3,4,...,n_j,\ j=1,2,...,k, \tag{4.7}$$

where $A_{ij} = [(i-1)/i]^{1/2} [Y_{ij} - \bar{Y}_{(i-1)j}] / S_{(i-1)j}$.

This transformation results in (*n-2k*) pooled *U* values obtained from all stress levels.

ii. CPIT 2 method

This method transforms from normal distribution to uniform one and takes into consideration the constancy of the scale parameter of the normal distribution at each stress level but neglects the assumption of the linear relationship between the location parameter and the transformed stress level. Using CPIT 2, the transformed U values are obtained by

$$U_{ij} = G_{v_{ij}}(A_{ij}), \text{ for } j = 1,\ i = 3, 4, \ldots, n_1, \text{ and}$$
$$\text{for } j = 2, 3, \ldots, k,\ i = 2, 3, \ldots, n_j, \quad (4.8)$$

where $A_{ij} = \left[(i-1)v_{ij}/i\right]^{1/2} \left[Y_{ij} - \bar{Y}_{(i-1)j}\right] / \left[\sum_{l=1}^{j-1} SS_{n_l l} + SS_{(i-1)j}\right]^{1/2}$.

This transformation results in (*n - k - 1*) pooled *U* values obtained from all stress levels.

iii. CPIT 3 method

This method considers all the assumptions of the linear regression (lognormal inverse power law AFT) model when transforming from normal distribution to uniform one. Under CPIT 3, the transformed U values are obtained using

$$U_{i-p} = G_{i-p}(A_i),\ i = p+1, p+2, \ldots, n, \quad (4.9)$$

where $A_i = (i-p)^{1/2} (y_i - x_i' b_i) / \left\{ \left[1 - x_i' (X_i' X_i)^{-1} x_i\right] S_i^2 - (y_i - x_i' b_i)^2 \right\}^{1/2}$,

and $x_i,\ i = 1, 2, \ldots, n,$ are the values of the transformed stress levels that correspond to each $y_i,\ i = 1, 2, \ldots, n,$ in Y_n as defined in (3.7). The transformation (4.9), results in (*n - 3*) pooled *U* values obtained from all stress levels.

By applying the U_{MOD}^2 statistic on the pooled sample of *U* values obtained by either (4.7), (4.8), or (4.9), the uniformity of these *U* values and accordingly the assumption of the lognormal distribution at each stress level can be validated.

5. A Simulation Study

5.1 Testing for the exponential distribution in AFT model

For testing the assumption of exponentiality in AFT models, the U_{MOD}^2 statistic, given by equation (4.4) is used, and is called U_{MOD-E}^2 in this case. The power of this statistic is then examined.

The simulation study is conducted under the following experiment

- There are k = 4 stress levels with values: $V_1 = 24, V_2 = 26, V_3 = 28,$ and $V_4 = 30$.
- Different sample sizes, n, and their division on the 4 stress levels, n_j, $j = 1, 2, 3, 4,$ are arbitrary chosen as shown in Table 1.
- Three different initial values of the parameters C and P in equation (4.1) are assumed to be $0.5, 1.5, 3.5$ and $0.1, 0.5, 0.9$, respectively. Then, we consider nine different combinations of these values. Three different values of significance levels $\alpha = 0.1, 0.05, 0.01$ are considered.
- For each combination of (C, P, n), 1000 samples are generated using Mathcad program from the following distributions

  1- Exponential with pdf given by equation (4.3).

  2- Weibull with pdf given by

$$f(t_{ij};\lambda_j,\eta) = \frac{\eta}{\lambda_j}\left(\frac{t_{ij}}{\lambda_j}\right)^{\eta-1} \exp\left\{-\left(\frac{t_{ij}}{\lambda_j}\right)^{\eta}\right\}, t_{ij} > 0, \lambda_j, \eta > 0,\ i = 1,2,...,n_j, \quad j = 1,2,...,k, \tag{5.1}$$

  where $t_{ij}, i = 1,2,...,n_j,\ j = 1,2,...,k,$ are the failure times. The shape parameter η is assumed to be independent of the stress levels and is taken to be 0.5. But, the scale parameter λ_j is assumed to be affected by the stress levels through the inverse power law relationship given by equation (4.1). After substituting (4.1) in (5.1), the pdf will be in the form

$$f(t_{ij};C,P,\eta) = \frac{\eta\, v_j^P}{C}\left(\frac{v_j^P t_{ij}}{C}\right)^{\eta-1} \exp\left\{-\left(\frac{v_j^P t_{ij}}{C}\right)^{\eta}\right\}, t_{ij} > 0, C, \eta > 0. \tag{5.2}$$

  3- Lognormal with pdf given by equation (4.6), with shape parameter $1/\sigma = 0.5$.

- For each sample, the MLE of the parameters of these distributions are obtained with tolerance value $\varepsilon = 0.00001$. Then, the U^2_{MOD-E} statistic is calculated and the power is estimated as:

  Power = Number of times rejecting H_0 /1000, where

  $H_0 : T_{ij}$ follows exponential distribution with pdf given by equation (4.3).

  $H_1 :$ Not H_0.

The estimated power values of U^2_{MOD-E} statistic in testing the exponential distribution in AFT model are given in Table 2. From this Table, it is seen that the U^2_{MOD-E} statistic is powerful for testing the exponential distribution versus both the Weibull and lognormal alternatives. This is true whatever the sample size. When the sample size increases, the power of this statistic becomes much better. Sometimes, the power reaches 1. Thus, it could be said that the CPIT method performs well when testing the exponentiality of the AFT model.

5.2 Testing for the lognormal distribution in AFT model

In this sub-section, we examine the power of the U^2_{MOD} statistic for testing the assumption of the lognormal distribution in AFT model. The U^2_{MOD} statistic computed based on the CPIT 1, CPIT 2 and CPIT 3 methods, given by equations (4.7), (4.8), and (4.9) are denoted by U^{2*}_{MOD-L}, U^{2**}_{MOD-L} and U^{2***}_{MOD-L}, respectively.

The simulation study is conducted under the same experiment and procedures as for the exponential distribution, but the values of n_j, $j = 1,...,4,$ are different. Since all the log-failure times occurring at the same stress level have the same transformed stress level value x_j, and since CPIT 3 is based on omitting the first 3 observations, then in order to use this transformation, the number of units at the first stress level n_1, should not exceed three test units. This is to avoid the problem of singularity of the matrix ($X'_i X_i$) computed from the matrix X. Thus, the total sample sizes used are redistributed to the 4 stress levels as indicated in Table 1. The distribution of the total sample sizes on the 4 stress levels is arbitrary chosen; taking into consideration that $n_1 = 3$. In this case it is desired to test the following hypotheses

$H_0 : T_{ij}$ follows lognormal distribution with pdf given by equation (4.6).

$H_1 :$ Not H_0.

The estimated power of U^{2*}_{MOD-L}, U^{2**}_{MOD-L} and U^{2***}_{MOD-L} statistics in testing the lognormal distribution versus the alternatives, exponential, Weibull and lognormal distributions, are given in Table 3 under different values of *C*, *P*, and *n*. From this Table, it is seen that there are small differences between the power of these statistics. In the majority of cases, U^{2***}_{MOD-L} is better than U^{2**}_{MOD-L}, which is better than U^{2*}_{MOD-L}. It is also seen that, the powers of these statistics improve as the sample gets larger. In general, and whatever the sample size, we can see that all these three statistics are powerful in testing whether the lifetime distribution under AFT model is lognormal versus both the exponential and Weibull alternatives.

6. A Real Life Example

In this Section, we apply the CPIT method to investigate the distribution of times to breakdown of an insulating fluid under three elevated voltage stress. The data used is referred to in Nair (1982). This data represents the times to breakdown of an insulating fluid under three elevated voltage stresses. The data is presented in Table 4 and constitutes the first sixty observations at the three voltage levels.

Nelson (1990) suggested the use of one of the exponential and lognormal distributions to describe the lifetime of insulating fluids. To check the appropriateness of using the lognormal distribution to represent the data given in Table 4, we will use the CPIT statistic, U^{2*}_{MOD-L}, which does not take into account neither the constancy of the shape parameter of the lognormal distribution nor the inverse power law relationship. First, the log-lifetimes are obtained and then the U^{2*}_{MOD-L} statistic is applied on these log-failures at each stress level separately. Second, the transformed *U* values are pooled together from all stress level, and then the U^{2*}_{MOD-L} statistic is applied on the pooled sample. It is found that, the computed value of U^{2*}_{MOD-L} statistic, (1.613) exceeds the corresponding critical value (0.187) at significance level $\alpha = 0.05$. This indicates that the lognormal distribution is not suitable to represent the lifetime distribution of the times to breakdown data given in Table 4.

To test for the exponential distribution without taking into consideration the inverse power law relationship, the CPIT statistic, U^{2}_{MOD-E} is applied on the failure times in the same way as the U^{2*}_{MOD-L}. The results of using this test indicates that the exponential distribution gives good fit for the data since the computed value of U^{2}_{MOD-E} statistic, (0.071) is less than the corresponding critical value (0.187) at significance level $\alpha = 0.05$.

7. Conclusions

This paper presents the application of the CPIT method to investigate the validity of the underlying distribution in AFT model under constant stress and complete sampling. The choice of this method is based on its capability to combine the failure times from all stress levels to reach a conclusion about the adequacy of a certain distribution at each stress level. The method is based on transforming the original variables at each stress level into i.i.d *U (0, 1)* random variables. Then by pooling these transformed variables from all stress levels as one sample and using an appropriate GOF technique to assess the uniformity of the transformed variables, the adequacy of the hypothesized distribution at each stress level can be validated. In this paper, the CPIT method is applied to test for the exponential and lognormal inverse power law models. The advantage of this method is that it tests for the underlying distribution regardless the other assumptions of the AFT model.

A simulation study is carried out to explore the power of the CPIT method in validating the underlying distributions in AFT model. First, the CPIT method is used to test for the exponential distribution. Second, three versions of this method are used in testing for the lognormal distribution. It is concluded that the CPIT method is a powerful test in both cases of exponential and lognormal distributions whatever the sample sizes.

Finally, the CPIT method is applied on a real life data that includes the times to breakdown of an insulating fluid to investigate the adequacy of both the lognormal and exponential inverse power law models. The results clarify that the exponential inverse power law model fits the times to breakdown of an insulating fluid better than the lognormal one.

To summarize, the CPIT method is used to assess the GOF of the log-location-scale family of distributions in AFT model. As examples, both the exponential and lognormal lifetime distributions are considered. It is concluded that the CPIT method performs well, so it is recommended to be used in the field of ALT. As a future work, an extension may be made to treat the same problem under different types of censoring.

Table 1: Total sample sizes, n and sub-samples, n_j, $j=1, ..., 4$ in the case of exponential and lognormal distributions

n	Lifetime Distribution							
	Exponential				Lognormal			
	n_j				n_j			
	n_1	n_2	n_3	n_4	n_1	n_2	n_3	n_4
33	3	5	10	15	3	5	10	15
63	13	15	17	18	3	15	20	25
103	18	20	30	35	3	25	35	40
203	35	45	55	68	3	50	70	80

Table 2: Estimated values of the power for testing the exponential distribution

			Alternative								
			Exponential			Weibull			Lognormal		
			α			α			α		
C	*P*	*n*	0.1	0.05	0.01	0.1	0.05	0.01	0.1	0.05	0.01
0.5	0.1	33	0.111	0.066	0.010	0.852	0.779	0.631	0.898	0.857	0.759
		63	0.112	0.059	0.014	0.993	0.983	0.946	0.993	0.992	0.980
		103	0.088	0.040	0.007	1.000	0.999	0.996	1.000	1.000	1.000
		203	0.094	0.050	0.012	1.000	1.000	1.000	1.000	1.000	1.000
0.5	0.5	33	0.107	0.056	0.015	0.844	0.781	0.624	0.909	0.875	0.796
		63	0.117	0.063	0.014	0.980	0.970	0.920	0.992	0.987	0.969
		103	0.107	0.061	0.013	0.998	0.998	0.992	1.000	1.000	1.000
		203	0.100	0.049	0.015	1.000	1.000	1.000	1.000	1.000	1.000
0.5	0.9	33	0.111	0.066	0.010	0.831	0.756	0.613	0.892	0.846	0.743
		63	0.097	0.041	0.012	0.984	0.965	0.915	0.995	0.992	0.979
		103	0.093	0.035	0.008	1.000	0.998	0.991	1.000	1.000	1.000
		203	0.102	0.053	0.019	1.000	1.000	1.000	1.000	1.000	1.000
1.5	0.1	33	0.091	0.048	0.010	0.841	0.772	0.636	0.912	0.865	0.758
		63	0.075	0.040	0.011	0.981	0.969	0.920	0.999	0.995	0.980
		103	0.096	0.051	0.012	1.000	1.000	0.995	1.000	1.000	1.000
		203	0.102	0.051	0.015	1.000	1.000	1.000	1.000	1.000	1.000
1.5	0.5	33	0.110	0.048	0.015	0.863	0.781	0.662	0.912	0.860	0.765
		63	0.098	0.052	0.012	0.988	0.980	0.933	0.996	0.993	0.982
		103	0.104	0.056	0.015	1.000	1.000	0.992	1.000	1.000	1.000
		203	0.103	0.046	0.015	1.000	1.000	1.000	1.000	1.000	1.000
1.5	0.9	33	0.113	0.058	0.017	0.837	0.777	0.632	0.917	0.863	0.761
		63	0.100	0.052	0.009	0.984	0.966	0.917	0.996	0.991	0.971
		103	0.109	0.052	0.012	1.000	0.999	0.998	1.000	1.000	1.000
		203	0.108	0.055	0.017	1.000	1.000	1.000	1.000	1.000	1.000
3.5	0.1	33	0.100	0.054	0.012	0.865	0.789	0.631	0.901	0.865	0.742
		63	0.093	0.045	0.013	0.981	0.968	0.926	0.996	0.996	0.983
		103	0.096	0.046	0.011	0.998	0.995	0.989	1.000	1.000	0.999
		203	0.095	0.045	0.017	1.000	1.000	1.000	1.000	1.000	1.000
3.5	0.5	33	0.106	0.053	0.012	0.822	0.768	0.629	0.894	0.846	0.755
		63	0.096	0.054	0.011	0.977	0.967	0.913	0.992	0.987	0.977
		103	0.102	0.047	0.009	1.000	1.000	0.994	1.000	1.000	0.999
		203	0.103	0.046	0.017	1.000	1.000	1.000	1.000	1.000	1.000
3.5	0.9	33	0.100	0.061	0.020	0.856	0.785	0.638	0.907	0.868	0.760
		63	0.100	0.058	0.016	0.987	0.978	0.936	0.996	0.992	0.982
		103	0.103	0.052	0.015	1.000	1.000	0.992	1.000	1.000	1.000
		203	0.097	0.054	0.008	1.000	1.000	1.000	1.000	1.000	1.000

Table 3: Estimated values of the power for testing the lognormal distribution

C	P	Alternative	n	U^{2*}_{MOD-L} α 0.1	U^{2*}_{MOD-L} α 0.05	U^{2*}_{MOD-L} α 0.01	U^{2**}_{MOD-L} α 0.1	U^{2**}_{MOD-L} α 0.05	U^{2**}_{MOD-L} α 0.01	U^{2***}_{MOD-L} α 0.1	U^{2***}_{MOD-L} α 0.05	U^{2***}_{MOD-L} α 0.01
0.5	0.1	Exponential	33	0.198	0.121	0.044	0.259	0.178	0.073	0.266	0.176	0.066
			63	0.349	0.252	0.132	0.407	0.286	0.160	0.460	0.343	0.179
			103	0.547	0.427	0.245	0.583	0.480	0.307	0.611	0.487	0.301
			203	0.854	0.765	0.591	0.864	0.803	0.631	0.877	0.805	0.666
		Weibull	33	0.190	0.104	0.043	0.237	0.149	0.067	0.280	0.187	0.068
			63	0.360	0.258	0.129	0.414	0.315	0.161	0.468	0.336	0.162
			103	0.533	0.420	0.261	0.591	0.477	0.293	0.615	0.507	0.301
			203	0.845	0.778	0.625	0.857	0.791	0.630	0.881	0.811	0.666
		Lognormal	33	0.079	0.040	0.012	0.111	0.055	0.015	0.102	0.048	0.015
			63	0.089	0.049	0.016	0.115	0.053	0.014	0.100	0.052	0.014
			103	0.090	0.045	0.012	0.117	0.053	0.016	0.083	0.047	0.013
			203	0.103	0.050	0.010	0.100	0.045	0.016	0.109	0.052	0.017
0.5	0.5	Exponential	33	0.207	0.131	0.050	0.276	0.180	0.083	0.249	0.155	0.056
			63	0.354	0.240	0.120	0.410	0.305	0.157	0.433	0.321	0.153
			103	0.557	0.435	0.266	0.571	0.459	0.282	0.606	0.506	0.297
			203	0.853	0.775	0.601	0.865	0.793	0.637	0.893	0.831	0.693
		Weibull	33	0.197	0.122	0.051	0.255	0.176	0.074	0.273	0.172	0.064
			63	0.364	0.246	0.116	0.409	0.289	0.142	0.474	0.360	0.194
			103	0.567	0.439	0.271	0.585	0.473	0.299	0.620	0.521	0.311
			203	0.849	0.758	0.602	0.855	0.789	0.642	0.865	0.789	0.637
		Lognormal	33	0.090	0.045	0.010	0.119	0.059	0.017	0.092	0.043	0.010
			63	0.105	0.055	0.012	0.114	0.060	0.020	0.095	0.039	0.012
			103	0.091	0.051	0.008	0.099	0.052	0.018	0.091	0.056	0.019
			203	0.090	0.049	0.017	0.109	0.052	0.011	0.093	0.055	0.014
0.5	0.9	Exponential	33	0.197	0.125	0.042	0.254	0.152	0.066	0.276	0.174	0.069
			63	0.357	0.248	0.112	0.413	0.311	0.150	0.430	0.316	0.155
			103	0.568	0.433	0.251	0.586	0.479	0.297	0.627	0.501	0.307
			203	0.847	0.782	0.613	0.871	0.800	0.630	0.906	0.834	0.674
		Weibull	33	0.201	0.124	0.036	0.248	0.160	0.071	0.249	0.168	0.064
			63	0.371	0.259	0.116	0.421	0.305	0.163	0.504	0.361	0.192
			103	0.566	0.435	0.276	0.601	0.487	0.300	0.621	0.509	0.309
			203	0.846	0.759	0.590	0.869	0.808	0.644	0.889	0.821	0.671
		Lognormal	33	0.095	0.041	0.010	0.107	0.052	0.017	0.098	0.053	0.009
			63	0.106	0.047	0.015	0.099	0.053	0.013	0.109	0.054	0.016
			103	0.092	0.045	0.012	0.106	0.052	0.015	0.106	0.052	0.016
			203	0.103	0.047	0.013	0.098	0.052	0.013	0.103	0.061	0.015

Table 3: Estimated values of the power for testing the lognormal distribution (Cont.)

C	P	Alternative	n	U^{2*}_{MOD-L} α 0.1	U^{2*}_{MOD-L} α 0.05	U^{2*}_{MOD-L} α 0.01	U^{2**}_{MOD-L} α 0.1	U^{2**}_{MOD-L} α 0.05	U^{2**}_{MOD-L} α 0.01	U^{2***}_{MOD-L} α 0.1	U^{2***}_{MOD-L} α 0.05	U^{2***}_{MOD-L} α 0.01
1.5	0.1	Exponential	33	0.196	0.128	0.050	0.246	0.156	0.066	0.252	0.165	0.073
			63	0.375	0.260	0.124	0.413	0.298	0.162	0.443	0.337	0.169
			103	0.527	0.397	0.226	0.598	0.479	0.294	0.637	0.500	0.323
			203	0.858	0.778	0.596	0.882	0.811	0.651	0.890	0.820	0.670
		Weibull	33	0.218	0.136	0.047	0.256	0.150	0.066	0.267	0.181	0.068
			63	0.354	0.238	0.115	0.403	0.286	0.150	0.485	0.380	0.202
			103	0.542	0.409	0.233	0.587	0.468	0.299	0.675	0.547	0.343
			203	0.865	0.790	0.620	0.881	0.814	0.653	0.866	0.796	0.645
		Lognormal	33	0.092	0.040	0.009	0.115	0.055	0.018	0.111	0.055	0.010
			63	0.111	0.059	0.014	0.099	0.055	0.017	0.089	0.037	0.010
			103	0.094	0.044	0.011	0.104	0.053	0.021	0.095	0.049	0.012
			203	0.106	0.051	0.017	0.099	0.062	0.016	0.090	0.047	0.015
1.5	0.5	Exponential	33	0.191	0.118	0.035	0.241	0.147	0.060	0.235	0.137	0.054
			63	0.346	0.232	0.113	0.405	0.291	0.140	0.456	0.322	0.142
			103	0.543	0.431	0.250	0.584	0.472	0.275	0.641	0.520	0.326
			203	0.843	0.762	0.595	0.875	0.799	0.637	0.902	0.843	0.694
		Weibull	33	0.207	0.125	0.052	0.230	0.153	0.056	0.279	0.170	0.078
			63	0.360	0.256	0.110	0.424	0.296	0.163	0.496	0.369	0.211
			103	0.584	0.414	0.239	0.594	0.474	0.289	0.656	0.547	0.347
			203	0.857	0.788	0.613	0.882	0.820	0.655	0.888	0.816	0.669
		Lognormal	33	0.096	0.046	0.013	0.118	0.056	0.013	0.092	0.047	0.012
			63	0.098	0.052	0.015	0.116	0.061	0.013	0.099	0.051	0.009
			103	0.108	0.058	0.010	0.112	0.053	0.018	0.091	0.040	0.014
			203	0.104	0.049	0.016	0.109	0.048	0.011	0.106	0.054	0.016
1.5	0.9	Exponential	33	0.201	0.134	0.051	0.256	0.169	0.069	0.262	0.174	0.065
			63	0.371	0.264	0.120	0.384	0.280	0.148	0.435	0.331	0.169
			103	0.528	0.412	0.241	0.583	0.456	0.273	0.654	0.523	0.325
			203	0.839	0.762	0.600	0.877	0.813	0.657	0.890	0.832	0.695
		Weibull	33	0.202	0.122	0.053	0.238	0.145	0.060	0.264	0.163	0.065
			63	0.356	0.255	0.121	0.392	0.279	0.136	0.493	0.347	0.179
			103	0.552	0.432	0.270	0.583	0.463	0.289	0.653	0.543	0.327
			203	0.843	0.759	0.590	0.874	0.809	0.646	0.870	0.791	0.632
		Lognormal	33	0.100	0.044	0.009	0.120	0.060	0.018	0.110	0.054	0.014
			63	0.108	0.054	0.013	0.102	0.049	0.014	0.086	0.044	0.010
			103	0.104	0.059	0.020	0.102	0.060	0.015	0.108	0.044	0.007
			203	0.099	0.041	0.007	0.108	0.057	0.013	0.103	0.044	0.011

Table 3: Estimated values of the power for testing the lognormal distribution (Cont.)

C	P	Alternative	n	U_{MOD-L}^{2*} α 0.1	U_{MOD-L}^{2*} α 0.05	U_{MOD-L}^{2*} α 0.01	U_{MOD-L}^{2**} α 0.1	U_{MOD-L}^{2**} α 0.05	U_{MOD-L}^{2**} α 0.01	U_{MOD-L}^{2***} α 0.1	U_{MOD-L}^{2***} α 0.05	U_{MOD-L}^{2***} α 0.01
3.5	0.1	Exponential	33	0.184	0.115	0.039	0.224	0.151	0.055	0.269	0.191	0.075
			63	0.355	0.251	0.122	0.385	0.283	0.143	0.441	0.326	0.172
			103	0.549	0.416	0.256	0.600	0.486	0.291	0.647	0.514	0.327
			203	0.834	0.757	0.588	0.888	0.817	0.625	0.888	0.831	0.686
		Weibull	33	0.176	0.100	0.040	0.259	0.178	0.073	0.267	0.173	0.061
			63	0.367	0.269	0.123	0.407	0.286	0.160	0.486	0.366	0.204
			103	0.545	0.436	0.262	0.583	0.480	0.307	0.650	0.535	0.358
			203	0.840	0.767	0.581	0.864	0.803	0.631	0.896	0.822	0.662
		Lognormal	33	0.092	0.049	0.013	0.111	0.060	0.014	0.091	0.050	0.011
			63	0.092	0.046	0.014	0.107	0.061	0.021	0.114	0.050	0.013
			103	0.092	0.047	0.015	0.095	0.048	0.010	0.108	0.054	0.013
			203	0.092	0.036	0.010	0.116	0.052	0.015	0.103	0.048	0.008
3.5	0.5	Exponential	33	0.175	0.111	0.040	0.259	0.178	0.073	0.267	0.175	0.067
			63	0.327	0.226	0.109	0.407	0.286	0.160	0.425	0.318	0.156
			103	0.555	0.438	0.247	0.583	0.480	0.307	0.615	0.489	0.290
			203	0.844	0.763	0.584	0.864	0.803	0.631	0.899	0.825	0.660
		Weibull	33	0.179	0.120	0.044	0.272	0.175	0.065	0.265	0.180	0.060
			63	0.330	0.230	0.115	0.416	0.308	0.155	0.486	0.370	0.209
			103	0.558	0.443	0.258	0.573	0.469	0.288	0.643	0.521	0.326
			203	0.834	0.754	0.608	0.865	0.788	0.644	0.879	0.808	0.665
		Lognormal	33	0.091	0.041	0.009	0.113	0.053	0.012	0.092	0.045	0.010
			63	0.092	0.046	0.011	0.104	0.058	0.014	0.101	0.049	0.016
			103	0.113	0.059	0.012	0.116	0.049	0.017	0.105	0.059	0.011
			203	0.099	0.049	0.010	0.115	0.055	0.011	0.093	0.046	0.011
3.5	0.9	Exponential	33	0.201	0.114	0.040	0.276	0.180	0.083	0.265	0.176	0.079
			63	0.344	0.254	0.114	0.403	0.295	0.163	0.464	0.337	0.184
			103	0.544	0.410	0.253	0.606	0.505	0.314	0.634	0.534	0.323
			203	0.847	0.755	0.603	0.868	0.806	0.647	0.888	0.825	0.687
		Weibull	33	0.197	0.127	0.050	0.284	0.181	0.079	0.262	0.173	0.064
			63	0.339	0.243	0.120	0.397	0.293	0.143	0.508	0.393	0.205
			103	0.535	0.438	0.247	0.605	0.482	0.304	0.660	0.553	0.360
			203	0.858	0.760	0.589	0.874	0.814	0.643	0.892	0.823	0.677
		Lognormal	33	0.103	0.048	0.012	0.113	0.061	0.015	0.083	0.038	0.009
			63	0.090	0.044	0.014	0.104	0.051	0.018	0.106	0.050	0.012
			103	0.096	0.047	0.012	0.098	0.047	0.013	0.091	0.045	0.012
			203	0.092	0.040	0.011	0.097	0.054	0.017	0.103	0.061	0.015

Table 4: Observed times to breakdown in minutes of an insulating fluid*

Voltage levels								
34 KV			35 KV			36 KV		
0.13	21.95	0.32	2.68	0.89	0.04	1.89	1.99	8.11
0.15	1.16	12.45	3.09	2.32	11.15	4.03	0.64	3.17
0.04	7.46	7.25	4.5	4.47	1.37	1.54	2.15	5.55
2.93	1.75	8.09	0.64	0.02	3.17	0.31	1.08	0.8
1.44	4.36	0.57	4.95	3.63	1.82	0.66	2.57	0.2
17.49	6.34	0.4	2.36	5.16	1.41	1.7	0.93	1.13
5.76	0.3	12.01	5.2	7.63	0.08	2.17	4.75	6.63
1.73	6.66	0.73	0.37	1.64	5.06	1.82	0.82	1.08
0.2	6.78	5.73	2.03	1.49	1.86	9.99	2.06	2.44
0.55	8.88	2.52	2.57	0.55	1.45	2.24	0.49	0.78
2.22	2.24	2.37	0.54	3.17	3.19	1.3	1.17	2.12
0.63	4.67	8.75	0	1.78	2.77	2.75	3.87	3.97
4.77	2.45	4.66	3.04	3.97	14.09	0	2.8	1.56
7.38	5.23	7.69	10.88	0.37	0	2.17	0.7	1.34
0.43	11.22	4.86	3.93	1.63	13.57	0.66	3.82	1.49
4.63	2.49	4.27	5.34	0.01	5.82	0.55	0.02	8.71
4.68	0	1.28	0.47	1.45	7	0.18	0.5	2.1
5.43	0.35	2.89	0.62	10.52	0.18	10.6	3.72	7.21
2.83	0.22	16.9	1.65	3.54	1.36	1.63	0.06	3.83
5.19	0.95	1.51	0.79	8.93	1.21	0.71	3.57	5.13

* Source: Nair (1982).

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