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Maximum Likelihood Estimation of Two Unknown Parameter of Beta-Weibull Distribution under Type II Censored Samples

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Abstract

In this paper, the maximumlikelihood estimates (mles) are obtained for the two unknown parameters of the Beta-Weibull(B-W)distribution under type II censored samples. Also,asymptotic variances and covariance matrix of the estimators aregiven. An iterative procedure is used to obtain the estimators numerically using MathCad Package. To study the properties of maximum likelihood estimators simulation results are included for different sample sizes.

Keywords: the Beta-Weibull distribution; censored type II;maximum likelihood estimates; variance covariance matrix

1. Introduction

The Weibull distribution, named after WallodiWeibull (1939) is one of the best known distributions and has wide applications in diverse disciplines (see Johnson *et al.* 1994 and Murthy *et al.* 2004). The Weibull family, characterized by a shape parameter and a scale parameter, has been extended in various ways to contain other distributions. Mudholkar and Srivastava (1993) introduced an exponentiated version of the Weibull model that included an additional shape parameter. The distribution has a closed form of probability density, survival, and hazard functions that are flexible and able to generate a wide variety of frequently observed hazard shapes, including unimodal and bathtub. Mudholkar*et al.* (1996) proposed another generalization of the Weibull model, which is able to generate similar types of hazard shapes as the exponentiated model; however, irregularities may arise as the support of the distribution becomes dependent on theparameter space. Famoye*et al.* (2005) introduced further extension of the exponentiatedWeibull distribution which they called the Beta- Weibull distribution.

Let $G(x)$ be the cumulative distribution function (cdf) of a random variable X. The cumulative distribution function for a generalized class of distributions for the random variable X was defined by Eugene *et al.* (2002) as follows:

Let $I_x(a,b) = B(a,b)^{-1} \int_0^x w^{a-1}(1-w)^{b-1} dw,$

denotes the incomplete beta function ratio (the cdf of the beta distribution with parameters $a > 0$ and $b > 0$) and $B(a,b)$ is the standard beta function,

then

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{a-1}(1-w)^{b-1} dw, \quad (1)$$

Jones (2004) noted that the parameters $a > 0$ and $b > 0$ have a role of the skewness and the tail weight. The distribution F well be called the beta G distribution and the corresponding probability density function for this generalized class of distribution $F(x)$ is given by

$$f(x) = \frac{1}{B(a,b)} \left(G(x)\right)^{a-1} \left(1 - G(x)\right)^{b-1} G'(x) \quad (2)$$

Singh *et al.* (1988) first introduced the family of distributions in (2) while considering applications in survival analysis of lung cancer data and they studied the case when $G(x)$ is the cdf of log-logistic distribution. Eugene *et al.* (2002) studied the case when $G(x)$ is the normal cumulative distribution function with parameters μ and σ and pointed out the relationship of the family in (2) with the distribution of logistic order statistics. Jones (2004) studied some general properties of this family of distributions and introduced the special cases of skewed t and log F distributions.

From (1), replacing $G(x)$ by the cdf of a Weibull distribution with parameters c and λ, the cdf of the beta Weibull distribution is obtained as follow:

Let

$$F(x) = I_{1-\exp\{-(\lambda x)^c\}}(a,b) \quad (3)$$

for $x > 0,\ a,\ b,\ c \ and\ \lambda > 0$.The corresponding probability density function (pdf) and the hazard rate function associated with (3) are:

$$f(x) = \frac{c\lambda^c}{B(a,b)} x^{c-1} exp\{-b(\lambda x)^c\}[1 - exp\{-(\lambda x)^c\}]^{a-1} \quad (4)$$

And

$$h(x) = \frac{c\lambda^c x^{c-1} exp\{-b(\lambda x)^c\}[1-exp\{-(\lambda x)^c\}]^{a-1}}{B_{exp\{-(\lambda x)^c\}}(b,a)}, \quad (5)$$

respectively.

Famoyee*et al.* (2005) studied some of the properties of an extension of the Weibull family which they called the beta-Weibull distribution.Lee *et al.* (2007) discussed some properties of the beta-Weibull model and providedformulas for the hazard function and discussed their properties under different values for the parameters. Cordeiro*et al.* (2008) gave an expansion for the cumulative distribution function, the moments and the moment generating function of the beta-Weibull distribution. They also discussed maximum likelihood estimation from complete samples and provided formulas for the elements of the fisher information matrix, as well as a demonstration of its usefulness on a real data set.Wahed*et al.* (2009) investigated the potential usefulness of the beta-Weibull distribution for modeling censored survival data from biomedical studies.

Special Cases:

The BW distribution contains several well-known distributions as special cases. For example,

(i) The Weibull distribution is clearly a special case when $a = b = 1$. Also if $a = 1$, the BW distribution becomes the Weibull distribution with parameters $\lambda b^{1/c}$ and *c*.

(ii) When *c*=-*k* the BW reduces to the beta- type 2 Extreme value distribution (beta-Fréchet distribution).

(iii) It simplifies to the beta- Rayleigh (BR) distribution when *c* = 2.

(iv) If *c* = 1, it reduces to the beta- exponential (BE) distribution.

(v) The exponentiatedWeibull (EW) distribution is also a special case when *b* =1.

(vi) If *c* = 2 and *b* = 1, it gives as special case the generalized Rayleigh (GR) distribution.

(vii) For $a = 2$, b$=1/\alpha$and $\lambda^c = \alpha\beta$, the BW distribution reduces to a weighted Weibull distribution (Shahbaz *et al.* (2011)).

(viii) When a (= *i*) and b are integers, the BW is the distribution of the i^{th} order statistic from a Weibull population with parameters *c* and λ.

(ix) When *b*=1 and *c*=2, the BW reduces to the two-parameter Burr type X distribution with density function

$$f(x) = 2a\lambda^2\left[1 - e^{-(\lambda x)^2}\right]^{a-1} e^{-(\lambda x)^2}.$$

In section (2), we consider maximum likelihood estimation (MLE) of two unknown parameters of beta–Wiebull distribution under type II censored samples. Section (3) gives the asymptotic variance covariance matrix. The results of simulation studies will be discussed in section (4). Tables and figures are displayed in the Appendix.

2. Maximum Likelihood Estimation

If a and b are known the problem of estimating c and λ becomes simply estimating the parameters of the standard Weibull distribution and it will not be discussed here. It shall be assumed that c and a are unknown and their ML estimates will be obtained.

Suppose that n items, whose life times follow beta-Weibull distribution (4) where the parameters b and λ are known, are put on test, the test is terminated when the rth item fails; the lifetimes of these first r failed items say $x_{(1)}, x_{(2)}, \dots \dots, x_{(r)}$ are observed. The likelihood function is given by:

$$L(c,a) = C \prod_{i=1}^{r} \frac{c\lambda^c}{B(a,b)} x_{(i)}{}^{c-1} exp\{-b(\lambda x_{(i)})^c\}[1 - exp\{-(\lambda x_{(i)})^c\}]^{a-1} \left[I_{e^{-(\lambda x_{(r)})^c}}(b,a)\right]^{n-r} \quad (6)$$

Taking logarithm of (4.1), the log-likelihood function is

$$lnL = lnC + rln\, c + rc \ln \lambda - rln\, B(a,b) + (c-1)\sum_{i=1}^{r} \ln(x_{(i)}) - b\sum_{i=1}^{r} (\lambda x_{(i)})^c +$$

$$(a-1)\sum_{i=1}^{r} ln[1 - exp\{-(\lambda x_{(i)})^c\}] + (n-r)\, ln\left[I_{e^{-(\lambda x_{(r)})^c}}(b,a)\right] \quad (7)$$

After taking partial derivatives of the log-likelihood in (7) with respect to c and a, and equating the derivatives to zero, the following equations are obtained:

$$\frac{\partial lnL}{\partial c} =$$
$$\frac{r}{c} + rln\, \lambda + \sum_{i=1}^{r} \ln(x_{(i)}) - b\sum_{i=1}^{r} (\lambda x_{(i)})^c\, ln(\lambda x_{(i)}) +$$
$$(a-1)\sum_{i=1}^{r} \frac{exp\{-(\lambda x_{(i)})^c\}(\lambda x_{(i)})^c ln(\lambda x_{(i)})}{[1 - exp\{-(\lambda x_{(i)})^c\}]} + (n-r)\frac{\frac{\partial}{\partial c} I_{exp\{-(\lambda x_{(r)})^c\}}(b,a)}{I_{exp\{-(\lambda x_{(r)})^c\}}(b,a)} = 0\,, \quad (8)$$

and

$$\frac{\partial lnL}{\partial a} = -r\psi(a) + r\psi(a+b) + \sum_{i=1}^{r} ln[1 - exp\{-(\lambda x_{(i)})^c\}] + (n-r)\frac{\frac{\partial}{\partial a} I_{exp\{-(\lambda x_{(r)})^c\}}(b,a)}{I_{exp\{-(\lambda x_{(r)})^c\}}(b,a)} \quad (9)$$

where $\psi(a) = \frac{\partial}{\partial a} ln\Gamma(a)$ which is called digamma function.

Now $\frac{\partial}{\partial c} I_{exp\{-(\lambda x_{(r)})^c\}}(b,a) = -\frac{1}{B(a,b)} (\lambda x_{(r)})^c ln(\lambda x_{(r)}) e^{-b(\lambda x_{(r)})^c} \left[1 - e^{-(\lambda x_{(r)})^c}\right]^{a-1}$,

and

$$\frac{\partial}{\partial a} I_{exp\left\{-(\lambda x_{(r)})^c\right\}}(b,a) = \frac{1}{B(a,b)}\left\{\int_0^{e^{-(\lambda x_{(r)})^c}} t^{b-1}(1-t)^{a-1}\ln(1-t)\,dt - \{\psi(a)-\psi(a+b)\}.B(a,b)I_{exp\left\{-(\lambda x_{(r)})^c\right\}}(b,a)\right\} \quad (10)$$

For computational purposes we rewrite the R.H.S of (10) as

$$\frac{1}{B(a,b)}\left\{\sum_{k=0}^{\infty}\frac{-1}{k+1}\int_0^{e^{-(\lambda x_{(r)})^c}} t^{b+k}(1-t)^{a-1}dt - \{\psi(a)-\psi(a+b)\}.B(a,b)I_{exp\left\{-(\lambda x_{(r)})^c\right\}}(b,a)\right\}$$

$$= \frac{1}{B(a,b)}\left\{\sum_{k=0}^{\infty}\frac{-1}{k+1} I_{exp\left\{-(\lambda x_{(r)})^c\right\}}(b+k+1,a) - \{\psi(a)-\psi(a+b)\}.B(a,b)I_{exp\left\{-(\lambda x_{(r)})^c\right\}}(b,a)\right\},$$

where$\ln(1-u) = \sum_{k=0}^{\infty} -\frac{u^{k+1}}{k+1}, |u| \le 1, u \ne 1$ and $\frac{\partial}{\partial a}B(a,b) = B(a,b)\{\psi(a)-\psi(a+b)\}$.

The L.H.S. of (10) can be obtained by using the relation between the incomplete beta function and the Guass hypergeometric function as follows:

$$\frac{\partial}{\partial a} I_{exp\left\{-(\lambda x_{(r)})^c\right\}}(b,a) = -\frac{\partial}{\partial a} I_{1-exp\left\{-(\lambda x_{(r)})^c\right\}}(a,b)$$
$$= \left[-\ln\left(1-exp\left\{-\left(\lambda x_{(r)}\right)^c\right\}\right) + \psi(a) - \psi(a+b)\right] I_{1-exp\left\{-(\lambda x_{(r)})^c\right\}}(a,b) + \frac{\left(1-exp\left\{-\left(\lambda x_{(r)}\right)^c\right\}\right)^a}{a^2\,B(a,b)}\ {}_3F_2\left(a,a,1-b;a+1,a+1;1-exp\left\{-\left(\lambda x_{(r)}\right)^c\right\}\right)$$

$$= \left[-\ln\left(1-exp\left\{-\left(\lambda x_{(r)}\right)^c\right\}\right) + \psi(a) - \psi(a+b)\right] I_{1-exp\left\{-(\lambda x_{(r)})^c\right\}}(a,b) + \frac{\Gamma(a)\Gamma(a+b)}{\Gamma(b)}\left(1-exp\left\{-\left(\lambda x_{(r)}\right)^c\right\}\right)^a\ {}_3\tilde{F}_2\left(a,a,1-b;a+1,a+1;1 - exp\left\{-\left(\lambda x_{(r)}\right)^c\right\}\right).$$

where ${}_p\tilde{F}_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right) = \frac{{}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right)}{\Gamma(b_1)\ldots\Gamma(b_q)}$ is regularized hypergeometric function (see reference [16]).

The equations (8) and (9) do not have analytical solution and they must be solved numerical. For this purpose, the popular numerical software package MathCad was used.

Using invariance property of maximum likelihood estimation (see Mood et al 1974), we can obtain the MLE of the hazard rate function of B-W distribution. From (5), replacing the parameters c and a by their MLEs $\hat{c}$ and $\hat{a}$ i.e,

$$\hat{h}(x) = \frac{\hat{c}\lambda^{\hat{c}}x^{\hat{c}-1}exp\{-b(\lambda x)^{\hat{c}}\}[1-exp\{-(\lambda x)^{\hat{c}}\}]^{\hat{a}-1}}{B_{exp\{-(\lambda x)^{\hat{c}}\}}(b,\hat{a})}.$$

3. The Asymptotic Variance Covariance Matrix

The asymptotic variance-covariance matrix for the estimators$\hat{c}$ and $\hat{a}$ can be obtained by inverting the information matrix with the elements that are negative of the expected values of the second order derivative of logarithms of the likelihood functions.

The second partial derivatives of the log- likelihood function are obtained in the following:

$$\frac{\partial^2 lnL}{\partial c^2} = -\frac{r}{c^2} - b\sum_{i=1}^{r}(\lambda x_{(i)})^c ln(\lambda x_{(i)})^2 + (a-1)\sum_{i=1}^{r}\left\{\frac{\left((\lambda x_{(i)})^c ln(\lambda x_{(i)})^2 e^{-(\lambda x_i)^c}\right)\left(1-(\lambda x_{(i)})^c\right)}{\left[1-e^{-(\lambda x_{(i)})^c}\right]} - \frac{\left[(\lambda x_{(i)})^c ln(\lambda x_{(i)}) e^{-(\lambda x_{(i)})^c}\right]^2}{\left[1-e^{-(\lambda x_{(i)})^c}\right]^2}\right\} + (n-r)\frac{\left\{I_{exp\{-(\lambda x_{(r)})^c\}}(b,a).\frac{\partial^2}{\partial c^2}I_{exp\{-(\lambda x_{(r)})^c\}}(b,a) - \left(\frac{\partial}{\partial c}I_{exp\{-(\lambda x_{(r)})^c\}}(b,a)\right)^2\right\}}{\left[I_{exp\{-(\lambda x_{(r)})^c\}}(b,a)\right]^2},$$

where

$$\frac{\partial^2}{\partial c^2}I_{exp\{-(\lambda x_{(r)})^c\}}(b,a) = \frac{-1}{B(a,b)}\left[\ln(\lambda x_{(r)})\right]^2 (\lambda x_{(r)})^c e^{-b(\lambda x_{(r)})^c}\left[1-e^{-(\lambda x_{(r)})^c}\right]^{a-2}\left\{(a-1)(\lambda x_{(r)})^c e^{-(\lambda x_{(r)})^c} + \left(1-e^{-(\lambda x_{(r)})^c}\right)(1-b(\lambda x_{(r)})^c)\right\}.$$

$$\frac{\partial^2 lnL}{\partial a\partial c} = \sum_{i=1}^{r}\frac{exp\{-(\lambda x_{(r)})^c\}(\lambda x_{(i)})^c ln(\lambda x_{(i)})}{\left[1-exp\{-(\lambda x_{(i)})^c\}\right]} + (n-r)\frac{\left[I_{exp\{-(\lambda x_{(r)})^c\}}(b,a)\right].\left(\frac{\partial^2}{\partial a\partial c}I_{exp\{-(\lambda x_{(r)})^c\}}(b,a)\right) - \left(\frac{\partial}{\partial c}I_{exp\{-(\lambda x_{(r)})^c\}}(b,a)\right).\left(\frac{\partial}{\partial a}I_{exp\{-(\lambda x_{(r)})^c\}}(b,a)\right)}{\left[I_{exp\{-(\lambda x_{(r)})^c\}}(b,a)\right]^2},$$

$$\frac{\partial^2}{\partial a\partial c}I_{exp\{-(\lambda x_{(r)})^c\}}(b,a) = \frac{1}{B(a,b)}[\psi(a)-\psi(a+b) - \ln\left(1-e^{-(\lambda x_{(r)})^c}\right)](\lambda x_{(r)})^c ln(\lambda x_{(r)})e^{-b(\lambda x_{(r)})^c}\left[1-e^{-(\lambda x_{(r)})^c}\right]^{a-1}.$$

$$\frac{\partial^2 lnL}{\partial a^2} = -r\psi'(a) + r\psi'(a+b) + (n-r)\left[\frac{\left(\frac{\partial^2}{\partial a^2} I_{exp\left\{-\left(\lambda x_{(r)}\right)^c\right\}}(b,a)\right)}{\left[I_{exp\left\{-\left(\lambda x_{(r)}\right)^c\right\}}(b,a)\right]} - \frac{\left(\frac{\partial}{\partial a} I_{exp\left\{-\left(\lambda x_{(r)}\right)^c\right\}}(b,a)\right)^2}{\left[I_{exp\left\{-\left(\lambda x_{(r)}\right)^c\right\}}(b,a)\right]^2}\right],$$

and

$$\frac{\partial^2}{\partial a^2} I_{exp\left\{-\left(\lambda x_{(r)}\right)^c\right\}}(b,a)$$
$$= \left\{\psi'(a) - \psi'(a+b) - \left[\ln\left(1 - e^{-\left(\lambda x_{(r)}\right)^c}\right) - \psi(a) + \psi(a+b)\right]^2\right\} I_{\left(1-e^{-\left(\lambda x_{(r)}\right)^c}\right)}(a,b) +$$

$$\frac{2\left[1 - e^{-\left(\lambda x_{(r)}\right)^c}\right]^a \left[ln\left(1 - e^{-\left(\lambda x_{(r)}\right)^c}\right) - \psi(a) + \psi(a+b)\right]}{a^2 B(a,b)} \; {}_3F_2\left(a, a, 1-b; a+1, a+1; 1 - e^{-\left(\lambda x_{(r)}\right)^c}\right)$$
$$- \frac{2\left[1 - e^{-\left(\lambda x_{(r)}\right)^c}\right]^a}{a^3 B(a,b)} \; {}_4F_3\left(a, a, a, 1-b; a+1, a+1, a+1; 1 - e^{-\left(\lambda x_{(r)}\right)^c}\right).$$

Cohen (1965) suggested the approximate variance covariance matrix may be obtained by replacing expected values by their MLEs.

4. Simulation Studies

A simulation is conducted to study properties of the MLE of the beta-Weibull distribution. The parameter sets for which the BW hazard rate function is bathtub, increasing and decreasing are simulated. For each simulated sample the absolute relative biases (ARbias), mean square errors (mse), relative mean square errors (Rmse) and relative root mean square errors (Rrmse) are computed.

We generate 10,000 random samples from B-W distribution with different sample sizes (20, 30, 50, 100 and 150) and with different cases of the uncensored percentage$\left(\frac{r}{n}\right)$% (70%, 80%, 90% and 100%). The samples are generated from BW distribution using the following transformation:

$$X_i = \frac{1}{\lambda}(-\ln(1 - B_i)^{1/c}, i = 1,2, \dots, n$$

where$B_1, \dots . B_n$ are random numbers from beta distribution in interval (0, 1).

Simulation results are summarized in Tables 1, 2, 3 and 4. Tables 1, 2 and 3 give theARbias, mse, Rmse and Rrmse of the estimators. The asymptotic variances and covariance matrix of the estimators are displayed in Table 4. The following are some observations from the simulation study

1. For the parameter set of a bathtub hazard function (Tables 1, 4) :
   - The ARbias of the mle$\hat{c}$ decreases when the sample size n increases and when the percentage of uncensored observation

$\left(\frac{r}{n}\right)\%$ increases. Also, the ARbias ofthe mle$\hat{a}$ decreases when the sample size n increases and when the percentage of uncensored observation $\left(\frac{r}{n}\right)\%$ increases except for (100) %.

- The mse, Rmse and Rrmsof the mles$\hat{c}$ and $\hat{a}$ decreasewhen the sample size n increases and when the percentage of uncensored observation $\left(\frac{r}{n}\right)\%$ increases.
- The asymptotic variance of $\hat{c}$ decreases when the sample size increases and the percentage of uncensored observation $\left(\frac{r}{n}\right)\%$ increases. Also, The asymptotic variance of $\hat{a}$ decreases when the sample size increases and when the percentage of uncensored observation $\left(\frac{r}{n}\right)\%$ increases except for small sample sizes (n=20 and n=30).

2. For the parameter set of a increasing hazard function (Tables 2, 4) :
   - The ARbias of the mle$\hat{c}$ decreases when the sample size n increases and when the percentage of uncensored observation $\left(\frac{r}{n}\right)\%$ increases. Also, the ARbias ofthe mle$\hat{a}$ decreases when the sample size n increases but in most cases it increases when the percentage of uncensored observation $\left(\frac{r}{n}\right)\%$ increases.
   - The mse, Rmse and Rrmseof the mles$\hat{c}$ and $\hat{a}$ decreasewhen the sample size n increases and they also decrease when the percentage of uncensored observation $\left(\frac{r}{n}\right)\%$ increases.
   - The asymptotic variance of $\hat{c}$ decreases when the sample size increases and the percentage of uncensored observation $\left(\frac{r}{n}\right)\%$ increases. Also, The asymptotic variance of $\hat{a}$ decreases when the sample size increases but when the percentage of uncensored observation $\left(\frac{r}{n}\right)\%$ decreases we have three cases: it increases for sample sizes (n=20, n=30), constant for n=50 and decreases slowly for n=100 and n=150.

3. For the parameter set of a decreasing hazard function (Tables 3, 4):
   - Similar results as increasing hazard function are observed.

4. For the parameter sets for which the BW hazard rate function is bathtub, increasing and decreasing, Tables (5, 6 and 7) and figures (1, 2 and 3) illustrate the MLE of hazard function at sample size n=50 and uncensored percentage 80%.

In general, the results for the parameter set of a bathtub hazard function are better than the results for the other two cases.

Table 1: ARbias, mse, Rmse and Rrmse of parameters c = 1.5 and a = 0.5 and with known parameters λ=0.5 and b=0.5 (bathtub hazard function)

n	$\left(\frac{r}{n}\right)$%	Parameter	ARbias	mse	Rmse	Rrmse
20	70%	c a	0.5290 0.0490	3.7151 0.0431	1.6510 0.1710	1.2851 0.4147
30		c a	0.3026 0.0399	1.4482 0.0262	0.6437 0.1046	0.8023 0.3237
50		c a	0.1411 0.0219	0.3698 0.0141	0.1644 0.0562	0.4054 0.2375
100		c a	0.0620 0.0110	0.0960 0.0066	0.0430 0.0260	0.2066 0.1625
150		c a	0.0559 0.0088	0.0559 0.0044	0.0248 0.0176	0.1576 0.1327
20	80%	c a	0.3131 0.0141	1.5340 0.0350	0.6820 0.1410	0.8257 0.3472
30		c a	0.1732 0.0134	0.5447 0.0205	0.2421 0.0820	0.4920 0.2864
50		c a	0.0865 0.0065	0.1566 0.0110	0.0696 0.0441	0.2638 0.2098
100		c a	0.0400 0.0034	0.0480 0.0052	0.0220 0.0210	0.1461 0.1442
150		c a	0.0259 0.0034	0.0301 0.0034	0.0134 0.0137	0.1157 0.1166
20	90%	c a	0.1761 0.0131	0.5099 0.0299	0.2266 0.1195	0.4761 0.3458
30		c a	0.1004 0.0066	0.1868 0.0172	0.0830 0.0687	0.2881 0.2623
50		c a	0.0310 0.0027	0.0700 0.0091	0.0310 0.0360	0.1764 0.1908
100		c a	0.0252 0.0028	0.0267 0.0044	0.0119 0.0176	0.1089 0.1327
150		c a	0.0164 0.0005	0.0170 0.0029	0.0075 0.0115	0.0869 0.1077
20	100%	c a	0.1029 0.0328	0.1787 0.0269	0.0794 0.1074	0.2818 0.3280
30		c a	0.0624 0.0176	0.0781 0.0153	0.0347 0.0611	0.1863 0.2474
50		c a	0.0360 0.0096	0.0360 0.0081	0.0160 0.0320	0.1265 0.1800
100		c a	0.0168 0.0061	0.0147 0.0040	0.0065 0.0152	0.0808 0.1265
150		c a	0.0106 0.0029	0.0094 0.0026	0.0042 0.0102	0.0646 0.1020

Table 2: ARbias, mse, Rmse and Rrmse of parameters c = 1.5 and a = 1.5 and with known parameters λ=0.5 and b=0.5 (increasing hazard function)

n	$\left(\frac{r}{n}\right)$%	Parameter	ARbias	mse	Rmse	Rrmse
20	70%	c	0.1145	0.1941	0.0862	0.2937
		a	0.0871	0.3432	0.1525	0.3906
30		c	0.0698	0.0910	0.0404	0.2011
		a	0.0557	0.1849	0.0822	0.2867
50		c	0.0389	0.0429	0.0191	0.1381
		a	0.0254	0.0863	0.0384	0.1959
100		c	0.0176	0.0185	0.0082	0.0907
		a	0.0127	0.0389	0.0173	0.1315
150		c	0.0121	0.0117	0.0052	0.0721
		a	0.0102	0.0261	0.0116	0.1077
20	80%	c	0.0841	0.1144	0.0508	0.2255
		a	0.0910	0.3326	0.1478	0.3845
30		c	0.0527	0.0593	0.0263	0.1623
		a	0.0583	0.1806	0.1806	0.2833
50		c	0.0289	0.0294	0.0131	0.1143
		a	0.0274	0.0848	0.0377	0.1941
100		c	0.0136	0.0134	0.0060	0.0772
		a	0.0136	0.0384	0.0171	0.1306
150		c	0.0093	0.0085	0.0038	0.0615
		a	0.0107	0.0259	0.0115	0.1073
20	90%	c	0.0646	0.0738	0.0328	0.1811
		a	0.0921	0.3191	0.1418	0.3766
30		c	0.0411	0.4110	0.0183	0.4274
		a	0.0595	0.1779	0.0791	0.2812
50		c	0.0229	0.0213	0.0095	0.0973
		a	0.0281	0.0838	0.0372	0.1930
100		c	0.0132	0.0101	0.0045	0.0670
		a	0.0138	0.0384	0.0171	0.1306
150		c	0.0071	0.0062	0.0028	0.0525
		a	0.0111	0.0257	0.0114	0.1069
20	100%	c	0.0506	0.0510	0.0226	0.1506
		a	0.0917	0.3097	0.1376	0.3710
30		c	0.0326	0.0298	0.0132	0.1151
		a	0.0599	0.1756	0.0781	0.2794
50		c	0.0181	0.0154	0.0069	0.0827
		a	0.0285	0.0830	0.0369	0.1921
100		c	0.0108	0.0074	0.0033	0.0574
		a	0.0140	0.0382	0.0170	0.1303
150		c	0.0056	0.0046	0.0021	0.0452
		a	0.0112	0.0256	0.0114	0.1067

Table 3: ARbias, mse, Rmse and Rrmse of parameters c = 0.5 and a = 1.5 and with known parameters λ=0.5 and b=0.5 (decreasing hazard function)

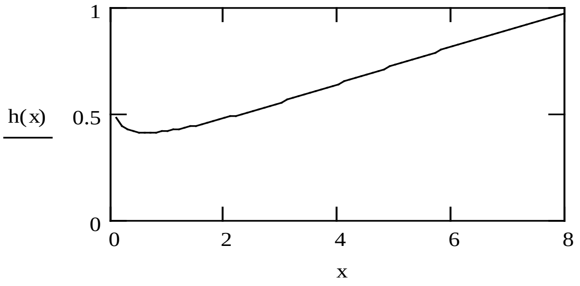
n	$\left(\frac{r}{n}\right)$ %	Parameter	ARbias	mse	Rmse	Rrmse
20	70%	c a	0.1127 0.0901	0.0231 0.3473	0.0924 0.1544	0.3040 0.3929
30		c a	0.0691 0.0571	0.0099 0.1746	0.0394 0.0776	0.1990 0.2786
50		c a	0.0407 0.0306	0.0048 0.0907	0.0194 0.0403	0.1386 0.2008
100		c a	0.0185 0.0110	0.0021 0.0384	0.0083 0.0171	0.0917 0.1306
150		c a	0.0121 0.0093	0.0013 0.0251	0.0051 0.0112	0.0721 0.1056
20	80%	c a	0.0834 0.0931	0.0127 0.3247	0.0506 0.1443	0.2254 0.3799
30		c a	0.0511 0.0604	0.0065 0.1720	0.0261 0.0764	0.1613 0.2765
50		c a	0.0305 0.0326	0.0034 0.0895	0.0134 0.0398	0.1166 0.1994
100		c a	0.0143 0.0119	0.0015 0.0379	0.0060 0.0169	0.0775 0.1298
150		c a	0.0103 0.0096	0.0010 0.0251	0.0038 0.0112	0.0616 0.1056
20	90%	c a	0.0642 0.0942	0.0083 0.3137	0.0330 0.1394	0.1822 0.3734
30		c a	0.0401 0.0616	0.0046 0.1704	0.0182 0.0757	0.1357 0.2752
50		c a	0.0237 0.0335	0.0024 0.0886	0.0096 0.0394	0.0980 0.1984
100		c a	0.0117 0.0122	0.0011 0.0376	0.0045 0.0167	0.0663 0.1293
150		c a	0.0082 0.0099	0.0007 0.0250	0.0028 0.0111	0.0529 0.1054
20	100%	c a	0.0501 0.0947	0.0057 0.3073	0.0228 0.1366	0.1510 0.3696
30		c a	0.0319 0.0619	0.0032 0.1688	0.0129 0.0750	0.1131 0.2739
50		c a	0.0192 0.0337	0.0018 0.0881	0.0072 0.0391	0.0849 0.1979
100		c a	0.0092 0.0125	0.0082 0.0374	0.0033 0.0166	0.1811 0.1289
150		c a	0.0070 0.0099	0.0005 0,0249	0.0021 0.0111	0.0447 0.1052

Table 4: Asymptotic variance covariance matrix of estimators under type II censored samples

n	$\left(\frac{r}{n}\right)$ %	(1.5, 0.5, 0.5, 0.5)		(1.5, 1.5, 0.5, 0.5)		(0.5, 1.5, 0.5, 0.5)	
		$\hat{c}$	$\hat{a}$	$\hat{c}$	$\hat{a}$	$\hat{c}$	$\hat{a}$
20	70%	0.1854	-0.0134	0.0793	-0.0160	0.0087	-0.0052
		-0.0134	0.0056	-0.0160	0.1459	-0.0052	0.1449
	80%	0.1410	-0.0150	0.0587	-0.0116	0.0065	-0.0038
		-0.0150	0.0086	-0.0116	0.1507	-0.0038	0.1530
	90%	0.1005	-0.0147	0.0445	-0.0088	0.0049	-0.0029
		-0.0147	0.0119	-0.0088	0.1531	-0.0029	0.1549
	100%	0.0647	-0.0113	0.0336	-0.0066	0.0037	-0.0022
		-0.0113	0.0136	-0.0066	0.1543	-0.0022	0.1564
30	70%	0.1356	-0.0156	0.0526	-0.0112	0.0058	-0.0037
		-0.0156	0.0067	-0.0112	0.1061	-0.0037	0.1072
	80%	0.1016	-0.0154	0.0390	-0.0080	0.0043	-0.0026
		-0.0154	0.0090	-0.0080	0.1070	-0.0026	0.1081
	90%	0.0687	-0.0123	0.0296	-0.0060	0.0033	-0.0020
		-0.0123	0.0101	-0.0060	0.1074	-0.0020	0.1085
	100%	0.0429	-0.0084	0.0222	-0.0045	0.0025	-0.0015
		-0.0084	0.0103	-0.0045	0.1076	-0.0015	0.1086
50	70%	0.0950	-0.0161	0.0316	-0.0070	0.0035	-0.0023
		-0.0161	0.0071	-0.0070	0.0661	-0.0023	0.0667
	80%	0.0652	-0.0123	0.0234	-0.0049	0.0026	-0.0016
		-0.0123	0.0073	-0.0049	0.0661	-0.0016	0.0667
	90%	0.0420	-0.0084	0.0177	-0.0036	0.0020	-0.0012
		-0.0084	0.0071	-0.0036	0.0661	-0.0012	0.0667
	100%	0.0260	-0.0053	0.0133	-0.0027	0.0015	-0.0009
		-0.0053	0.0067	-0.0027	0.0661	-0.0009	0.0666
100	70%	0.0530	-0.0110	0.0157	-0.0052	0.0018	-0.0012
		-0.0110	0.0048	-0.0052	0.0343	-0.0012	0.0342
	80%	0.0340	-0.0071	0.0117	-0.0025	0.0013	-0.0008
		-0.0071	0.0043	-0.0025	0.0341	-0.0008	0.0340
	90%	0.0209	-0.0045	0.0089	-0.0018	0.0010	-0.0006
		-0.0045	0.0039	-0.0018	0.0340	-0.0006	0.0339
	100%	0.0126	-0.0028	0.0066	-0.0014	0.0007	-0.0005
		-0.0028	0.0035	-0.0014	0.0339	-0.0005	0.0338
150	70%	0.0359	-0.0077	0.0105	-0.0024	0.0012	-0.0008
		-0.0077	0.0035	-0.0024	0.0231	-0.0008	0.0231
	80%	0.0224	-0.0049	0.0078	-0.0016	0.0009	-0.0006
		-0.0049	0.0030	-0.0016	0.0230	-0.0006	0.0230
	90%	0.0140	-0.0031	0.0059	-0.0012	0.0007	-0.0004
		-0.0031	0.0026	-0.0012	0.0229	-0.0004	0.0229
	100%	0.0084	-0.0019	0.0044	-0.0009	0.0005	-0.0003
		-0.0019	0.0238	-0.0009	0.0229	-0.0003	0.0228

Table 5: Maximum Likelihood Estimation of Hazard Function when c and a are unknown at n=50 and uncensored percentage 80% (bathtub hazard function)

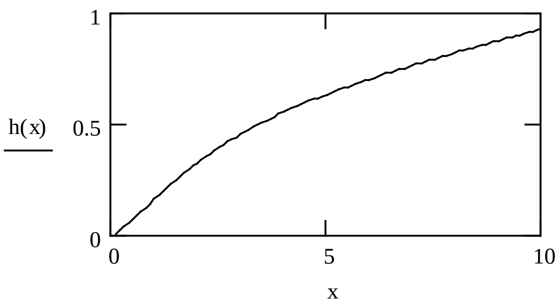
x	$h(x)$	x	$h(x)$	x	$h(x)$	x	$h(x)$
0.1	0.4833	2.6	0.5221	5.1	0.7371	7.6	0.9444
0.2	0.4416	2.7	0.5301	5.2	0.7458	7.7	0.9522
0.3	0.4245	2.8	0.5381	5.3	0.7545	7.8	0.9600
0.4	0.4163	2.9	0.5462	5.4	0.7632	7.9	0.9677
0.5	0.4125	3.0	0.5545	5.5	0.7718	8.0	0.9754
0.6	0.4114	3.1	0.5628	5.6	0.7804	8.1	0.9830
0.7	0.4120	3.2	0.5712	5.7	0.7890	8.2	0.9907
0.8	0.4139	3.3	0.5796	5.8	0.7975	8.3	0.9982
0.9	0.4167	3.4	0.5882	5.9	0.8060	8.4	1.0058
1.0	0.4202	3.5	0.5967	6.0	0.8144	8.5	1.0133
1.1	0.4244	3.6	0.6054	6.1	0.8229	8.6	1.0208
1.2	0.4290	3.7	0.6171	6.2	0.8312	8.7	1.0283
1.3	0.4340	3.8	0.6228	6.3	0.8396	8.8	1.0357
1.4	0.4394	3.9	0.6315	6.4	0.8479	8.9	1.0431
1.5	0.4451	4.0	0.6403	6.5	0.8561	9.0	1.0505
1.6	0.4511	4.1	0.6491	6.6	0.8643	9.1	1.0578
1.7	0.4574	4.2	0.6579	6.7	0.8725	9.2	1.0651
1.8	0.4638	4.3	0.6667	6.8	0.8807	9.3	1.0724
1.9	0.4705	4.4	0.6755	6.9	0.8888	9.4	1.0796
2.0	0.4774	4.5	0.6844	7.0	0.8968	9.5	1.0868
2.1	0.4845	4.6	0.6932	7.1	0.9049	9.6	1.0940
2.2	0.4917	4.7	0.7020	7.2	0.9128	9.7	1.1012
2.3	0.4991	4.8	0.7108	7.3	0.9208	9.8	1.1083
2.4	0.5067	4.9	0.7196	7.4	0.9287	9.9	1.1154
2.5	0.5143	5.0	0.7284	7.5	0.9366	10.0	1.1225



Figuer1: Hazard function (bathtub)

Table 6:Maximum Likelihood Estimation of Hazard Function when c and a are unknown at n=50 and uncensored percentage 80% (increasing hazard function)

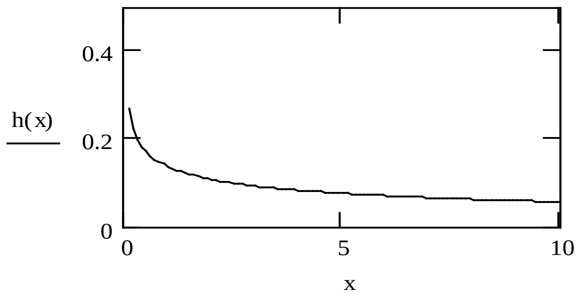
x	$h(x)$	x	$h(x)$	x	$h(x)$	x	$h(x)$
0.1	0.0080	2.6	0.4050	5.1	0.6383	7.6	0.7969
0.2	0.0205	2.7	0.4172	5.2	0.6455	7.7	0.8026
0.3	0.0352	2.8	0.4292	5.3	0.6526	7.8	0.8083
0.4	0.0514	2.9	0.4408	5.4	0.6596	7.9	0.8139
0.5	0.0684	3.0	0.4522	5.5	0.6665	8.0	0.8195
0.6	0.0861	3.1	0.4632	5.6	0.6733	8.1	0.8251
0.7	0.1041	3.2	0.4632	5.7	0.6801	8.2	0.8306
0.8	0.1223	3.3	0.4739	5.8	0.6867	8.3	0.8361
0.9	0.1406	3.4	0.4844	5.9	0.6933	8.4	0.8415
1.0	0.1587	3.5	0.4946	6.0	0.6999	8.5	0.8470
1.1	0.1766	3.6	0.5046	6.1	0.7063	8.6	0.8524
1.2	0.1943	3.7	0.5143	6.2	0.7127	8.7	0.8578
1.3	0.2118	3.8	0.5238	6.3	0.7191	8.8	0.8631
1.4	0.2289	3.9	0.5422	6.4	0.7253	8.9	0.8684
1.5	0.2456	4.0	0.5511	6.5	0.7316	9.0	0.8737
1.6	0.2620	4.1	0.5598	6.6	0.7377	9.1	0.8790
1.7	0.2781	4.2	0.5683	6.7	0.7438	9.2	0.8842
1.8	0.2937	4.3	0.5766	6.8	0.7499	9.3	0.8894
1.9	0.3089	4.4	0.5848	6.9	0.7559	9.4	0.8946
2.0	0.3238	4.5	0.5929	7.0	0.7619	9.5	0.8998
2.1	0.3383	4.6	0.6008	7.1	0.7679	9.6	0.9049
2.2	0.3523	4.7	0.6085	7.2	0.7737	9.7	0.9100
2.3	0.3660	4.8	0.6162	7.3	0.7796	9.8	0.9151
2.4	0.3794	4.9	0.6237	7.4	0.7854	9.9	0.9202
2.5	0.3923	5.0	0.6311	7.5	0.7912	10.0	0.9252



Figuer2: Hazard function (increasing)

Table 7: Maximum Likelihood Estimation of Hazard Function when c and a are unknown at n=50 and uncensored percentage 80% (decreasing hazard function)

x	$h(x)$	x	$h(x)$	x	$h(x)$	x	$h(x)$
0.1	0.2689	2.6	0.0979	5.1	0.0753	7.6	0.0638
0.2	0.2234	2.7	0.0965	5.2	0.0747	7.7	0.0635
0.3	0.1992	2.8	0.0952	5.3	0.0741	7.8	0.0631
0.4	0.1830	2.9	0.0939	5.4	0.0736	7.9	0.0628
0.5	0.1711	3.0	0.0927	5.5	0.0730	8.0	0.0625
0.6	0.1617	3.1	0.0916	5.6	0.0725	8.1	0.0621
0.7	0.1540	3.2	0.0905	5.7	0.0720	8.2	0.0618
0.8	0.1476	3.3	0.0894	5.8	0.0714	8.3	0.0615
0.9	0.1420	3.4	0.0884	5.9	0.0709	8.4	0.0612
1.0	0.1371	3.5	0.0874	6.0	0.0705	8.5	0.0609
1.1	0.1328	3.6	0.0865	6.1	0.0700	8.6	0.0606
1.2	0.1290	3.7	0.0855	6.2	0.0695	8.7	0.0603
1.3	0.1255	3.8	0.0846	6.3	0.0690	8.8	0.0600
1.4	0.1223	3.9	0.0838	6.4	0.0686	8.9	0.0597
1.5	0.1194	4.0	0.0830	6.5	0.0682	9.0	0.0594
1.6	0.1168	4.1	0.0822	6.6	0.0677	9.1	0.0591
1.7	0.1143	4.2	0.0814	6.7	0.0673	9.2	0.0588
1.8	0.1120	4.3	0.0806	6.8	0.0669	9.3	0.0585
1.9	0.1098	4.4	0.0799	6.9	0.0665	9.4	0.0583
2.0	0.1078	4.5	0.0792	7.0	0.0661	9.5	0.0580
2.1	0.1059	4.6	0.0785	7.1	0.0657	9.6	0.0578
2.2	0.1041	4.7	0.0778	7.2	0.0653	9.7	0.0575
2.3	0.1024	4.8	0.0772	7.3	0.0649	9.8	0.0572
2.4	0.1008	4.9	0.0765	7.4	0.0645	9.9	0.0570
2.5	0.0993	5.0	0.0759	7.5	0.0642	10.0	0.0567



Figuer3: Hazard function (decreasing)

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