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Bayesian Inference from the Kumaraswamy-Weibull Distribution with Applications to Real Data

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Abstract

In this article, we introduce a Bayesian analysis for the Kumaraswamy-Weibull (Kum-W) distribution. Approximate Bayes estimates are obtained under the assumptions of non-informative priors using the Gibbs sampling procedure. This procedure allows for generating samples from posterior distributions. Also, using Bayesian framework, the predictive density for a single future response, a bivariate future response, and several future responses are derived. The predictive means, standard deviations, highest predictive density (HPD) intervals, and the shape characteristics for a single future response are determined. Finally, applications to real data sets are utilized to illustrate the potentiality of the Bayesian analysis and the predictive results.

Keywords: Kumaraswamy-Weibull (Kum-W) distribution; Bayesian approach; predictive inference

1. Introduction

A suitable parametric model is often of interest in the analysis of survival data, as it provides insight into characteristics of failure times and hazard functions that may not be available with non-parametric methods. The Weibull distribution is one of the most commonly used families for modeling such data. However, only monotonically increasing and decreasing hazard functions can be generated from the classic two-parameter Weibull distribution. As such, this two-parameter model is inadequate when the true hazard shape is of unimodal or has bathtub nature. Many extensions of the Weibull distribution have been proposed to enhance its capability to fit diverse life time data. Here, we will discuss one of these extensions which is called Kumaraswamy-Weibull distribution. A review of this distribution will be discussed in section 2.

The Bayesian predictive approach is growing in popularity. New practical applications in the fields of health sciences, social sciences, and environmental sciences, among others are appearing frequently. This approach, which is used for the design and analysis of survival research studies in the health sciences, is now widely used to reduce healthcare cost and to successfully allocate healthcare resources. Predictive inference has been discussed by Khan *et al*. (2003), Khan (2012), Khan *et al.* (2013), among others. Additional applications of the Bayesian approach to predictive inference for breast cancer survival data have been discussed by Khan *et al.* (2014a) and Khan *et al.* (2014b).

In this article, a review of the Kumaraswamy-Weibull Distribution will be discussed in section 2, approximate Bayes estimates are obtained using the Gibbs sampling procedure in section 3. In section 4 the predictive density for a single future response, a bivariate future response, and several future responses are derived. Finally, two real data sets are considered in section 5 to illustrate the potentiality of the Bayesian analysis and the predictive results.

2. The Kumaraswamy-Weibull Distribution

Starting with the Kumaraswamy's distribution (Kum distribution) on the interval [0, 1] which has the cdf: $F(x) = 1 - \{1 - x^{a}\}^{b},\ a > 0,\ b > 0$ as an alternative to the beta distribution in generated-beta distributions, Cordeiro and de Castro (2011) introduced a class of Kum generalized (Kum-G) distributions. From an arbitrary cdf $G(x)$, the cdf $F(x)$ of the Kum-G distribution is defined by

$$F(x) = 1 - \{1 - G(x)^{a}\}^{b}\ , \tag{2.1}$$

where $a > 0$ and $b > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weights and the corresponding pdf of this family of distributions has a very simple form

$$f(x) = abg(x)G(x)^{a-1}\{1 - G(x)^{a}\}^{b-1}. \tag{2.2}$$

Note that: the basic difference (except for a scale multiplier) between the pdf of Kum-G distributions and the pdf of the beta-G distributions is the power of G(x) inside the braces and for $b = 1$ both densities are identical. By taking the cdf $G(x) = 1 - e^{(\lambda x)^{c}}$ of the Weibull distribution with shape parameter c > 0 and scale parameter λ > 0, the cdf and pdf of this distribution are obtained from equations (2.1) and (2.2) as

$$F(x) = 1 - \left\{1 - \left[1 - e^{-(\lambda x)^{c}}\right]^{a}\right\}^{b}, \tag{2.3}$$

and

$$f(x) = abc\lambda^c x^{c-1} e^{-(\lambda x)^c} \left[1 - e^{-(\lambda x)^c}\right]^{a-1} \left\{1 - \left[1 - e^{-(\lambda x)^c}\right]^a\right\}^{b-1} \qquad (2.4)$$

respectively.
The hazard rate function for Kum-W distribution is

$$h(x) = \frac{abc\lambda^c x^{c-1} e^{-(\lambda x)^c}\left[1-e^{-(\lambda x)^c}\right]^{a-1}}{1-\left[1-e^{-(\lambda x)^c}\right]^a}.$$

The Weibull, exponentiated Weibull (EW) and exponentiated exponential (EE) distributions are the most important sub-models of (2.4) for a = b = 1, b = 1, and c = b = 1, respectively. For more details about other sub-models of the Kum-W distribution see Cordeiro *et al.* (2010). Also, it can be noted that the Kum-W distribution has three shape parameters, *a*, *b* and *c*. These three shape parameters allow for a high degree of flexibility of the Kum-W distribution and also, allow for all five major hazard shapes: constant, increasing, decreasing, bathtub and unimodal failure rates.

Given a random sample $x_1, x_2, \ldots, x_n$, the log-likelihood function $l = l(a,b,c,\lambda)$ for the model parameters of the Kum-W distribution can be written from (2.4) as

$$l = n ln(abc\lambda^a) + (c-1)\sum_{i=1}^{n} \ln(x_i) - \sum_{i=1}^{n} (\lambda x_i)^c + (a-1)\sum_{i=1}^{n} \ln(w_i)] + (b-1)\sum_{i=1}^{n} \ln[1 - \mathrm{w}_i{}^a], \qquad (2.5)$$

where $w_i = [1 - \exp(-(\lambda x_i)^c].$

The MLEs $\hat{\lambda}, \hat{c}, \hat{a}, and\ \hat{b}$ are obtained from the nonlinear equations $\frac{\partial l}{\partial c} = 0$, $\frac{\partial l}{\partial \lambda} = 0$, $\frac{\partial l}{\partial a} = 0\ and\ \frac{\partial l}{\partial b} = 0$ using iterative procedures.

3. Bayesian Analysis for the Kum-W Distribution

Here, approximate Bayes estimates are performed under the assumptions of non-informative priors using the Gibbs sampling procedure. This procedure allows for generating samples from the posterior distributions. We consider the Kum-W model with density function (2.4) and a non-informative joint prior distribution for a, b, c and λ given by:

$$\pi_0(a,b,c,\lambda) \propto \frac{1}{abc\lambda}, \qquad (3.1)$$

where $a, b, c\ and\ \lambda > 0$. The joint posterior distribution for these parameters can be written as

$$\pi(a,b,c,\lambda \mid x) \propto \pi_0(a,b,c,\lambda) exp\{l(x; a,b,c,\lambda)\} \qquad (3.2)$$

where $l(x; a,b,c,\lambda)$ as given by (2.5).

Consider the reparametrization $\rho_1 = \log(a)$ and $\rho_2 = \log(b)$, $\rho_3 = \log(c)$ and $\rho_4 = \log(\lambda)$. We obtain from (3.1) a non-informative prior for ρ_1, ρ_2, ρ_3 and ρ_4, namely

$$\pi(\rho_1, \rho_2, \rho_3, \rho_4) = costant, \quad where \ -\infty < \rho_1, \rho_2, \rho_3 and\ \rho_4 < \infty.$$

The convergence of the Gibbs sampling algorithm depends upon the choice of the values of hyper-parameters of the uniform priors.

Using the above reparamertization, the joint posterior distributions for ρ_1, ρ_2, ρ_3 and ρ_4 reduces to

$$\pi(\rho_1, \rho_2, \rho_3, \rho_4 \mid x) \propto \pi(\rho_1, \rho_2, \rho_3, \rho_4) \exp\Big\{n\rho_3 + n\rho_1 + n\rho_2 + n.exp(\rho_3)\rho_4 + (\exp(\rho_3) - 1)\sum_{i=1}^{n} \ln(x_i) - \sum_{i=1}^{n} (\exp(\rho_4)\, x_i)^{\exp(\rho_3)} + (\exp(\rho_1) - 1)\sum_{i=1}^{n} \ln[1 - e^{-(\exp(\rho_4)x_i)^{\exp(\rho_3)}}\Big] + (\exp(\rho_2) - 1)\sum_{i=1}^{n} \ln[1 - \left(1 - e^{-(\exp(\rho_4)x_i)^{\exp(\rho_3)}}\right)^{\exp(\rho_1)}]\} \tag{3.3}$$

If we assume the prior $\pi(\rho_1, \rho_2, \rho_3, \rho_4) = costant$, the conditional posterior distributions used in the Gibbs sampling algorithm are given by:

$$\pi(\rho_1 \mid \rho_2, \rho_3, \rho_4, x) \propto \exp\Big\{\mathrm{n}\rho_1 + (\exp(\rho_1) - 1)\sum_{i=1}^{n} \ln[1 - e^{-(\exp(\rho_4))^{\exp(\rho_3)}}\Big] + (\exp(\rho_2) - 1)\sum_{i=1}^{n} \ln[1 - \left(1 - e^{-(\exp(\rho_4)x_i)^{\exp(\rho_3)}}\right)^{\exp(\rho_1)}]\},$$

$$\pi(\rho_2 \mid \rho_1, \rho_3, \rho_4, x) \propto \exp\{n\rho_2 + (\exp(\rho_2) - 1)\sum_{i=1}^{n} \ln[1 - \left(1 - e^{-(\exp(\rho_4)x_i)^{\exp(\rho_3)}}\right)^{\exp(\rho_1)}]\},$$

$$\pi(\rho_3 \mid \rho_1, \rho_2, \rho_4, x) \propto \exp\{n\rho_3 + n exp(\rho_3)\rho_4 + (\exp(\rho_3) - 1)\sum_{i=1}^{n} \ln(x_i) - \sum_{i=1}^{n} (\exp(\rho_4)x_i)^{\exp(\rho_3)} + (\exp(\rho_1) - 1)\sum_{i=1}^{n} \ln\left[1 - e^{-(\exp(\rho_4)x_i)^{\exp(\rho_3)}}\right] + (\exp(\rho_2) - 1)\sum_{i=1}^{n} \ln[1 - \left(1 - e^{-(\exp(\rho_4)x_i)^{\exp(\rho_3)}}\right)^{\exp(\rho_1)}]\},$$

and

$$\pi(\rho_4 \mid \rho_1, \rho_2, \rho_3, x) \propto \exp\{n.exp(\rho_3)\rho_4 - \sum_{i=1}^{n} (\exp(\rho_4)\, x_i)^{\exp(\rho_3)} + (\exp(\rho_1) - 1)\sum_{i=1}^{n} \ln[1 - e^{-(\exp(\rho_4)x_i)^{\exp(\rho_3)}}] + (\exp(\rho_2) - 1)\sum_{i=1}^{n} \ln[1 - \left(1 - e^{-(\exp(\rho_4)x_i)^{\exp(\rho_3)}}\right)^{\exp(\rho_1)}]\}.$$

Posterior summaries of interest can be performed using the WinBUGS software which requires only the specification of the joint distribution for the data and the prior distributions for the model parameters.

4. The Bayesian Prediction Model

Now, Predictive density for a single future response, bivariate future response and m future responses are derived.

4.1 Predictive Density for a Single Future Response

Let z be a single future response from the model given by (2.4), where z is independent of the observed data. Then, the predictive density for a single future response (z) given $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ is

$$p(z|\boldsymbol{x}) = \int_{b=0}^{\infty}\int_{a=0}^{\infty}\int_{\lambda=0}^{\infty}\int_{c=0}^{\infty} p(z|a,b,c,\lambda)\,\pi(\,a,b,c,\lambda \mid \boldsymbol{x}\,)\,dcd\lambda dadb,$$

where $p(z|a,b,c,\lambda)$ may be defined from model (2.4), see Khan *et al.* (2013). Thus, the predictive density for a single future response is given by

$$p(z|\boldsymbol{x}) = \begin{cases} \eta_1(x)\int_{b=0}^{\infty}\int_{a=0}^{\infty}\int_{\lambda=0}^{\infty}\int_{c=0}^{\infty}(abc)^n\,\lambda^{c(n+1)-1}z^{c-1}e^{-(\lambda z)^c}\left[1-e^{-(\lambda z)^c}\right]^{a-1}\left[1-\left[1-e^{-(\lambda z)^c}\right]^a\right]^{b-1}\times \\ e^{-\sum_{i=1}^{n}(\lambda x_i)^c}\prod_{i=1}^{n} x_i^{\,c-1}\left[1-e^{-(\lambda x_i)^c}\right]^{a-1}\left[1-\left[1-e^{-(\lambda x_i)^c}\right]^a\right]^{b-1} dadbd\lambda dc, for\ z \ge 0; c,\lambda,a, and\ b > 0, \\ 0 \qquad eleswhere, \end{cases} \tag{4.1}$$

where $\eta_1(x)$ is a normalizing constant.

The predictive estimates for a future response will be discussed separately based on two real data sets. A numerical integration procedure “NIntegrate” in Mathematica software version 8.0, Wolfram Research (2012), is applied to plot the predictive density graph. Also the Mathematica Package is utilized to carry out all related calculations such as the predictive means, standard deviation, predictive intervals, and the measures of skewness and kurtosis.

4.2 Predictive Density for a Bivariate Future Response

Let z_1 and z_2 be two independent future responses from model (2.4). To derive the joint predictive density of z_1 and z_2, we utilize the posterior density $\pi(\,a,b,c,\lambda \mid x\,)$ specified by (3.2). Thus, the predictive density for a bivariate future response is given by

$$p(z_1,z_2|\boldsymbol{x}) = \begin{cases} \eta_2(x)\int_{b=0}^{\infty}\int_{a=0}^{\infty}\int_{\lambda=0}^{\infty}\int_{c=0}^{\infty}(abc)^{n+1}\,\lambda^{c(n+2)-1}e^{-\sum_{i=1}^{2}(\lambda z_i)^c}\prod_{i=1}^{2} z_i^{\,c-1}\left[1-e^{-(\lambda z_i)^c}\right]^{a-1}\left[1-\left[1-e^{-(\lambda z_i)^c}\right]^a\right]\times \\ e^{-\sum_{i=1}^{n}(\lambda x_i)^c}\prod_{i=1}^{n} x_i^{\,c-1}\left[1-e^{-(\lambda x_i)^c}\right]^{a-1}\left[1-\left[1-e^{-(\lambda x_i)^c}\right]^a\right]^{b-1} dadbd\lambda dc, for\ z_i \ge 0; c,\lambda,a, and\ b > 0, \\ 0 \qquad eleswhere, \end{cases} \tag{4.2}$$

where $\eta_2(x)$ is a normalizing constant.

4.3 Predictive Density for m Future Responses

Let $z_1, z_2, \dots, z_m$ be the m future responses from model (2.4). Thus,

$p(z_1, z_2, \ldots, z_m|\boldsymbol{x}) =$

$$\begin{cases} \eta_m(x) \int_{b=0}^{\infty} \int_{a=0}^{\infty} \int_{\lambda=0}^{\infty} \int_{c=0}^{\infty} (abc)^{n+m-1} \lambda^{c(n+m)-1} e^{-\sum_{i=1}^{m} (\lambda z_i)^c} \prod_{i=1}^{m} z_i^{c-1} \left[1 - e^{-(\lambda z_i)^c}\right]^{a-1} \left[1 - \left[1 - e^{-(\lambda z_i)^c}\right]^a\right] \\ e^{-\sum_{i=1}^{n} (\lambda x_i)^c} \prod_{i=1}^{n} x_i^{c-1} \left[1 - e^{-(\lambda x_i)^c}\right]^{a-1} \left[1 - \left[1 - e^{-(\lambda x_i)^c}\right]^a\right]^{b-1} \, da db d\lambda dc, for\ z_i \geq 0; c, \lambda, a, and\ b > 0, \\ 0 \qquad eleswhere, \end{cases}$$

(4.3)

where $\eta_m(x)$ is a normalizing constant. For m = 1, equation (4.3) reduces to the predictive density for a single future response obtained in (4.1); when m = 2, equation (4.3) reduces to the predictive density for a bivariate future response obtained in (4.2); and so on.

5. Applications to Real Data

Here, two real data sets were considered. Through likelihood ratio test and Kolmogorov-Smirnov test, Cordeiro *et al.* (2010) mentioned that the data sets studied by Meeker and Escobar (1998, p. 383) and by Murthy *et al*. (2004, p. 154) were fitted to the Kum-W distribution.

Data Set 1 (voltage data): This data gives the times of failure and running times for a sample of devices from a field-tracking study of a larger system. At a certain point in time, 30 units were installed in normal service conditions. The times (Thousands of cycles) are: 275, 13, 147, 23, 181, 30, 65, 10, 300, 173, 106, 300, 300, 212, 300, 300, 300, 2, 261, 293, 88, 147, 28, 143, 300, 23, 300, 80, 245, 266. Note that: data were censored at 300.

They considered this data as complete data and obtained the maximum likelihood estimates for the Kum-W distribution as follow:

$$\hat{c} = 7.7026, \ \hat{\lambda} = 0.0043, \qquad \hat{a} = 0.0516, \qquad and \qquad \hat{b} = 0.2288.$$

Data Set 2 (test stopped data): This data represents failure times and are taken from Murthy *et al* (2004, p. 154). The data set is: 0.0014, 0.0623, 1.3826, 2.0130, 2.5274, 2.8221, 3.1544, 4.9835, 5.5462, 5.8196, 5.8714, 7.4710, 7.5080, 7.6667, 8.6122, 9.0442, 9.1153, 9.6477, 10.1547and 10.7582.

They considered this data as complete data and obtained the maximum likelihood estimates for the Kum-W distribution as follow:

$$\hat{c} = 4.4200, \ \hat{\lambda} = 0.1744, \qquad \hat{a} = 0.0663, \qquad and \qquad \hat{b} = 0.1725.$$

Now, We consider the Kum-W distribution with density (3.2) under the reparametrization $\rho_1 = \log(a)$,$\rho_2 = \log(b)$, $\rho_3 = \log(c)$ and $\rho_4 = \log(\lambda)$. We assume approximate non-informative prior uniform U(0,2), U(0,0.01), U(0,0.01) and U(-4,-3) distributions for ρ_1, ρ_2, ρ_3 and ρ_4 respectively. A set of 9000 Gibbs samples was generated after a “burn-in-sample” of size 1000 to eliminate the initial values considered for the Gibbs sampling algorithm. All the calculations are performed using the WinBUGS software. The following tables list the posterior descriptive summaries of interest for the Kum-W model. The posterior kernel densities for the parameters are given in figures 1-2.

Table1: Summary results for the posterior parameters in the case of Kum-W model based on 30 data points (voltage data)

Parameter	Estimate	Standard Deviation	MC error	95% Credible Interval
a	2.524	0.464200	0.004882	(1.705, 3.506)
b	1.004	0.002864	3.324E-5	(1.000, 1.010)
c	1.004	0.002781	4.245E-5	(1.000, 1.010)
λ	0.01889	5.547E-4	9.655E-6	(0.01833, 0.0204)

Table2: Summary results for the posterior parameters in the case of Kum-W model based on 20 data points (test stopped data)

Parameter	Estimate	Standard Deviation	MC error	95% Credible Interval
a	1.042	0.0402	6.627E-4	(1.001, 1.149)
b	1.005	0.002914	3.457E-5	(1.000, 1.010)
c	1.005	0.002907	3.222E-5	(1.000, 1.010)
λ	0.04845	0.001291	2.193E-5	(0.04499, 0.04975)

An HPD interval is the interval which includes the most probable values of a given density at a given significance level, subject to the condition that the density function has the same value at both end points. The HPD interval $[g_1, g_2]$ for a single future response, z, must simultaneously satisfy the following two conditions:

$$P_r(g_1 \leq z \leq g_2) = 1 - \alpha$$

and

$$p(g_1|\boldsymbol{x}) = p(g_2|\boldsymbol{x}),$$

where g_1 and g_2 are to be arbitrary chosen so that $p(g_1|\boldsymbol{x}) = p(g_2|\boldsymbol{x})$. For more details about HPD intervals, see Box and Tiao (1973).

We estimated the predictive inference for a future response and their results are given in table (3). We determined certain levels of HPD interval for a single future response given a complete sample which are specified by g_1 and g_2, and their results are reported in table (4). The predictive densities for the future response are given in figures 3-4.

Table (3): Summary Results of the Predictive Inference for a Single Future Response

Data set 1 (n=20)		Data set 2 (n=30)	
Raw Moments	**Central Moments**	**Raw Moments**	**Central Moments**
$\acute{\mu}_1 = 20.6137$	$\mu_2 = 64.5201$	$\acute{\mu}_1 = 110.75100$	$\mu_2 = 1773.770$
$\acute{\mu}_2 = 489.446$	$\mu_3 = 71.8853$	$\acute{\mu}_2 = 14039.5000$	$\mu_3 = 8330.690$
$\acute{\mu}_3 = 12821.2$	$\mu_4 = 11014.2$	$\acute{\mu}_3 = 1.956 \times 10^6$	$\mu_4 = 8.2997 \times 10^6$
$\acute{\mu}_4 = 362000$		$\acute{\mu}_4 = 2.9298 \times 10^8$	
Skewness & Kurtosis $\beta_1 = 0.0192396$ $\beta_2 = 2.645840$ $\gamma_1 = \sqrt{\beta_1} = 0.13871$ $\gamma_2 = \lvert\beta_2 - 3\rvert = 0.354162$		**Skewness & Kurtosis** $\beta_1 = 0.0124358$ $\beta_2 = 2.63797$ $\gamma_1 = \sqrt{\beta_1} = 0.11152$ $\gamma_2 = \lvert\beta_2 - 3\rvert = 0.36203$	
Mean = 20.6137 **Standard deviation** = 8.03244 **Coefficient of Skewness** = 0.13871 **Coefficient of Kurtosis** = 0.35416		**Mean** = 110.7510 **Standard deviation** = 42.1161 **Coefficient of Skewness** = 0.11152 **Coefficient of Kurtosis** = 0.36203	

Table (4): Summary Results of the Highest Predictive Density (HPD) Intervals with different levels for a Single Future Response

Data Set 1: HPD Intervals		Data Set 2: HPD Intervals	
90%	(7.07854, 33.5074)	90%	(39.8427, 178.579)
95%	(5.10813, 35.7896)	95%	(29.1782, 190.343)
98%	(3.25.778, 38.3968)	98%	(19.0238, 203.724)
99%	(2.28423, 40.1602)	99%	(13.5896, 212.737)

Posterior Densities:

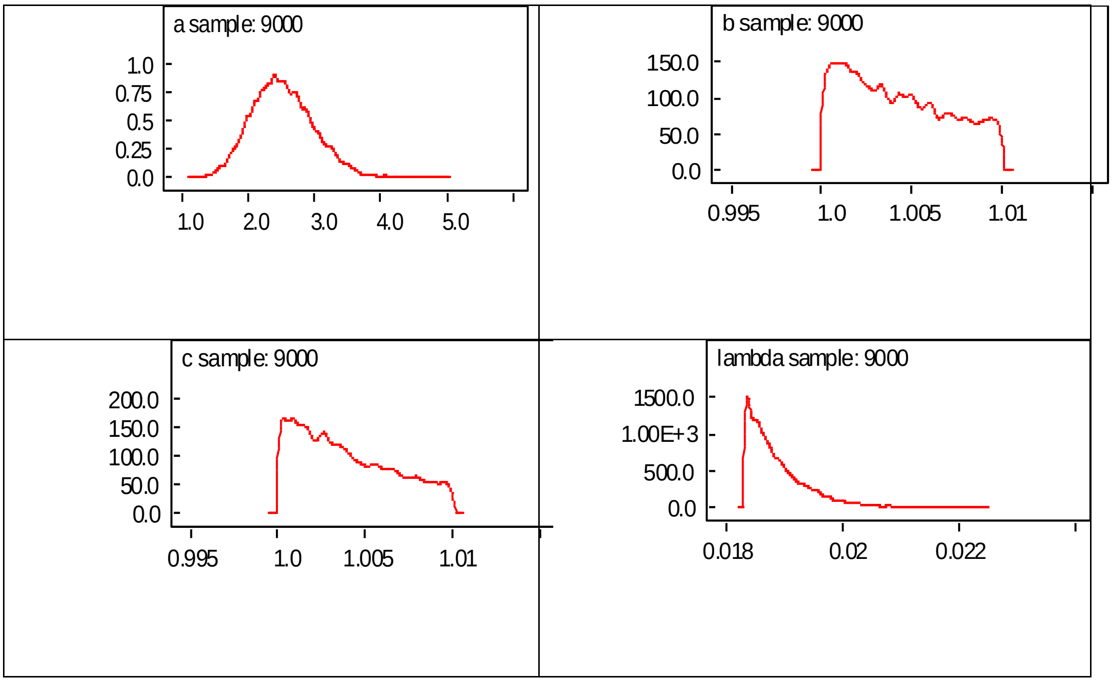

Figure1: Posterior kernel density for the parameters in the case of Kum-W model based on 30 voltage data

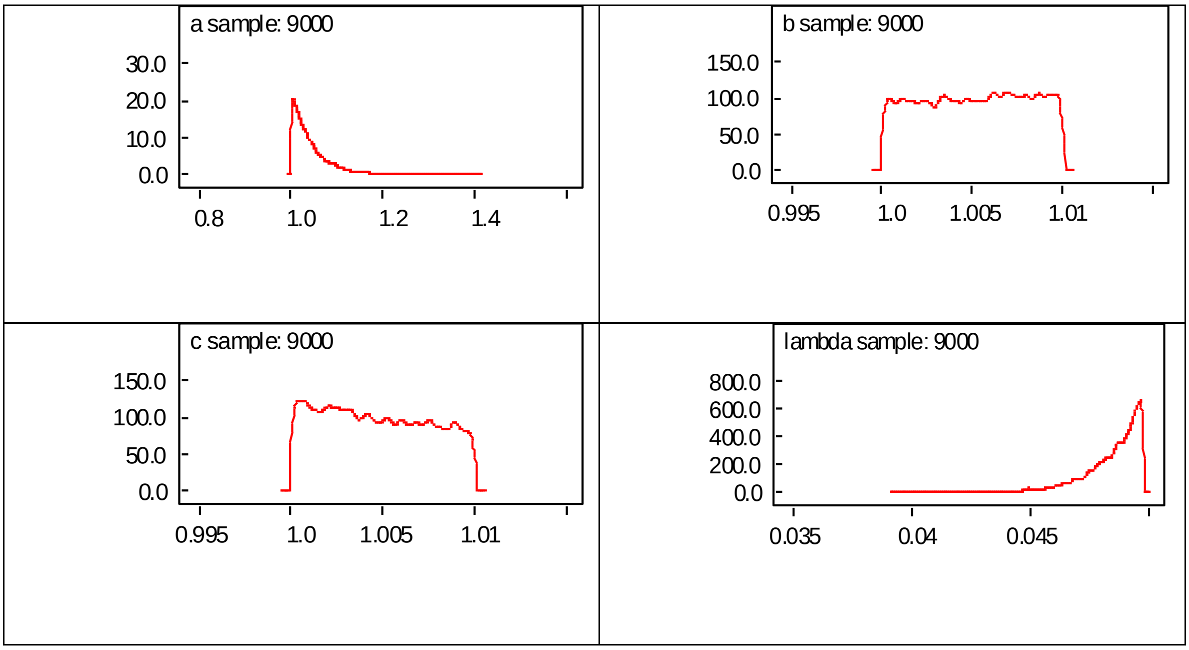

Figure2: Posterior kernel density for the parameters in the case of Kum-W model based on 20 test stopped data

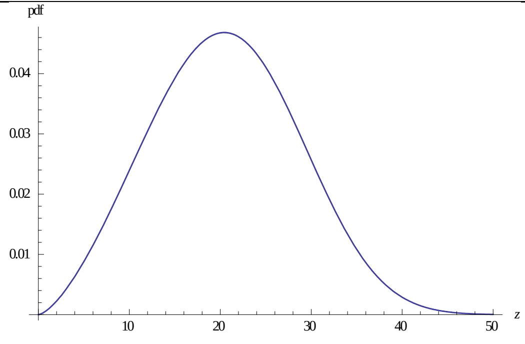

Figure3: Predictive density function based on 20 data points (data set 1)

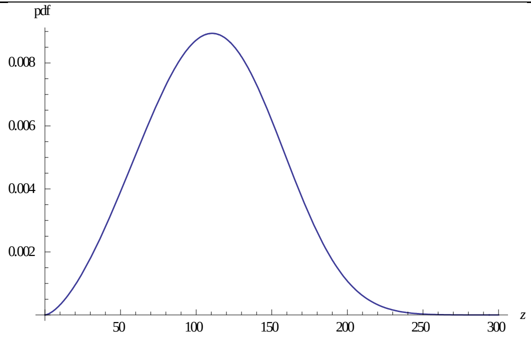

Figure4: Predictive density function based on 30 data points (data set 2)

6. Conclusion

Approximate Bayes estimates are obtained using the Gibbs sampling procedure. The posterior kernel densities are plotted for each parameter and the summary results are given. Using the Bayesian approach predictive densities for a single future response, a bivariate future response, and several future responses from the Kum-W model are discussed. Two real data sets are used to illustrate the predictive results in the case of a single future response. The normalizing constant for each of the predictive density is estimated to plot the predictive density. The first four raw moments and the central moments are computed for each of the predictive density. Estimated values of the measures of skewness and kurtosis of the predictive are reported. Based on these measures one can be noted that the predictive density has minor positive skewness. Finally, the highest predictive density intervals (90%, 95%, 98%, and 99%) are also computed.

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