

Journal of Statistical Computation and Simulation

Publication details, including instructions for authors and subscription information:

<http://www.tandfonline.com/loi/gscs20>

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Published online: 08 Apr 2013.

To cite this article: Amitava Mukherjee, A.M. Abd-Elfattah & Barendra Pukait (2014) A rule of thumb for testing symmetry about an unknown median against a long right tail, *Journal of Statistical Computation and Simulation*, 84:10, 2138-2155, DOI: [10.1080/00949655.2013.784316](https://doi.org/10.1080/00949655.2013.784316)

To link to this article: <http://dx.doi.org/10.1080/00949655.2013.784316>

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A rule of thumb for testing symmetry about an unknown median against a long right tail

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(Received 19 October 2012; final version received 7 March 2013)

A rule of thumb for testing symmetry of an unknown univariate continuous distribution against the alternative of a long right tail is proposed. Our proposed test is based on the concept of exceedance statistic and is ad hoc in nature. Exact performances of the proposed rule are investigated in detail. Some results from an asymptotic point of view are also provided. We compare our proposed test with several classical tests which are practically applicable and are known to be exact or nearly distribution free. We see that the proposed rule is better than most of the existing tests for symmetry and can be applied with ease. An illustration with real data is provided.

Keywords: adaptive test; distribution free; long right tail; Monte-Carlo study; positively skewed; power; test of symmetry

AMS Subject Classification: 62G10

1. Introduction

Natural presence of arsenic in groundwater with variable levels of concentration in several parts of the world is a well-known environmental hazard. This has become a severe problem in recent years; specially where deep tube-wells are used for water supply as in the Ganges Delta, in India and in Bangladesh. It is continuously causing serious arsenic poisoning to large numbers of people. It was found that over 137 million people in more than 70 countries are probably affected by arsenic poisoning from drinking water during 2007. Arsenic contamination of groundwater is found in many countries throughout the world, including the USA. Several incidents of groundwater arsenic contamination are reported from all over the world of which four major occurrences are in Asia, including locations in Thailand, Taiwan and Mainland China. South American countries such as Argentina and Chile are also under the risk. It is seen that the groundwater arsenic concentrations in different regions of the USA are higher than the allowable limit of 10 parts per billion, a standard adopted by the Environmental Protection Agency in 2001. Millions of private wells in the USA have unknown arsenic levels, and in some areas of the USA, nearly 20% or more of wells may contain levels that are not safe.

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Arsenic is a carcinogen which causes diseases of the circulatory and respiratory system. It affects kidney and liver apart from causing many cancers including that of skin, lung and bladder. Many researchers are actively involved in studying various aspects of the arsenic contamination problem all over the world. Mukherjee et al. [1] discussed the global perspective of arsenic contamination in groundwater with emphasis on the Asian scenario. Twarakavi and Kaluarachchi [2] assessed the arsenic problems in the shallow ground waters of conterminous USA along with associated health risks. In an earlier study, Chowdhury et al. [3] reported groundwater arsenic contamination in Bangladesh and West Bengal, India. A detailed account of the arsenic contamination and related problems are provided by Bandyopadhyay et al. [4]. For further information, interested readers may see [5–7].

Amidst several statistical problems associated with groundwater arsenic contamination, a key issue is to study the shape of the distribution of the contamination level. Rahman and Hossain [8], from the perspective of Bangladesh, showed that the distribution of arsenic concentration is symmetric in regions of very high mean concentration, while in the regions of low and moderate contamination, distribution is highly skewed in nature. Bhuyan [9] showed that the empirical distribution of the arsenic contamination level in some parts of North-East India differs from the ideal Gaussian. Hossain et al. [10] investigated that the log-transformation of arsenic concentrations (in logarithm of parts per billion concentration) resulted in a highly plausible Gaussian and symmetric distribution about a near-zero mean. Their study area was restricted to the Bangladesh region. Distributions of arsenic concentrations in shallow wells in Bangladesh were reported to be highly skewed by Gaus et al. [11] and Shamsudduha [12]. In all existing literature, when significant skewness is observed, it is seen to be a positive skewness. The lognormal distribution, which is a very popular consideration for modelling arsenic concentration in groundwater, is well known to be positively skewed. In general, distributions are found to be bell shaped and geologists are often interested to detect quickly, using some statistical tests, if the distribution is close to symmetry or it is positively skewed with a long right tail.

It is observed in several areas that the arsenic concentration in highly contaminated zones sometimes gets reduced during the monsoon season possibly due to recharge from rainfall and also by transmission through groundwater channels. As a result, nearby less-contaminated zones may get enriched with arsenic. Regions with symmetric nature of distribution of arsenic contamination have very few or no highly contaminated underground arsenic pits. On the other hand, regions where the distribution of arsenic has a long right tail certainly contain some highly concentrated arsenic pits. There are possibilities that post-monsoon, nearby low-concentrated zones may get affected. Preventive measures to check the adverse effects of such transmission are necessary. Clearly, it is important to know the nature of the arsenic distribution pattern in different regions.

To meet this requirement of the geologists, in the present work, we consider an ad hoc test for testing the symmetric bell-shaped nature of a univariate continuous distribution. In the light of the above discussions, clearly the alternative of interest is to detect the presence of a long right tail. Let X be any continuous real-valued random variable. We assume that X follows F_X or simply F ; where $F \in \mathcal{F}$, where \mathcal{F} is the class of bell-shaped distributions with moderate to high order of contact at tails. Further, let μ be the median of the distribution of X . That is,

$$F_X(\mu) = 0.5. \quad (1)$$

We may realize two distinct possibilities, namely, μ known and μ unknown. One may think of testing the symmetry of an underlying population distribution for both these situations. In the present context, we consider a more general case and assume that μ is unknown. Note that, during field survey on arsenic contamination in a region, mean level is usually unknown a priori. Even if prior information is available, it is often found to be incorrect as the contamination level often changes with time, specially through transmission in monsoon. We develop a simple rule of thumb assuming that μ is unknown. Nevertheless, such a rule can easily be used when μ is known.

For testing the symmetry, we may write the null hypothesis as

$$H_0 : F_X(\mu + x) = 1 - F_X(\mu - x); \quad \text{for all } x > 0.$$

In general, such a null hypothesis may be tested against one of the three possible alternative hypotheses; namely, the distribution is positively skewed or the distribution is negatively skewed or the distribution is asymmetric.

If the distribution is positively skewed, that is, if there is a long right tail we have

$$H_1 : F_X(\mu + x) < 1 - F_X(\mu - x); \quad \text{for all } x > 0.$$

If the distribution is negatively skewed, that is, if there is a long left tail we have

$$H_2 : F_X(\mu + x) > 1 - F_X(\mu - x); \quad \text{for all } x > 0.$$

Finally, if the alternative is general asymmetry, we have

$$H_3 : F_X(\mu + x) \neq 1 - F_X(\mu - x); \quad \text{for all } x > 0.$$

Obviously, H_1 and H_2 are one-sided alternatives and H_3 is a two-sided alternative. In the light of our motivating problem of arsenic contamination, in the present treaties, we mainly consider testing H_0 against H_1 .

For the various alternatives, tests for symmetry of the distribution of a random variable have been considered by a host of researchers. The literature on this topic is extensive and can be divided into two problems, the first in which the median of the distribution is unknown and the second in which the median is assumed to be known. For the interests of the general readers, we include a large number of references of both situations in a separate section.

The purpose of this paper is to establish a rule of thumb for testing symmetry of an unknown univariate bell-shaped continuous distribution against positive skewness, usually characterized by a long right tail. We assume that population median is unknown. We only consider the class of symmetric distributions that are unimodal, bell-shaped and have reasonably high order of contact at the tails under our null hypothesis. Needless to say, such a class contains most of the well-known univariate symmetric populations including Normal, Logistic, Laplace and even heavy-tailed Cauchy distribution.

The rest of the sections are organized as follows. A brief review of classical literature is presented in Section 2. Some classical tests along with their suitable adaptation for unknown median and one-sided alternative whenever necessary are discussed in the various subsections of Section 2. The proposed rule of thumb is introduced in Section 3. Section 4 contains the simulation results based on Monte-Carlo that includes a study of the level actually attained by various tests at a nominal level of 5%, some limitations of the existing classical test procedures and a comparative study of power performance for different sample sizes. An illustration with real data is presented in Section 5. Finally, Section 6 concludes with some remarks.

2. Some popular existing tests – an overview

The problem of testing the symmetry of an underlying unknown population distribution is one of the classical inference problems, which remains popular among the research workers in Statistics even in the present era. In fact, the history of evolution of the tests of symmetry is enriched over more than seven decades. We come across several articles related to the age-old problem of testing the symmetry in various contexts in the last decade and there is no doubt about the liveliness of

the problem. It is immensely difficult to recall all articles in this area. Here, we have made an attempt to provide a list for ready references for the researchers and for the general interest of the readers.

At the very outset, one may note that Hajek et al. [13] provided a brief but very sound account of the various tests of symmetry. They mentioned several well-known linear rank tests and Kolmogorov-type tests. Among the linear rank tests, we have Fraser test, van der Waerden-type test, Wilcoxon one-sample test and the old and famous sign test. The sign test is locally optimal when the underlying population is of Laplace type. Dixon and Mood [14] and Waerden van der and Nievergelt [15] have discussed this test in detail. For further details, one may see Gastwirth [16]. Wilcoxon [17] introduced a test, which is locally optimum for a logistic-type underlying population. Similarly, Fraser test as well as van der Waerden-type test are optimal when the underlying density is normal and were studied by Fraser [18], Klotz [19] and van Eeden [20]. Kolmogorov-type tests are attributable to [21,22]. While Butler [21] used a simple random walk model, Chatterjee and Sen [22] utilized the martingale characterization. Kolmogorov-type tests are well established to be exact and distribution free.

Besides the above, there are also many other statistics for the goodness-of-fit test for symmetry under the univariate situation; among them test statistics attributable to [23,24, Section 22,25–27] are well known. Finch [28] considered the robust univariate test for symmetry. Gupta [29] considered an asymptotically nonparametric test for symmetry while Doksum et al. [30] considered tests of the hypothesis that the distribution is symmetric about an unknown median. Hill and Rao [31] proposed a test for symmetry based on Cramer–von Mises statistics. Davis and Quade [32] and Randles et al. [33] suggested independently a test based on triples. Bhattacharya et al. [34] considered two modifications of Wilcoxon tests for symmetry about an unknown location parameter. Boos [35] introduced a test for asymmetry associated with the Hodges–Lehmann estimator. Antille and Kersting [36] and Antille et al. [37] proposed some other tests for symmetry. Many of those tests are mentioned in Hollander [38]. Cabilio and Masaro [39] also proposed a simple test of symmetry about an unknown median. McWilliams [40] considered a distribution-free test for symmetry based on a runs statistic. Empirical distribution is considered in many of the above listed articles. In an interesting development, Ahmad and Li [41] proposed a test best on kernel density which performs wonderfully against a large class of alternatives. In a short time, the test became very popular among econometricians. Chirstofides and Stengos [42] used that test to study the symmetry of wage-change distribution in studying income dynamics. Dette et al. [43] also consider the approach of Ahmad and Li [41] in testing the symmetry of a nonparametric regression model. Neumeyer et al. [44] and Neumeyer and Dette [45] discuss the testing for symmetric error distribution in different regression models. Franch and Contreras [46] considered the application of Pearson’s test for symmetry in the Spanish business cycle. In the recent past, we have come across various notable contributions towards testing symmetry. Among them to name a few are the tests by Mira [47], Ekström and Jammalamadaka [48] and Holgersson [49].

In the following subsections, we present some existing popular tests for symmetry in detail.

2.1. Triplet test based on U -statistics

Randles et al. [33] and Davis and Quade [32] independently proposed an asymptotically distribution-free test for testing the symmetry of a random variable about an unknown point. The test has also been discussed by Kochar [50] under convex ordering. Their test is popularly known as the ‘triples test’ and is based on the U -statistic estimator $\hat{\eta}$ given by

$$\hat{\eta} = \binom{n}{3}^{-1} \sum_{i < j < k} f^*(X_i, X_j, X_k),$$

where

$$f^*(X_i, X_j, X_k) = \frac{[\text{sign}(X_i + X_j - 2X_k) + \text{sign}(X_j + X_k - 2X_i) + \text{sign}(X_k + X_i - 2X_j)]}{3},$$

where X_i, X_j and X_k are three independent observations from X and $\text{sign}(u) = -1, 0, 1$ according as $u < \text{or} = \text{or} > 0$. The null hypothesis of symmetry (about an unknown point) is rejected against a one-sided alternative of a long right tail at level α , if $\hat{\eta} > C_1$, where C_1 is such that

$$\text{Prob}_{H_0}[\hat{\eta} > C_1] \leq \alpha.$$

Asymptotic normality of a normalized triplet test statistic is well known and is explicitly given in Randles et al. [33]. They selected six symmetric members of the generalized lambda family as the basic random variables for the study. They show that the level condition holds reasonably well for all six of the symmetric distributions under a two-sided test if variance estimator is appropriately chosen. Unfortunately, non-normalized triplet test statistic fails to maintain its nonparametric characteristics and that may result in higher type-I error rate for a broad class of symmetric distributions. We illustrate that in Section 4.2.

2.2. Antille–Kersting-type sign test based on spacing

Antille and Kersting [36] introduced a sign test statistic for tests for symmetry of F about an unknown value μ , based on spacing of first order defined as

$$D_i = X_{(i+1)} - X_{(i)}, \quad i = 1, \dots, n-1.$$

Define

$$U_i = \begin{cases} 0.5 & \text{if } D_i - D_{n-i} \leq 0, \\ -0.5 & \text{otherwise} \end{cases}$$

and denote

$$V = \sum_{i=1}^{[(n-1)/2]} U_i.$$

A test at level α against a long right tail may be given by: reject H_0 if $V > C_2$, where C_2 is such that

$$\text{Prob}_{H_0}[V > C_2] \leq \alpha.$$

We study the exact performances through computations and results are not encouraging for small samples. Recently, Ekström and Jammalamadaka [48] consider an improved test based on spacing but those are also more effective from an asymptotic point of view.

2.3. Gastwirth's type sign test of symmetry

Let us modify Gastwirth's [16] sign test for symmetry assuming an unknown median. For $I = 1, 2, \dots, [n/2]$, introduce the indicator function

$$I_i = \begin{cases} 1 & \text{if } X_{(n+1-i)} > 2\tilde{\mu} - X_{(i)}, \\ 0 & \text{if } X_{(n+1-i)} \leq 2\tilde{\mu} - X_{(i)} \end{cases}$$

and define

$$S = \sum_{i=1}^{[n/2]} I_i.$$

Consequently, a test at level α against a long right tail may be given by: reject H_0 if $S > C_3$, where C_3 is such that

$$\text{Prob}_{H_0}[S > C_3] \leq \alpha.$$

It is known that Gastwirth's type sign test does not retain its distribution-free characteristics when the median is assumed to be unknown and is estimated from the observed sample. Also, power performance of such a test is very poor.

2.4. *Butler-type test based on Smirnov's statistics*

Butler [21] designed a test for symmetry based on Smirnov's statistics assuming a known median. We may consider the modified Butler test replacing the known median by its estimate. Let

$$D_n = \max_{i: X_i < \tilde{\mu}} [1 - F_n(X_i) - F_n(2\tilde{\mu} - X_i)].$$

Consequently, a test at level α against a long right tail may be given by: reject H_0 if $D_n > C_4$, where C_4 is such that

$$\text{Prob}_{H_0}[D_n > C_4] \leq \alpha.$$

Like Gastwirth's type sign test, the modified Butler-type test also does not retain its distribution-free characteristics for a small number of samples when the median is assumed to be unknown. However, null distribution in this case with a moderately large sample size is not highly affected compared to the previous case as the statistic is based on the empirical distribution function. Unfortunately, power performance of such a test is not good when the sample size is small.

2.5. *Modified Rothman–Woodrooffe-type test of symmetry*

Rothman and Woodrooffe [23] introduced a test for symmetry based on Cramer–von Mises statistics. In the present context, we modify it for the unknown median and incorporate weight for enhanced performance for detecting possible presence of a long right tail. Define,

$$R_n = \sum_{i=1}^n a_i \left[F_n(2\tilde{\mu} - X_{(i)}) - \frac{2n - 2i + 1}{2n} \right]^2,$$

where $a_1 > a_2 > \dots > a_n > 0$ is a set of suitable weights.

Consequently, a test at level α against a long right tail may be given by: reject H_0 if $R_n > C_5$, where C_5 is such that

$$\text{Prob}_{H_0}[R_n > C_5] \leq \alpha.$$

For testing the null hypothesis against positive skewness $a_j = 1/j$, for $j = 1, 2, \dots, n$, is a good choice. Later, we shall see that the power performance of this modified test is relatively closer to the proposed test.

2.6. *Modified McWilliams-type test based on runs*

McWilliams [40] test introduced a test for symmetry based on runs. We consider the modified version of it, for detecting a long right tail. Let $Y_{(1)}, Y_{(2)}, \dots, Y_{(n)}$ denote the sample values ordered

from smallest to largest according to absolute value of $X_{(i)} - \tilde{\mu}$ (signs are retained), and let S_1, S_2, \dots, S_n denote indicator variables designating the signs of the $Y_{(i)}$ values such that

$$S_i = \begin{cases} 1 & \text{if } Y_{(i)} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The statistic used for testing symmetry is simply R^* , the number of successive runs of 1 in the $\{S_i\}$ sequence. As in McWilliams [40] R^* can also be expressed as

$$R^* = I_2 + I_3 + \dots + I_n,$$

where

$$I_k = \begin{cases} 1 & \text{if } S_k = S_{k-1} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We reject the null hypothesis H_0 at level α against a long right tail iff $R^* > C_6$, where C_6 is such that

$$\text{Prob}_{H_0}[R^* > C_6] \leq \alpha.$$

The performance of this test is also not very encouraging compared to the proposed test.

It is seen that most of the classical tests are robust against two-sided alternatives for a large sample. Some of the tests are even robust for a one-sided alternative when the sample size is large. In most of the cases, however, determination of cut-off points at a given level is not easy unless a large sample approximation is considered. Moreover, there are no unique best tests for detecting symmetry of a bell-shaped population against a long right tail specially when numbers of observations are not so large and the population median is unknown. With this background, we introduce a rule of thumb in the next section, which will fill the gap.

3. An ad hoc test based on exceedances and image point of first-order statistics

Let X_1, X_2, \dots, X_n be the random sample of size n from F . Suppose that the corresponding order statistics are $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ and define the observed (sample) median by $\tilde{\mu}$. Recall that under H_0 , $(\tilde{\mu} - X_{(1)})$ and $(X_{(n)} - \tilde{\mu})$ are expected to be nearly equal. In other words, we can argue that under H_0 , $X_{(n)}$ will lie in the vicinity of $(2\tilde{\mu} - X_{(1)})$. On the other hand, if there exists a long right tail, one can expect that $X_{(n)}$, along with few more higher-order statistics, will exceed $(2\tilde{\mu} - X_{(1)})$. Let $x_{(j)}$ be the observed value of the order statistics $X_{(j)}$, $j = 1, 2, \dots, n$. Then, we can frame the decision rule at a given level α as: Reject H_0 if $x_{(t)} > 2\tilde{\mu} - x_{(1)}$, where t is such that

$$\max_{[(n+1)/2] < t \leq n} \text{Prob}_{H_0}[X_{(t)} > 2\tilde{\mu} - X_{(1)}] \leq \alpha$$

where $[\cdot]$ denotes the largest integer contained in it. Clearly, the proposed test may be looked upon as an exceedance-type test for symmetry. An exceedance statistic is nothing but the reverse of more well-known precedence statistics. The precedence-type test for the location parameter was introduced by Nelson [51] and was subsequently explored by many researchers in the past five decades. In this context, interested readers may see Balakrishnan and Ng [52] for a detailed discussion on precedence/exceedance tests for location shift.

3.1. Justification for large n

Define \mathcal{F} as a class of absolutely continuous distribution F having continuous derivative of first order with $dF(x) = f(x) dx$, where f is the probability density function. Further, assume that f

has a high order of contact at the extremes (diminishing tails of bell-shaped curves) in support of F , in the sense that

$$\lim_{n \rightarrow \infty} \sup_{X_{(1)} - 2|\mu - \tilde{\mu}| < x < X_{(1)} + 2|\mu - \tilde{\mu}|} \sqrt{n} f(x) = 0. \quad (2)$$

The assumption is plausible for a wide range of bell-shaped distributions with high order of contact at the tails.

Result 3.1 For any absolutely continuous distribution F in \mathcal{F} and for moderately large sample size $n \geq N$, suppose ad hoc order statistics $X_{(t)}$ ensures a nominal level of significance $\alpha \in [0, 1)$. Then, t can be approximated by $[n(\alpha^{1/n})]$, where $[q]$ is the largest integer contained in q .

Proof Note that, $\alpha = \text{Prob}_{H_0}[X_{(t)} > 2\tilde{\mu} - X_{(1)}] = \text{Prob}_{H_0}[F_n(X_{(t)}) > F_n(2\tilde{\mu} - X_{(1)})]$, where F_n stands for the empirical distribution function corresponding to X and therefore $F_n(X_{(t)}) = t/n$. Further, as F_n converges to F in probability, we have,

$$\alpha = \text{Prob}_{H_0}[F_n(X_{(t)}) > F_n(2\tilde{\mu} - X_{(1)})] \xrightarrow{D} \text{Prob}_{H_0}\left[F(2\tilde{\mu} - X_{(1)}) < \frac{t}{n}\right]. \quad (3)$$

By symmetry of F under H_0 , we have R.H.S. of Equation (3) = $\text{Prob}_{H_0}[F(2(\mu - \tilde{\mu}) + X_{(1)}) > 1 - t/n]$. Since first-order derivative of F exists, for some ξ such that

$$\min(X_{(1)}, 2(\mu - \tilde{\mu}) + X_{(1)}) < \xi < \max(X_{(1)}, 2(\mu - \tilde{\mu}) + X_{(1)}),$$

applying mean value theorem, we get

$$F(2(\mu - \tilde{\mu}) + X_{(1)}) = F(X_{(1)}) + 2(\mu - \tilde{\mu})f(\xi).$$

Note that sample median ($\tilde{\mu}$) is always a consistent estimator of population median (μ) and therefore $(\mu - \tilde{\mu})$ tends to 0 in probability, while from Equation (2) we see $(\xi) = o(1/\sqrt{n})$. Combining the above arguments along with Equation (2) we get, for large n ,

$$\alpha = \text{Prob}_{H_0}\left[F(X_{(1)}) + 2(\mu - \tilde{\mu})f(\xi X_{(1)}) > 1 - \frac{t}{n}\right] \approx \text{Prob}_{H_0}\left[F(X_{(1)}) > 1 - \frac{t}{n}\right] = \left(\frac{t}{n}\right)^n.$$

Hence, the result follows. ■

It is interesting to note that intensive computation shows when $\alpha = 0.05$, $t = n - 3$ for all $n > 3$. We check computationally that the level condition is fairly accurate for all $n = 4, 5, \dots$, at least up to 10^6 . Therefore, for most of the practical purposes where sample size lies between 4 and 10^6 , we can reject H_0 at 5% level of significance if $X_{(n-3)} > 2\tilde{\mu} - X_{(1)}$.

Similarly, we see that when $\alpha = 0.01$, $t = n - 4$ for $n = 5, 6, \dots, 15$, and $t = n - 5$ for $n > 15$. This makes the decision rule as simple as a rule of thumb. Not only so, the rule can be applied even if the sample size is as small as 10. Further discussions on computational results are provided in Section 5.

3.2. Unbiasedness

Note that under a long right tail, essentially,

$$\text{Prob}_{H_0}\left[F(2\tilde{\mu} - X_{(1)}) < \frac{t}{n}\right] < \text{Prob}_{H_0}\left[1 - F(2(\mu - \tilde{\mu}) + X_{(1)}) < \frac{t}{n}\right].$$

Therefore, following a similar logic as in the proof of Result 3.1 we can conclude that the proposed test is unbiased for testing symmetry against a long right tail.

3.3. Dealing with ties in practice

The probability of a tie when observations are recorded from an absolute continuous population is essentially zero. Nevertheless, owing to the limitations of measuring instruments often observations are recorded with a pre-fixed number of decimal places. This may give rise to a few ties in practice. Note that the test only counts the number of exceedances over $2\tilde{\mu} - X_{(1)}$ and checks whether $X_{(t)}$ exceeds $2\tilde{\mu} - X_{(1)}$ or not. Clearly, the proposed test will remain unaffected if there are ties in places other than either $X_{(1)}$ or $X_{(t)}$. If there exists more than one minimum, we should, without loss of generality, consider the common minimum of the observations as observed $X_{(1)}$. This assumption will not affect the number of exceedances over $2\tilde{\mu} - X_{(1)}$. Similarly, if there are k tied observations that exceed $X_{(r)}$, and $r < t \leq r + k$, we may assign $X_{(r+1)} = X_{(r+2)} = \dots = X_{(r+k)}$ and determine $X_{(t)}$ accordingly. We shall only compare $X_{(t)}$ with $2\tilde{\mu} - X_{(1)}$, and naturally if $X_{(t-1)} = X_{(t)}$ or $X_{(t)} = X_{(t+1)}$, ties will not influence the decision. Therefore, the performance of the proposed test will remain unaffected in the presence of ties.

3.4. Right tail weight quotient

A simple measure of the right tail weight may be given by the right tail weight quotient (RTWQ). We may define RTWQ as the proportion of observations that exceeds $(2\tilde{\mu} - X_{(1)})$. A very high median of arsenic contamination along with very high RTWQ indicates that groundwater should be strictly avoided for drinking without proper filtration. It is easy to see that the efficacy of the measure depends on the power performance of the proposed test. One can draw an analogy between RTWQ with the p -value traditional tests; of course they are not the same. Since the proposed test is ad hoc and finding actual p -value is not possible, RTWQ can be used instead.

4. Numerical results based on Monte-Carlo simulation

In this section, we present numerical results based on the Monte-Carlo simulation. We first examine the exact performance of the proposed test (rule of thumb) for symmetry against a long right tail based on Monte-Carlo simulations using R.2.15.0 software. We compute the nominal level (Level Actually Attained) at 5% level of significance of the proposed test based on exceedance and image point of first-order statistic for various symmetric bell-shaped distributions for different sample sizes n . We consider n as 20 and 30 to investigate situations under small sample size, choose n as 50 to examine performances under moderate sample size and finally take n as 100 and 200 to study the performance of the proposed rule under larger sample size. Additionally, the performance under very large sample size is studied taking n as 500. In the class of symmetric bell-shaped distributions, we consider the standard normal model as the representative of thin-tailed distribution, standard Laplace and t distribution with 5 degrees of freedom (d.f.) as slightly heavier-tailed distribution, standard logistic model as moderately heavier-tailed distribution and standard Cauchy model as a very thick-tailed distribution. As all these models have unbounded support, we also consider the beta distribution of the first kind with both parameters as 5 as the representative of symmetric bell-shaped distributions with bounded support. In each case, 50,000 replicates of the Monte-Carlo study have been performed. Detailed results are presented in Section 4.1.

In the same subsection, we also examine the power performance of the proposed tests. Power performances of the proposed rule are studied for different sample sizes as before. To this end, we consider a class of distributions which are bell shaped but have a long right tail, namely, Gamma (shape = α_1 , scale = 1), for 3 combinations of $\alpha_1 = 1.0, 5.0$ and 0.5 ; Lognormal($0, \alpha_2$) for 3 combinations of $\alpha_2 = 1.0, 5.0$ and 0.5 ; chi-square with 3 d.f.; chi-square with 5 d.f.; Beta(1,3);

Beta(2,5); Weibull(0.5,1) and Weibull(2,1). Such a set of distributions covers a wide range of very small to large positive skewness and different types of Kurtosis and forms a nice representative of various possible alternative models.

In Section 4.2, we compute the nominal level (Level Actually Attained) at 5% level of significance of the competitive tests described under Sections 2.1–2.6. To this end, we consider the same set of symmetric bell-shaped distributions that are used for computing level actually attained by the proposed test and for the same set of values of sample size n used earlier except $n = 500$. At $n = 500$ large sample results hold for almost all the tests and therefore such a very high value of n is not of much interest. The study divulges some limitations of the classical test procedures.

Finally, in Section 4.3, we compare the power performance of the proposed test along with the classical test procedures described under Sections 2.1–2.6. For comparison purposes, we consider $n = 20$ to represent small sample size, $n = 50$ to represent moderate sample size and $n = 100$ to characterize larger sample size. For comparative power study, we choose the same set of bell-shaped right-tailed distributions as before.

4.1. Exact performance of the proposed test

In this subsection, we examine the exact performance of the proposed test (rule of thumb) for symmetry against a long right tail based on Monte-Carlo simulations. Results based on 50,000 replicates of the Monte-Carlo study are summarized in Table 1. Table 1 shows that the ad hoc test meets the level condition nicely in almost all cases. Only negligible upward biases are found with Beta(5,5) for a certain n . Note that Beta(1,1) is uniform and is not unimodal and bell shaped. Beta(2,2) is unimodal, bell shaped but order of contact at the tails is reasonably poor. Beta(5,5) displays only a moderate order of contact at the tails and therefore marginally overestimates the type-I error. Simulation results in Table 1 clearly show that in every case where the distribution is unimodal bell-shaped with reasonably high order of contact at the tails ad hoc test satisfies the level condition. It is worth mentioning that for Normal, Cauchy, Logistic and Laplace distributions, level actually attained depends on the sample size but not on the parameters. This is because tail behaviour of those models is not parameter specific. On the other hand, for the t and Beta distributions where tail behaviour changes with parameter, the level actually attained also varies. We omit the detail for brevity.

Subsequently, we examine the power performances of the proposed test under various representative alternative bell-shaped distributions with the right tail described before. We present the numerical results in Table 2. From Table 2, we observe that the proposed test is highly consistent and displays very good power performance even with $n \geq 20$. We see that with $n \geq 100$, in almost all cases, the powers are close to unity except for the gamma distribution where $\alpha_1 = 5$ and Weibull(2,1). Note that with an increase in the shape parameter, a gamma distribution tends to become more and more symmetric and therefore such a finding is reasonable. Similarly, the right tail of Weibull(2,1) is very short compared to Weibull(0.5, 1). Nevertheless with $n \geq 200$, perfect detection is almost assured for a large class of alternatives.

Table 1. Nominal level (level actually attained) at 5% level of significance of the proposed test.

n	20	30	50	100	200	500
Normal(0,1)	0.027	0.036	0.038	0.040	0.037	0.035
$t(5)$	0.019	0.027	0.029	0.030	0.031	0.031
Cauchy(0,1)	0.016	0.024	0.029	0.031	0.030	0.032
Logistic(0,1)	0.021	0.026	0.029	0.031	0.031	0.031
Laplace(0,1)	0.015	0.021	0.025	0.027	0.028	0.030
Beta(5,5)	0.036	0.047	0.054	0.055	0.053	0.049

Table 2. Power performance of the proposed test at the 5% level under various alternatives.

n	20	30	50	100	200	500
Gamma(1,1)	0.604	0.904	0.997	1	1	1
Gamma(5,1)	0.142	0.297	0.572	0.914	0.998	1
Gamma(0.5,1)	0.872	0.993	1	1	1	1
Lognormal(0,1)	0.669	0.945	0.999	1	1	1
Lognormal(0,2)	0.955	0.999	1	1	1	1
Lognormal(0,0.5)	0.260	0.547	0.873	0.997	1	1
chi-square(3)	0.427	0.767	0.978	1	1	1
chi-square(5)	0.263	0.538	0.868	0.998	1	1
Beta(1,3)	0.425	0.739	0.963	1	1	1
Beta(2,5)	0.170	0.345	0.632	0.946	0.999	1
Weibull(0.5,1)	0.954	0.999	1	1	1	1
Weibull(2,1)	0.126	0.254	0.501	0.870	0.996	1

4.2. A limitation of the competitive tests

In this subsection, we study the nominal levels of the competitive tests described under Sections 2.1–2.6 based on Monte-Carlo simulations. In Section 2, we mention that some of the modified tests, like Modified Gastwirth or Modified Butler-type tests do not retain their distribution-free characteristics when the median is assumed to be unknown and is estimated from the observed sample. This leads to a serious practical problem in implementation of those tests. We illustrate that problem with two examples.

Consider Randle's test statistic $\hat{\eta}$ as described in Section 2.1. One can easily compute that

$$\text{Prob}_{H_0}[\hat{\eta} > 0.066 \mid X \sim \text{Normal}(0, 1), n = 20] \approx 0.049.$$

It is well known that the standard normal distribution is perfectly symmetric and bell shaped. Therefore, when $n = 20$, one may think of using 0.066 as a cut-off point at 5% level of significance to test null hypothesis of symmetric and bell-shaped characteristics against a long right tail. However, we see that

$$\text{Prob}_{H_0}[\hat{\eta} > 0.066 \mid X \sim \text{Cauchy}(0, 1), n = 20] \approx 0.168.$$

That is, even though standard Cauchy is symmetric and bell shaped in nature, Randle's test will reject true null hypothesis in approximately 16.8% cases. In other words, the test no longer ensures the 5% level of significance if we choose the cut point obtained using standard normal distribution. Clearly, $\hat{\eta}$ is not exhibiting distribution-free characteristics for small samples. In fact, tail behaviour of $\hat{\eta}$ depends on the parent distribution of sample observations free even for moderately large sample size. Table 3 displays the differences in level actually attained in Randle's type test if the cut-off point C_1 is determined using a standard Cauchy distribution and a Beta(2,2) distribution.

Further, consider Antille's test statistic V as described in Section 2.2. One can easily see that

$$\text{Prob}_{H_0}[V > 8 \mid X \sim \text{Laplace}(0, 1), n = 200] \approx 0.050.$$

The standard Laplace distribution is perfectly symmetric and bell shaped. Therefore, when $n = 200$, one consider 8 as a cut-off point at 5% level of significance to test null hypothesis of symmetric and bell-shaped characteristics against a long right tail. However, we see that

$$\text{Prob}_{H_0}[V > 8 \mid X \sim \text{Cauchy}(0, 1), n = 200] \approx 0.127.$$

That is, Antille's type test will reject true null hypothesis in approximately 12.7% cases even with 200 available samples. Clearly, V does not exhibit distribution-free characteristics even for larger

Table 3. Level actually attained at the 5% cut-off point for adaptive Randle-type test.

<i>n</i>	Cut-off points based on Cauchy(0,1)					Cut-off points based on Beta(2,2)				
	20	30	50	100	200	20	30	50	100	200
Cut off	0.116	0.090	0.067	0.051	0.032	0.066	0.051	0.039	0.026	0.019
Normal(0,1)	0.003	0.004	0.002	0.0	0.004	0.049	0.055	0.052	0.060	0.074
<i>t</i> (5)	0.003	0.005	0.008	0.001	0.013	0.086	0.074	0.078	0.079	0.108
Cauchy(0,1)	0.038	0.045	0.043	0.033	0.046	0.168	0.165	0.175	0.200	0.193
Logistic(0,1)	0.004	0.002	0.003	0.001	0.002	0.070	0.068	0.067	0.079	0.106
Laplace(0,1)	0.007	0.008	0.006	0.004	0.006	0.085	0.094	0.079	0.084	0.111
Beta(5,5)	0.001	0.001	0.001	0.001	0	0.051	0.051	0.041	0.047	0.060

sample sizes. Table 4 displays the differences in level actually attained for Antille’s type test if the cut-off point C_2 is determined using a standard Cauchy distribution and a Beta(2,2) distribution.

The above examples indicate that the value of the cut-off points $C_i, i = 1(1)6$ may be different even for the same n depending on the underlying model in \mathcal{F} . Certainly, this is not desirable for testing symmetry and bell-shaped nature of a distribution. However, as a remedial measure, one may consider,

$$C_i^*(\alpha) = \max_{F \in \mathcal{F}} C_i(\alpha).$$

It is therefore possible to determine $C_i^*(\alpha)$ for a given class of $F \in \mathcal{F}$. Such adaptation will give rise to an adaptive Gastwirth’s sign test or adaptive Butler-type test etc. We have seen through a large-scale simulation study that $C_i^*(0.05)$ [the cut-off points at a nominal level of 5%] is same as the $C_i(0.05)$ determined on the basis of Beta(2,2) distribution. This is true for all four tests except for Randle’s triplet type test and Antille’s type test based on spacing (i.e. for, $i = 3, 4, 5$ and 6). Such cut-off points along with the level actually attained under various symmetric and bell-shaped models for the tests described in Sections 2.3–2.6 are presented in Table 5.

From Table 3. we see that Randle’s type test satisfies level conditions when $C_1^*(0.05)$ is obtained with respect to the heavy-tailed standard Cauchy model. However, the similar cut-off points return very low values of level actually attained in the case of other distributions. On the other hand, if cut-off points are obtained on the basis of a Beta(2,2) distribution, we see that levels actually attained are much more than the specified level 0.05 in the case of very heavy-tailed Cauchy models or moderately heavy-tailed Laplace, logistic or *t*-distributions with lower d.f. Therefore, more serious type-I error creeps in. Similar findings for Antille’s type test can be seen in the details of Table 4. However, in this case, the cut-off points based on the Beta(2,2) model shows that Antille’s type test-based spacing has better control over type-I error except for very heavy-tailed Cauchy distribution.

4.3. Power comparison

In this subsection, we investigate the power performances of the six modified or adaptive type tests presented in Sections 2.1–2.6 and compare them with the proposed test. Power performances of the proposed rule are studied for different sample sizes as in Section 4.1. As before, we consider 50,000 replicates of the Monte-Carlo simulation. It is worth mentioning that such a high number of replicates of Monte-Carlo experiment give very stable results.

Table 6 displays the differences in Power in Randle’s type test if the cut-off point C_1 is determined using a standard Cauchy distribution and a Beta(2,2) distribution. Certainly, when C_1 is based on a Beta(2,2) distribution power performance of the test is exceptionally good. Nevertheless, those cut-off points can return very high type-I error and have to be avoided in practice. Power of the test drops significantly when we use cut-off points based on a standard Cauchy

Table 4. Level actually attained at the 5% cut-off point for adaptive Antille–Kersting–Ekström–Jammalamadaka-type test.

<i>n</i>	Cut-off points based on Cauchy(0,1)					Cut-off points based on Beta(2,2)				
	20	30	50	100	200	20	30	50	100	200
Cut off	3	4.5	5.5	8	12	3	3.5	4.5	6	8
Normal(0,1)	0.015	0.004	0.007	0.006	0.005	0.015	0.023	0.024	0.033	0.043
<i>t</i> (5)	0.018	0.006	0.010	0.009	0.008	0.018	0.027	0.029	0.040	0.051
Cauchy(0,1)	0.047	0.025	0.042	0.045	0.042	0.047	0.069	0.083	0.107	0.128
Logistic(0,1)	0.017	0.005	0.010	0.008	0.006	0.017	0.024	0.027	0.037	0.047
Laplace(0,1)	0.020	0.006	0.011	0.011	0.007	0.020	0.028	0.032	0.043	0.050
Beta(5,5)	0.013	0.004	0.007	0.007	0.004	0.013	0.021	0.023	0.031	0.040

Table 5. Level actually attained at the 5% cut-off point.

<i>n</i>	20	30	50	100	200	20	30	50	100	200
	Adaptive Gastwirth’s type sign test					Adaptive Butler-type test				
Cut off	9	14	24	49	98	0.200	0.200	0.180	0.130	0.095
Normal(0,1)	0.034	0.026	0.017	0.010	0.023	0.032	0.020	0.032	0.025	0.026
<i>t</i> (5)	0.028	0.021	0.012	0.007	0.017	0.027	0.016	0.024	0.018	0.021
Cauchy(0,1)	0.021	0.015	0.010	0.004	0.011	0.017	0.010	0.016	0.010	0.014
Logistic(0,1)	0.032	0.021	0.013	0.007	0.020	0.027	0.016	0.025	0.018	0.023
Laplace(0,1)	0.026	0.017	0.009	0.005	0.012	0.017	0.011	0.016	0.010	0.012
Beta(5,5)	0.034	0.025	0.019	0.010	0.031	0.038	0.025	0.036	0.029	0.034
	Adaptive Rothman–Woodrooffe-type test					Modified McWilliams-type test				
Cut off	0.110	0.087	0.062	0.034	0.018	6	9	15	29	56
Normal(0,1)	0.026	0.023	0.019	0.015	0.013	0.020	0.020	0.017	0.019	0.019
<i>t</i> (5)	0.018	0.015	0.011	0.008	0.006	0.016	0.020	0.015	0.016	0.022
Cauchy(0,1)	0.013	0.010	0.006	0.004	0.003	0.014	0.018	0.014	0.016	0.020
Logistic(0,1)	0.019	0.016	0.011	0.008	0.007	0.017	0.020	0.015	0.019	0.020
Laplace(0,1)	0.012	0.009	0.006	0.003	0.003	0.015	0.015	0.013	0.016	0.018
Beta(5,5)	0.031	0.030	0.025	0.022	0.019	0.022	0.024	0.022	0.020	0.022

Table 6. Power performance under various alternatives for adaptive Randle-type test based on 5% cut-off points.

<i>n</i>	Cut-off points based on Cauchy(0,1)					Cut-off points based on Beta(2,2)				
	20	30	50	100	200	20	30	50	100	200
Gamma(1,1)	0.484	0.803	0.980	1.000	1.000	0.879	0.974	0.999	1.000	1.000
Gamma(5,1)	0.031	0.075	0.222	0.472	0.934	0.340	0.512	0.681	0.926	0.998
Gamma(0.5,1)	0.867	0.984	0.999	1	1	0.986	0.997	1	1	1
Lognormal(0,1)	0.697	0.929	0.995	1	1	0.953	0.995	0.999	1	1
Lognormal(0,2)	0.983	1	1	1	1	0.999	1	1	1	1
Lognormal(0,0.5)	0.138	0.339	0.662	0.940	1	0.622	0.798	0.936	0.998	1
chi-square(3)	0.272	0.530	0.869	0.991	1	0.766	0.901	0.981	1	1
chi-square(5)	0.104	0.261	0.566	0.896	1	0.580	0.749	0.921	0.996	1
Beta(1,3)	0.139	0.318	0.653	0.942	1	0.632	0.794	0.941	1	1
Beta(2,5)	0.017	0.054	0.162	0.343	0.892	0.325	0.448	0.622	0.866	0.996
Weibull(0.5,1)	0.980	1	1	1	1	0.998	1	1	1	1
Weibull(2,1)	0.014	0.039	0.095	0.214	0.743	0.256	0.365	0.541	0.793	0.970

distribution. In Table 7, we compare the power performances of the proposed test and the six possible competitors for three different sample sizes. From Table 7, we see that the proposed test outperforms all the six other tests for moderate to large sample sizes. For smaller sample size, the proposed test outperforms five out of six other tests, except the Randle’s triplet test which

Table 7. Power comparison of different tests under small, moderate and large sample sizes.

Distribution	Proposed test	Randle-type test	Antille-Kersting-type test	Gastwirth's sign test	Butler-type test	Rothman-Woodroofe-type test	Modified McWilliams-type test
<i>n</i> = 20							
Gamma(1,1)	0.604	0.484	0.285	0.092	0.408	0.429	0.144
Gamma(5,1)	0.142	0.031	0.071	0.065	0.097	0.103	0.024
Gamma(0.5,1)	0.872	0.867	0.526	0.108	0.728	0.734	0.402
Lognormal(0,1)	0.669	0.697	0.362	0.104	0.457	0.487	0.154
Lognormal(0,2)	0.955	0.983	0.691	0.125	0.857	0.862	0.516
Lognormal(0,0.5)	0.260	0.138	0.137	0.076	0.160	0.180	0.038
chi-square(3)	0.427	0.272	0.192	0.086	0.265	0.289	0.080
chi-square(5)	0.263	0.104	0.121	0.076	0.162	0.179	0.039
Beta(1,3)	0.425	0.139	0.163	0.078	0.270	0.293	0.083
Beta(2,5)	0.170	0.017	0.067	0.065	0.110	0.119	0.027
Weibull(0.5,1)	0.954	0.980	0.684	0.122	0.865	0.867	0.571
Weibull(2,1)	0.126	0.014	0.056	0.062	0.084	0.092	0.020
<i>n</i> = 50							
Gamma(1,1)	0.997	0.980	0.778	0.070	0.853	0.890	0.490
Gamma(5,1)	0.572	0.222	0.229	0.050	0.173	0.219	0.046
Gamma(0.5,1)	1	0.999	0.964	0.083	0.993	0.995	0.914
Lognormal(0,1)	0.999	0.995	0.872	0.074	0.908	0.944	0.530
Lognormal(0,2)	1	1	0.995	0.092	1	1	0.971
Lognormal(0,0.5)	0.873	0.662	0.449	0.059	0.373	0.475	0.103
chi-square(3)	0.978	0.869	0.603	0.061	0.626	0.718	0.254
chi-square(5)	0.868	0.566	0.403	0.066	0.358	0.461	0.105
Beta(1,3)	0.963	0.653	0.524	0.061	0.608	0.687	0.305
Beta(2,5)	0.632	0.162	0.206	0.045	0.194	0.255	0.063
Weibull(0.5,1)	1	1	0.995	0.092	1	1	0.981
Weibull(2,1)	0.501	0.095	0.167	0.045	0.132	0.172	0.038
<i>n</i> = 100							
Gamma(1,1)	1	1.000	0.983	0.056	0.995	0.999	0.864
Gamma(5,1)	0.914	0.472	0.469	0.040	0.276	0.458	0.088
Gamma(0.5,1)	1	1	0.999	0.068	1	1	0.999
Lognormal(0,1)	1	1	0.996	0.057	0.999	1	0.881
Lognormal(0,2)	1	1	1	0.075	1	1	1
Lognormal(0,0.5)	0.997	0.940	0.795	0.043	0.659	0.862	0.262
chi-square(3)	1	0.991	0.922	0.049	0.923	0.981	0.580
chi-square(5)	0.998	0.896	0.749	0.054	0.624	0.837	0.257
Beta(1,3)	1	0.942	0.864	0.050	0.899	0.965	0.621
Beta(2,5)	0.946	0.343	0.437	0.040	0.309	0.510	0.123
Weibull(0.5,1)	1	1	1.000	0.075	1	1	1
Weibull(2,1)	0.870	0.214	0.353	0.035	0.188	0.344	0.077

performs marginally better in certain situations. We further see that the modified Gastwirth and McWilliams's tests return relatively poorer power performances compared to other tests.

In Figure 1(a)–(d), we compare the power performance of the proposed test along with Randle's test, Antille's test, Butler's test and Rothman's test. We plot smoothed power curves against increasing sample size. We choose the Gamma(1,1), Lognormal(0,1), Chi-square with 3 d.f. and Beta(1,3) distributions, respectively, as the alternative for Figure 1(a)–(d). We see that the proposed test clearly shows better power performances in three out of the four cases, except for the Lognormal(0,1) distribution. For the Lognormal(0,1) model, Randle's test is better when sample size is less than or equal to 20. However, if sample size exceeds 20, both the proposed test and Randle's test perform almost similarly, the proposed test being marginally better. Further, the advantage of the proposed test is its simplicity. Moreover, the proposed test is free of ambiguity related to choice of the exact 5% cut-off point.

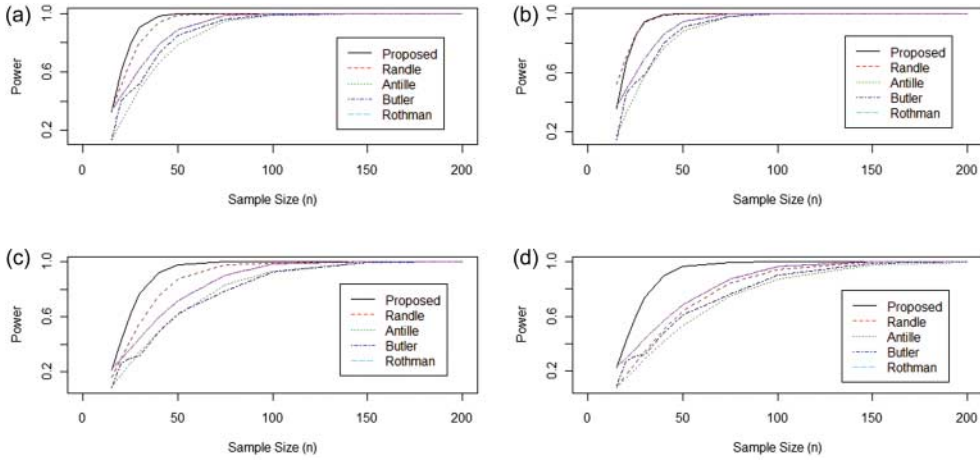


Figure 1. (a) Power comparisons for Gamma(1,1) distribution, (b) power comparisons for Lognormal(0,1) distribution, (c) power comparisons for chi-square with 3 d.f. and (d) power comparisons for Beta(1,3) distribution.

Table 8. Observed arsenic contaminations in different blocks (samples in mg/l).

English Bazaar	0.30; 0.18; 0.25; 0.15; 0.04; 0.18; 0.15; 0.07; 0.18; 0.15; 0.15; 0.00; 0.00; 0.03; 0.01; 0.20; 0.18; 0.25; 0.07; 0.09; 0.09; 0.07; 0.20; 0.01; 0.07
Kaliachak (Block-1)	0.42; 0.10; 0.45; 0.10; 0.50; 0.47; 0.20; 0.03; 0.30; 0.06; 0.10; 0.15; 0.50; 0.25; 0.10; 0.15; 0.20; 0.04; 0.00; 0.08; 0.30; 0.15; 0.48; 0.01; 0.55
Kaliachak (Block-2)	0.08; 0.18; 0.15; 0.15; 0.15; 0.18; 0.07; 0.30; 0.05; 0.04; 0.45; 0.40; 0.30; 0.25; 0.06; 0.18; 0.07; 0.20; 0.55; 0.10; 0.06; 0.20; 0.55; 0.45; 0.07
Kaliachak (Block-3)	0.03; 0.08; 0.15; 0.07; 0.25; 0.20; 0.30; 0.08; 0.07; 0.40; 0.35; 0.25; 0.55; 0.04; 0.04; 0.05; 0.08; 0.55; 0.45; 0.09; 0.03; 0.25; 0.50; 0.07; 0.09

5. Illustrative example

In this section, we illustrate our proposed test with real data. Table 8 represents the random samples of size 25 that were collected from tube wells of four blocks of Malda District of West Bengal, India, during an arsenic Contamination monitoring programme. Locations of sampling within each block were widely apart from one other. Further, the groundwater sources were found to vary in terms of depth, dissolved oxygen etc., which ensured randomness of the observed samples.

It is easy to see that for the English Bazaar block observed first-order statistics $X_{(1)} = 0.00$; the median $\tilde{\mu} = X_{(13)} = 0.15$ and observed $X_{(n-3)} = X_{(22)} = 0.20$. This implies that observed $x_{(22)} < (2\tilde{\mu} - x_{(1)}) = 0.30$. Therefore, we find no evidence at the 5% level to conclude that there is a long right tail in the distribution of arsenic contamination. Now, in Kaliachack-I, we see, $X_{(1)} = 0.00$; the median $\tilde{\mu} = X_{(13)} = 0.15$ and observed $X_{(n-3)} = X_{(22)} = 0.48$. This implies that observed $x_{(22)} \gg (2\tilde{\mu} - x_{(1)}) = 0.30$. Therefore, unlike English Bazaar, data provide certain evidences at the 5% level to conclude that there is a long right tail in the distribution of arsenic contamination in Kaliachack-I. Further, we see that there are seven observations in this block with contamination more than 0.30 mg/l. Therefore, we can say that RTWQ, as in Section 3.4, is 0.28 for Kaliachack-I. Again, in Kaliachack-II, we see, $X_{(1)} = 0.04$; the median $\tilde{\mu} = X_{(13)} = 0.18$ and observed $X_{(n-3)} = X_{(22)} = 0.45$. This implies that observed $x_{(22)} \gg (2\tilde{\mu} - x_{(1)}) = 0.32$. Therefore, in Kaliachack-II also data provides certain evidences at the 5% level to conclude that there is a long right tail in the distribution of arsenic contamination. Further, we see that there are five observations in this block with contamination more than 0.32 mg/l. Therefore, we can say that RTWQ is 0.20 for Kaliachack-II. Finally, in Kaliachack-III, we see, $X_{(1)} = 0.03$; the median $\tilde{\mu} = X_{(13)} = 0.09$

and observed $X_{(n-3)} = X_{(22)} = 0.45$. This implies that observed $x_{(22)} \gg (2\tilde{\mu} - x_{(1)}) = 0.15$. This indicates that in Kaliachak-III also data provide certain evidences at the 5% level to conclude that there is a long right tail in the distribution of arsenic contamination. Further, we see that there are 12 observations in this block with contamination more than 0.15 mg/l. Therefore, we can say that RTWQ is 0.48 for Kaliachak-III. Certainly, arsenic contamination distribution of Kaliachak-III has the longest right tail as indicated by RTWQ. The median level of contamination in all four blocks are found to be greater than the admissible limit and, therefore, unlike old claims, we see that even in the high-contaminated zones arsenic distribution may be symmetric or right skewed.

6. Concluding remarks

To summarize, an ad hoc test for testing symmetry and bell-shaped nature of a continuous probability distribution against a long right tail is suggested in the present article. This is ad hoc in the sense that this can be used for a particular purpose of testing the symmetry of a bell-shaped model against a long right tail and this is not a traditional test of symmetry versus asymmetry. Moreover, the test is not evolved from traditional likelihood ratio, Neyman Pearson lemma or other traditional approaches of construction of statistical tests. Our proposed test does not belong to the class of adaptive tests as well. However, one can consider this as a heuristic test. Cut-off points at a nominal level and power performances of the proposed test are studied via Monte-Carlo simulations. The proposed ad hoc test is not only the quickest among several other competitive tests but also returns high power in most of the situations. Moreover, the proposed test retains much better non-parametric characteristics compared to several other similar tests when true median is known. Unlike other tests, for a smaller number of observations, determining the exact cut-off points at a given level is the most convenient for the proposed test. When asymptotic results are in operation all the tests are, however, consistent. All the competitive tests are as good as the proposed test under certain large sample situations.

Our computational studies based on Monte-Carlo essentially serve three purposes. The first and foremost purpose is to show that the ad hoc test described in Section 3 satisfies level conditions very well for a large class of symmetric as well as bell-shaped distributions. It also returns very good power against a large class of distributions having a long right tail. Second, it confirms that the most of the well-known test statistics for symmetry do not show up good tail behaviour. That results in over- or under-estimation of type-I error in different situations. Finally, our study establishes that the power performance of the ad hoc test is better than or almost as good as most of the traditional competitors.

Acknowledgements

The authors wish to thank the Editor for his encouragement. They are also grateful to the Associate Editor and an anonymous reviewer for various comments and suggestions upon an earlier version of this work that led to significant improvement. They are also grateful to Dr S. Chowdhury of Army Institute of Management, Kolkata, India, for his careful reading of the manuscript and his help in language editing.

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