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ORIGINAL PAPER

On the "$\hat{kp}$-operator", new extension of the KdV6 to $(m+1)$-dimensional equation and traveling waves solutions

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Abstract Very recently, the KdV6 was extended to a (2 + 1)- dimensional equation by a direct way. Here, by using the "$\hat{kp}$-operator" the extension of that equation is done in a rigorous mathematical formulation. Polynomial solutions to the closed form equation are obtained via the unified method. It is found that via nonlinear interactions of traveling waves the solutions of the constructed equation may not be unique. Also, these solutions may show solitary or explosive shock waves.

Keywords A method of extension of the KdV6 to higher-dimensional equation · Solitary · Soliton · Elliptic · Shock waves · Non-uniqueness of the solutions · The unified method

1 Introduction

The dynamics of shallow water waves had been studied in the literature via many basic equations, namely the Korteweg de Vries (KdV), Kadomtsev–Petviashvili (KP), Boussinesq and primitively by using the equations derived from the linear theory, and they are complete integrability. For studying the solvability of these equation, many different methods were presented [1–13]. We mention that by using the Painleve' analysis. An equation is tested for complete integrability or non-integrability. We distinguish between complete integrability and solvability. This may be argued to the fact that an equation that may be not integrable but an exact solution can be found, which means that there is no equivalence between complete integrability and solvability. To this aim, in view of the unified method UM (Sect. 2) we suggest the following propositions.

1. A PDE is completely solvable for polynomial solutions if there exist k, k_0, so that the balance and consistency conditions, namely $n := n(k)$, $and\, k \leq k_0$, $k_0 \geq 2$ hold (cf (2.3)).
2. A PDE is completely solvable for rational solutions if there exist $n = r = 1$ k, k_0, so that the balance and consistency conditions, namely $k \leq k_0$, $k_0 \geq 1$. (cf (2.4))
3. If a PDE is completely solvable for polynomial and rational solutions. then complete solvability and integrability may be equivalent.

The notions of complete integrability and exact solvability of a PDE are equivalent if they have the same number of free parameters, which is equal to its order.

On the other side, the author presented the extended and the generalized (UM) to study the solvability of evolution equations with space(or time)-dependent coefficients and to find multi-solitons solutions, respectively.

The main results of the UM method are resumed in what follows. Traveling wave solutions are obtained via either polynomials (namely of degree n) or rational

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in an auxiliary function $\varphi(z)$ say. This function satisfies an auxiliary equation. An equation between n and k is obtained by using the leading analysis, and the equation for k is found by the consistency condition. This will be illustrated later on in this work. It is worth mentioning that polynomial and rational functions solutions may be interpreted as solutions that result from direct and indirect nonlinear interactions of basic traveling waves solutions, respectively. These later solutions result by solving the auxiliary equations. They may be solitary, soliton, elliptic waves or wave-like patterns. It is worth mentioning that a PDE has periodic (or elliptic) waves if it is invariant under the transformation; $x \to -x$, $t \to -t$ and $u \to -u$.

Two-layer fluids have been studied via many modeling equations [14,17] which are mainly coupled KdV or complex KdV equations. Solvability of these coupled equations occupies a wide area in the literature [18,19], and they are studied by using a variety of methods [2–13]. A class of higher-order nonlinear wave equations, among the KdV6 equation [20], are studied to inspect completely integrability. Many works were done to find the solutions of the KdV6 equation [21–25].

In general, the solutions of nonlinear equations with constant coefficient are classified to be traveling or self-similar solutions (SSS) if they exist. When the coefficients are time or space dependent, they may be semi-self-similar or self-similar ones. Here we use the unified method [21], while the extended unified method enables us to find the solutions to evolute equations with variable coefficients [26].

2 Extension of the KdV6 to $(m + 1)$-dimensional equation

2.1 Direct extension

Here, we follow the work done in [11] for the extension of the KdV6 to a (2+1)-dimensional equation. We start by the KdV6, namely $F = \hat{f}_1\, S = 0$, where

$$\hat{f}_1 = (\partial_x^3 + 8v\, \partial_x + 4v_x),\ S = u_t + vxx + 6v^2\, u = v_x. \tag{1}$$

$$F = \hat{f}_1\, S := F(u_t,\, u_x, u_{xx},\, u_x^{(3)}, \ldots u_x^{(6)}), \tag{2}$$

The equation $F = \hat{f}_1\, S$ where $F(\ldots)$ is the result of acting by $\hat{f}_1$ on S. The extension of (2) is done directly by [11].

$$\partial_x F(\ldots) + \sigma^2 u_{yy} = 0 \tag{3}$$

2.2 Indirect extension and formulation of the method

Here, we suggest a more sophisticated method and define the $\hat{k_{dv}}$ and $\hat{k_p}$ associated with the KdV and KP equations, respectively, by

$$\hat{k}_{dv} = (\partial_t + \partial_{xxx}^3 + u\, \partial_x), \tag{4}$$
$$\hat{k}_p = (\partial_t + \partial_{xxx}^3 + u\, \partial_x)_x + \sigma^2 \partial_{yy}^2, \tag{5}$$

It is worth noting that $\hat{k}_{dv}(u) = 0$ and $\hat{k_p}(u) = 0$ give rise to the solutions of KdV and KP equations, respectively. Further, we have the following relations

$$\hat{k}_{dv}(u_1 + u_2) = \hat{k}_{dv}(u_1) + \hat{k}_{dv}(u_2) + (u_1 u_2)_x \tag{6}$$
$$\hat{k}_p(u_1 + u_2) = \hat{k}_p(u_1) + \hat{k}_p(u_2) + (u_1 u_2)_{xx},$$
$$u_j = u_j(x, y, t) \tag{7}$$

Last Eqs. (6) and (7) show that the two operators are not additive.

Now the method that we suggest here is based on defining the sets of operators M_1 and M_2 which are given by (4) and (5), respectively, and we define a continuously differentiable transformation T: $M_1 \to M_2$, $T(\hat{k}_{dv}) = \hat{k}_p$, which extends the KdV equation to $(2 + 1)$-dimensional KP equation. It is worth mentioning that the transformation T cannot be used to perform the extension of a general operator. This suggests to redefine a transform $T_g : M_1 \cup K_1 \cup K_2 \to M_2$, where

$$K_1 = \{\hat{s},\ \hat{s} = \hat{k}_{dv} + \hat{s}_0\},\ K_{2l} = \{\hat{p} = \hat{p}_l,\ \hat{p}_l = \hat{k}_{dv}\,\hat{p}_0\}, \tag{8}$$

where $\hat{s}_0$ and $\hat{p}_0$ are not of the $\hat{k}_{dv}$ type. A similar definition holds for $\hat{p}_r$. The properties of T_g are taken

$$T_g(\hat{k}_{dv} + \hat{s}_0) = T(\hat{k}_{dv}) + \hat{s}_0,\ T_g(\hat{p}_l) = T(\hat{k}_{dv})\hat{p}_0 \tag{9}$$

There is no justification to (9), but here no matter as we are only concerned with extending an equation of dimension $[(m - 1) + 1]$ to $(m + 1)$ which is also based on the fact that this extension is not unique. Thus, we arrive to the result that the works here and in [21] are different.

3 The New extension of the KdV6 to $(2+1)$-dimensional equation

By returning to Eq. (2), we find that it has a direct first integral and is reduced to

$$S_{xx} = \frac{S_x^2}{2S} - 4vS - \frac{A(t)}{S}, \tag{10}$$

where $A(t)$ is an arbitrary function. By taking into account of mild transverse dispersion in Eq. (3), S_{xx} is replaced by $\tilde{S}_{xx}$ in Eqs. (3) and (4). Finally, the extended KdV6 is (when $m=2$ and $A(t)=0$)

$$\begin{aligned} \frac{S_x^2}{2S} - 4v\,S - \frac{A(t)}{S} &= (v_t + v_{xxx} + 12vv_x)_x + \sigma^2 v_{yy} \\ &= \tilde{S}_{xx}, \\ S &= (u_t + v_{xx} + 6v^2),\ v = u_x. \end{aligned} \tag{11}$$

For higher-dimensional extensions, in (11) we replace $\sigma^2 v_{yy}$ by $\sum_{j=1}^{m-1} \sigma_j^2 v_{y_j y_j}$.

It is worth mentioning that, in the original variables, the equations in (11) are rewritten

$$\begin{aligned} &12v_x^2 + v_{xt} + 12vv_{xx} + v_{xxxx} + \sigma^2 v_{yy} = 0, \\ &\quad vv_t + v_x(30v^2 + u_t + v_{xx}) \\ &\quad + 2vv_{xxxx} = 0,\ v = u_x. \end{aligned} \tag{12}$$

We proceed the study that will be carried in the next section and mention that the shock wave solution is not just a matter to find exact solutions and claim that they exhibit wave breaking without justification. In fact, many traveling wave solutions may show blowup of waves or wave breaking (non-smooth wave). But the characteristics lines, planes or hyperplanes (or surfaces), may not intersect mutually. A necessary condition for shock waves to be produced is that the characteristics should mutually intersect. We think that this condition is not sufficient. The breaking time in this case is evaluated as the minimal time, where minimization is taken over all time values evaluated when pairs of characteristics intersect, namely $\tau_b = min\,\{\tau_{i\,j}, C_i \cap C_j \neq \varnothing\}$, where $C_i \cap C_j$ is the set of points of intersections of the $i', s\ and\ j', s$ characteristics. It is worthy mentioning that to estimate accurately the breaking time one has to solve an initial value problem [28]. Thus, an asymptotic analysis for finding the breaking time, rather than being inconvenient, it is so far from being accurate or rather relevant.

4 Traveling wave solutions of the extended KdV6

Here, for convenience we shall be concerned with finding the traveling wave solutions TWS of Eq. (11) [or (12)] in the polynomial form by using the unified method UM [21] (see also [26]). To this aim, we write Eq. (11) in a closed form in the original wave function $u(x, y, t)$. For instance, we write $u(x, y, t) = u_0(z)$, $v(x, y, t) = v_0(z)$, $S_b(x, y, t) = S_0(z)$, and $z = \alpha x + \beta y + \gamma t$. We think that one of the main results is to emphasize the existence of shock wave by checking that the condition mentioned in the above holds.

For simplicity, we take $A(t) = 0$ in (10) (or (11)). Thus Eq. (9) reduces to

$$\begin{aligned} S_0'' &= \left(-\beta^2\sigma^2 v_0'' + \frac{S_0'^2}{2S_0} - 4v_0\right)/\alpha^2 \\ &= (\gamma v' + \alpha^3 v^{(3)} + 12vv')' + \beta^2\sigma^2 v''/\alpha^2 \\ S_0 &= (\gamma u_0' + \alpha^2 v_0'' + 6v_0^2),\ v_0 = \alpha u_0',\ u' = \frac{\mathrm{d}u}{\mathrm{d}z}. \end{aligned} \tag{13}$$

We remark that by eliminating v_0'' between the first and second equations and we find that for each solution $S_0(z)$ there exist two solutions $v_0(z)$.

We mention that the closed form equation in $u_0(z)$ is too lengthy to be written here. We search for polynomial solutions, namely

$$\begin{aligned} u_0(z) &= \sum_{j=0}^{j=n} a_j\,(g(z))^j, \\ g'(z)^p &= \sum_{j=0}^{j=k} c_j\,(g(z))^j,\ p = 1, 2,\ k \geq 2 \end{aligned} \tag{14}$$

We mention that when $p = 1$, the solution of the auxiliary equation gives rise to (explicit or implicit) solutions in elementary functions. But when $p = 2$ and the solution in the first equation holds, explicit solutions in Jacobi-elliptic, Weierstrass functions or to implicit elliptic integrals of the first, second or third kinds. Unfortunately this does not hold in the present case.

From the leading analysis, we find that $n = k - 1$ and the consistency condition is $k < 17/3$. Thus, polynomial solutions exist when $k = 2, 3, 4, 5$. Indeed, we have outlined a proof to a theorem that confirms this result, but it will not be produced here.

The following cases are considered when $p = 1$. In Eq. (13), we take the following cases;

1. When $k = 2$ and $n = 1$, in this case the solution of (11) is a soliton wave
2. When $k = 3$ and $n = 2$, calculations yield

$$
\begin{aligned}
a_2 &= -2c_3\alpha,\ a_1 = -4\alpha/3,\\
c_1 &= (2c_2^3 + 27c_3^2c_0)/(9c_2c_3),\\
\gamma &= \alpha^3(-4c_2^6 + 216^3c_2c_3^2c_0\\
&\quad -2916c_3^4c_0^2)/(81c_2^2c_3^2)
\end{aligned}
\tag{15}
$$

For convenience, we introduce a new parameter s and we get

$$
\begin{aligned}
s &= \frac{2(c_2^3 - 27c_3^2c_0)}{9c_2c_3},\\
u_0(z) &= \frac{(-9a_0c_3(c_2 - 3c_3e^{s\,z}) + c_2(-2c_2^2\hat{}2 + 6c_2e^{s\,z} + 9c_3s)\alpha)}{9c_3(-c_2 + 3c_3e^{s\,z})},
\end{aligned}
\tag{16}
$$

and

$$
\gamma = -s^2\alpha^3,\ c_o = (c_2(2c_2^2 - 9c_3s))/(54c_3^2). \tag{17}
$$

From (15), we find that when $c_2^3 \neq 27c_3^2c_0$, the solution of (12) shows an explosive shock waves when $c_2c_3 > 0$, and it is solitary wave solution when $c_2c_3 < 0$. But when $c_2^3 = 27c_3^2c_0$, there is no wave solution.

The results in (15) are shown in Fig. 1a, b for the first case.

In Fig. 1a $a_0 = 5$, $\alpha = 1.6$, $c_2 = 4$, $c_3 = 1.5$, $\sigma = 1$, $\beta = 0.4$, $s = 2.2$, $c_3 = 1.5$. The solution is displayed against t for discrete values of y and x, while in Fig. 1b, the characteristic curves are shown in the $y-t$ plane for discrete values of α, $(1.6 < \alpha < 2.2)$ and x.

Figure 1b shows that the characteristic curves in the yt plane. On the other hand from Eq. (13), the parameter s plays a significant role, which it can be interpreted as the rate of variation of the solution on the characteristic plane (namely $z = const.$).

It is worth mentioning that the forth equation in (15) for γ is of great interest as it may be interpreted as a "nonlinear dispersion equation."

3. When $k = 4$ and $n = 3$, we find the following results

$$
\begin{aligned}
a_3 &= -3c_4\alpha,\ a_2 = -\frac{9c_3\alpha}{4},\\
a_1 &= \frac{9\alpha(c_3^2\alpha - 16c_2c_4)}{80c_4},\\
c_1 &= -\frac{c_3^3\hat{}3 - 4c_2c_3c_4}{8c_4^2},\\
c_0 &= \frac{(19c3\hat{}4 - 368c2c3\hat{}2c4 + 1024c_2^2c_4^2)}{6400c_4^3},\\
\gamma &= -\frac{\alpha(\alpha^2(243c_3^6 - 1944c_2c_3^4c_4 + 5184c_2^2\hat{}2c_3^2c_4^2 - 4608c_2^3c_4^3))}{4000c_4^4},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
z &= \frac{40\sqrt{10}c4^2(\mathrm{ArcTan}\,(h\,(z)) - 2\mathrm{ArcTan}(2h\,(z))}{(3(3c_3^2 - 8c_2c_4))^{3/2}},\\
h\,(z) &= \frac{(\sqrt{5/2}(c_3 + 4c_4g(z)))}{2\sqrt{-3c_3^2 + 8c_2c_4)}},\\
u_0(z) &= ao + \frac{9(c_3^2 - 16c_2c_4)\alpha g(z)}{80c4}\\
&\quad - \frac{9c_3\alpha g(z)^2}{4} - 3c_4\alpha g(z)^3.
\end{aligned}
\tag{19}
$$

It is worthy mentioning that the first equation in (18) can be written by

$$
z = \frac{40\sqrt{10}c4^2\mathrm{ArcTan}\,(k\,(z)}{(3(3c_3^2 - 8c_2c_4))^{3/2}},
$$

$$
k(z) = -\frac{\mathrm{Tan}\,((h\,(z))(-3 + \mathrm{Tan}\,(h\,(z))}{-1 + 3\mathrm{Tan}\,(h\,(z))}
$$

$$
c_2 = \frac{(9c_3^2 - 206^{1/3}c_4^{4/3}s)}{24c_4},\ \gamma = -(4s^3\alpha^3), \tag{20}
$$

Indeed, the solution in (18) and (19) are in the parametric form. We distinguish two cases.

When $c_4 < 3c_3^2/8c_2$, $c_2 > 0$, the solution in (18) and (19) shows solitary waves. When $c_4 > 3c_3^2/8c_2$, $c_2 > 0$, the solution is displayed in Fig. 2a, b. Fig. 1b shows the mutual intersections of characteristic curves. Thus this solution is a shock solution. The breaking time could be estimated by $\tau_b \approx 0.1$.

In Fig. 2a: $a_0 = 5$, $\alpha = 1.6$, $c_2 = 4$, $c_3 = 1.5$, $c_4 = 0.5$, $\sigma = 1$, $\beta = 0.4$

4. When $k = 5$ and $n = 4$, the calculations give rise to

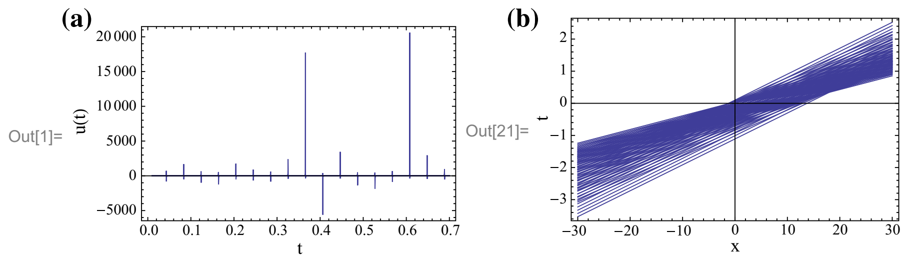

Fig. 1 **a** $a_0 = 5, \alpha = 1.6, c_2 = 4, c_3 = 1.5, \sigma = 1, \beta = 0.4, s = 2.2, c_3 = 1.5$. The solution is displayed against t for discrete values of y and x, **b** the characteristic curves in the $y - t$ plane for discrete values of α, $(1.6 < \alpha < 2.2)$ and x

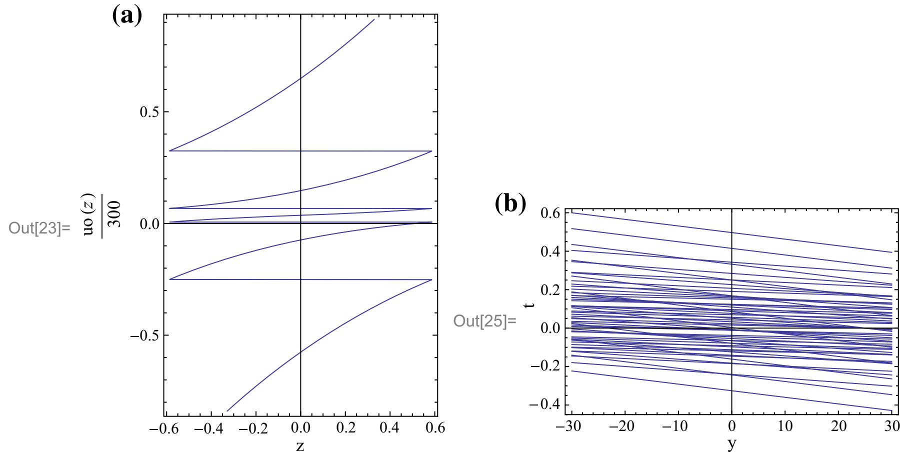

Fig. 2 **a** $a_0 = 5, \alpha = 1.6, c_2 = 4, c_3 = 1.5, c_4 = 0.5, \sigma = 1, \beta = 0.4$

$$
\begin{aligned}
a_4 &= -4c_5\alpha,\ a_3 = -\frac{16c_4\alpha}{5},\\
a_2 &= -\frac{8\alpha(-c_4^2+25c_3c_5)}{75c5},\\
a_1 &= -\frac{((8c_4(82c_4^2-100c_3c_5+675c_5^{3/2}s)\alpha}{2625c5\hat{}2}\\
c_1 &= \frac{(2c_4^4+45c_4^2c_5^{3/2}s+125c_5^3s^2)}{250c_5^3},\\
c_0 &= \frac{(2c_4^5+75c_4^3c_5^{3/2}s+625c_4c_{53}s^2)}{6250c_5^4},\\
c &= \frac{(4c_4^3+45c_4c_5^{3/2}s)}{50c5^2},\ \gamma = -\alpha(s\hat{}4\alpha^2+\mu)
\end{aligned}
\tag{21}
$$

$$
u_0(z) = \frac{9c_4(625a_0c_5^3+(2c_4^2+25c_5^{3/2}s)^2\alpha)e^{s^2z}+625a_oc_5^3\hat{}3+4c_4^2\hat{}(c_4^2+25c_5^{3/2}s)\alpha}{625c_{53}(1+9c_4e^{s^2z})}, \tag{22}
$$

and this solutions shows solitary waves.

5 Quadratic invariant

In fact the quadratic invariants QI give rise, in particular, to the Hamiltonian (H) function which is of two degrees of freedom. Here, we are concerned with finding the QI for TWS [27]. To this issue, we write the kinetic equations [see (12)]

$$
\begin{aligned}
v_0'' &= \frac{(\alpha Q^2-6\alpha v_0^2-\gamma v_0'-\alpha\mu v_0)}{\alpha^3},\\
Q'' &= -\frac{(A\,\alpha^3-\beta^2\sigma^2Q\,v_0(\gamma+\alpha\mu+6\alpha v_0)+Q^4(\alpha\beta^2\sigma^2+4\alpha^3v_0))}{2\alpha^5Q^3},
\end{aligned}
\tag{23}
$$

where $S_0 = Q^2$.

We assume that the QI are given by $a_1(v_0)\,(Q')^2 + a_2(Q)\,(v_0')^2 + a_3(v_0, Q)\,Q'v_0' + a_4[(v_0, Q)\,Q' + a_5(v_0, Q)v_0' + a_0(v_0, Q) = 0$, where $a_i(v_0, Q)$ are arbitrary functions. By carrying the total derivative w.r.t. z, the calculation gives rise to the QI;

$$
\begin{aligned}
&-A\,k_3\alpha^2+2A_{00}\alpha^4S_0-2k_0\alpha^2S_0^2v_0\\
&\quad+\alpha^4Q^2(A_{11}(S_0')^2)/2+S_0^2(k_3\beta^2\sigma^2\\
&\quad+4k_0\alpha^2v_0^3+k_0\alpha^4\,(v_0')^2)=0,
\end{aligned}
\tag{24}
$$

where $A,\ k_3,\ k_0,\ldots$ are arbitrary constants. Apart of the first term in (28), the H function is given by setting

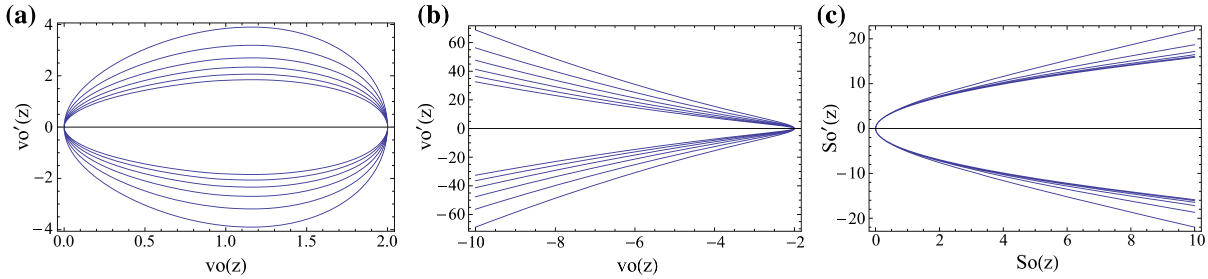

Fig. 3 In (**a–c**) discrete values of $0.9 < \alpha < 2$ with steps 0.2 are considered and for the same caption for the parameters namely: $k_3 = -5, \beta = 0.4, \sigma = 1, k_0 = 0.2, S_0(z) = 1.5, A_{00} = -6, A = 0, A_{11} = 1$. In (**b**) the same caption holds but the bifurcation "line" is $S \backslash S_0(z) = 8$

$A\,k_3\alpha^2 = H$. By solving Eq. (24) for v_0' and on the bifurcation orbits which are given by

$$S_0' = \sqrt{2}\sqrt{\frac{A\,k_3\alpha^2 - 2A_{00}\alpha^4 So - k_3\beta^2\sigma^2 S_0^2)}{\sqrt{A11}\alpha^2}}. \quad (25)$$

The phase portrait in the plane v_0-v_0' is shown in Fig. 3a–c. In Fig. 3c, the bifurcation curves are shown, while in Fig. 3a, b the phase portraits are shown when $0 < vo < 0.707So$ and $vo < -0.707So$, respectively.

In these figures, discrete values of $0.9 < \alpha < 2$ with steps 0.2 are considered and for the same caption for the parameters namely: $k_3 = -5,\ \beta = 0.4, \sigma = 1,\ k_0 = 0.2,\ S_0(z) = 1.5,\ A_{00} = -6,\ A = 0,\ A_{11} = 1$. In (3b), the same caption holds, but the bifurcation "line" is $S \backslash S_0(z) = 8$.

After the this diagram, we find that (a) corresponds to homoclinic orbits (periodic solutions), while in (b) parabolic trajectories may correspond to solitary or shock waves and hence the node is stable or not.

6 Conclusions

A method for extending the KdV6 to an equation in (2+1)-dimensional equation was suggested. It was shown the extension is not unique and extension to higher-dimensional equation may be established. The traveling wave solutions of the extended equation may show, solitary, soliton, explosive shock waves or non-unique solutions. Finally, the bifurcation diagram and the pase portraits of trajectories are shown.

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