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Exact Solutions of Space Dependent Korteweg–de Vries Equation by The Extended Unified Method

Hamdy I. ABDEL-GAWAD*, Nasser S. ELAZAB†, and Mohamed OSMAN‡

Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt

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Recently the unified method for finding traveling wave solutions of nonlinear evolution equations was proposed by one of the authors (HIAG). It was shown that, this method unifies all the methods being used to find these solutions. In this paper, we extend this method to find a class of formal exact solutions to Korteweg–de Vries equation with space dependent coefficients.

KEYWORDS: exact solution, extended unified method, Korteweg–de Vries equation, variable coefficients

1. Introduction

We consider the following evolution equation

$$f\left(x,t,u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x\partial t},\frac{\partial^2 u}{\partial x^2},\dots,\frac{\partial^m u}{\partial x^m}\right)=0,\quad m\geqslant 1,\qquad(1)$$

where f is a polynomial in its arguments. With relevance to the Korteweg–de Vries (KdV) equation we write

$$H\left(x,t,u,\frac{\partial u}{\partial x},\dots\right)\equiv\frac{\partial u}{\partial t}+f_0(x,t)\frac{\partial^m u}{\partial x^m}+f_1\left(x,t,u,\frac{\partial u}{\partial x}\right)=0.\qquad(2)$$

When Eq. (1) does not depend explicitly on x and t, it can be reduced to a subclass of ordinary differential equations by using the Lie groups for partial differential equations[1] or by using similarity transformations. Among them, the equation for traveling waves is

$$g(u,u',u'',\dots u^{(m)})=0,\quad u'=\frac{du}{dz},\quad z=x-ct,\qquad(3)$$

which results due to the translation symmetry of (1). The Painleve' analysis was used to testing the integrability of partial differential equations, which has been developed in Ref. 2 The exact solutions of (2) for completely or partially integrable were dealt with them by the auto-Bäcklund transformation. This was done by truncating Painleve' expansion.[3–9] Recently the auto-Bäcklund transformation that was extrapolated in Refs. 10–14 and the homogeneous balance method in Refs. 15–19 were used to find solutions for evolution equations with variable coefficients in the form

$$u(x,t)=\frac{\partial^{m-2}}{\partial x^{m-2}}(a(\phi)\phi_x)+u^{(0)}(x,t),$$

where ϕ is the base function.

2. The Extended Unified Method

Explicit solutions of Eq. (2) are, in fact, particular solutions. In this respect, these solutions can be mapped to polynomial or rational solutions through an "auxiliary" function and with appropriate auxiliary equations. The later equations may be solved to elementary of elliptic functions. In Ref. 20 the notion of unified method was implemented and detailed necessary conditions for the existence of polynomial or rational solutions were established. In the present paper, we extend this method to handle the evolution equations with variable coefficients of the type (2).

2.1 Polynomial solutions

Here, we assume that, we search for solutions of (2) which are in $C^s(\mathbb{R}\times\mathbb{R}^+)$ (the class of continuously partially differentiable functions up to order s), and we define the set of functions

$$S=\{\phi:\mathbb{R}\times\mathbb{R}^+\to\mathbb{K}\subset\mathbb{R},\phi_t^q=P_k^t(\phi),(\phi_x)^p=P_k^x(\phi)\},$$
$$P_k^t(\phi)=\sum_{i=0}^k b_i(x,t)\phi^i(x,t),\quad P_k^x(\phi)=\sum_{i=0}^k c_i(x,t)\phi^i(x,t).\qquad(4)$$

Indeed the set S contains elementary or elliptic functions for some particular values of q, p and k when $p-q=1$ or $p=q\geq 2$ respectively. We mention here that when $p=q=1$, the set S is closed under addition and multiplication by a real number. Here, we shall confine ourselves to the case $p=q=1$. In the present case, the mapping method asserts that there exist a positive integer n and a mapping

$$M:C^s(\mathbb{R}\times\mathbb{R}^+)\to\Omega,$$
$$\Omega=\left\{v,v=\sum_{i=0}^{s_0}a_i(x,t)\phi^i,\phi\in S,s_0\leq s\right\}$$

such that $M(u)=P_n(\phi)$, $n<s_0<s$ which satisfies the properties

$$M(\alpha_1u_1+\alpha_2u_2)=\alpha_1M(u_1)+\alpha_2M(u_2),$$
$$M(u_1u_2)=M(u_1)M(u_2),$$
$$M(u_t)=(M(u))_t,$$
$$M(u_x)=(M(u))_x.$$

Thus M is a ring homomorphism that conserves differentiation. By using (4) we find that,

$$M(u_t)=P^t_{(n-1+k)}(\phi)\in\Omega,M(u_x)=P^x_{(n-1+k)}(\phi)\in\Omega.$$

By using the properties of M and the last results and as $H\equiv f(x,t,u,u_t,\dots)$ is a polynomial in its arguments, we find that $M(H)$ is a polynomial. So that there exists $s_0\leq s$ such that $M(H)=P_{s_0}(\phi)\in\Omega$. It is worthy to notice that all these polynomials have different coefficients.

More simply the mapping M assigns to u and H with two auxiliary equations, the polynomials $P_n(\phi)$ and $P_{s_0}(\phi)$ respectively. In the case of Eq. (2) $s_0=n-m+mk$. The utility of the above presentation is that it helps us to give arguments to the statements of the conditions in Lemma 2.1. Also, we think that it allows for constructing more generalization and it is more appropriate when (2) is a vector equation.

We substitute for $u, u_t, u_x, \ldots, (\partial^m/\partial x^m)u$ as polynomials in ϕ, so that the function H is a polynomial in ϕ, together with two auxiliary equations. In the applications we may write directly $u = P_n(\phi)$, and $H = P_{s_0}(\phi)$, namely

$$u = \sum_{i=0}^{n} a_i(x,t)\phi^i, \tag{5}$$

and the auxiliary equations are

$$\phi_t = \sum_{i=0}^{k} b_i(x,t)\phi^i,\ \phi_x = \sum_{i=0}^{k} c_i(x,t)\phi^i, \tag{6}$$

together with the compatibility equation

$$\phi_{xt} = \phi_{tx}. \tag{7}$$

When substituting from (5) and (6) into (2) we find that it is transformed to $P_{s_0}^{(H)}(\phi) \equiv 0$ that gives rise to

$$\sum_{i=0}^{s_0} h_i(a_{r_0}(x,t), b_{r_1}(x,t), c_{r_2}(x,t), a_{r_0t}(x,t), a_{r_0x}(x,t),\ldots)\phi^i \equiv 0,$$
$$r_0 = 0,1,\ldots,n \quad \text{and} \quad r_1, r_2 = 0,\ldots,k. \tag{8}$$

By equating the coefficients of $\phi^i, i = 0,1,\ldots,s_0$ to zero, we get a set of (s_0+1) algebraic equations (or partial differential equations). The equations obtained in the functions a_i, b_i and c_i are called the principle equations. On the other hand the equations that result from (7) count: $2k-1$, $k \geq 2$. We mention that the unknown functions count: $n+2k+3$.

In the Eq. (2), if $u^j u_x$ is the highest nonlinear term then we get the balancing condition; $s_0 = nj + n + k - 1 = n - m + mk$. When solving for n, we find that it depends on m, j and k. From the last result and the number of the principles and compatibility equations, we can determine if the equations to be solved are over-determined or under-determined. The number of the determining equations, balances the number of unknowns, is over-determined or is under-determined when the difference, namely $(n - m + (m+2)k) - (n + 2k + 3)$ is equal to 0, >0, or <0 respectively. From the last condition, we may determine the consistency condition that will be identified in the lemma. In what follows necessary conditions for the existence of polynomial solutions will be stated.

Lemma 2.1. *For polynomial-solutions of (2) (as a polynomial in ϕ) to exist it is necessary that*

(i) $(m-1)(k-1)/j (:= n)$ is a positive integer

(ii) $m(k-1) - 3 \leq m$ when the equation (2) in the absence of x, and t passes the Painleve' test. Otherwise m is replaced by 2.

We notice that the first and the second conditions in Lemma 2.1 are the balancing and the consistency conditions respectively. For details see Ref. 20.

2.2 *The rational function solutions*

Here, also we search for solutions of Eq. (2) which are in $C^s(\mathbb{R} \times \mathbb{R}^+)$ and in the form of rational function in the auxiliary function ϕ.

To this end, we define the space of functions; $\Omega_R = \{v, v = P_n(\phi)/Q_r(\phi), \phi \in S\}$ and $Q_r(\phi)$ has no zeros in $\mathbb{K} \subset \mathbb{R}$, where S is defined in Sect. 2.1. The definitions in the above and the GMM for rational function solutions assert that there exists a mapping

$$M_R : C^s(\mathbb{R} \times \mathbb{R}^+) \to \Omega_R,$$
$$M_R(u) = P_n(\phi)/Q_r(\phi),$$
$$\phi \in S.$$

The properties of this mapping are the same properties of the mapping $M(u)$ which are given in Sect. 2.1. By bearing in mind these properties and from (6) and (7) we find that

$$M_R(u_t) = P^t_{(n-1+k+r)}(\phi)/Q_r^2(\phi) \in \Omega_R, M_R(u_x) = P^x_{(n-1+k+r)}(\phi)/Q_r^2(\phi) \in \Omega_R, \tag{9}$$

and by induction $M_R(\partial^i u/\partial x^i) \in \Omega_R$, $i = 1,\ldots,m$. By using the properties of M_R and equation (9), there exists $s_1 \leq s$ such that $M_R(H) = P^H_{s_1}(\phi)/Q_r^{m+1}(\phi)$ and it holds that $M_R(H) \in \Omega_R$. Indeed s_1 depends on n, r, k and also on m, where in the case mentioned in the above $s_1 = n - m + mk + mr$.

In the applications, we write simply;

$$u = \frac{\sum_{i=0}^{n} a_i(x,t)\phi^i}{\sum_{i=0}^{r} d_i(x,t)\,\phi^i}. \tag{10}$$

So that the Eq. (2) is transformed to $P^H_{s_1}(\phi) \equiv 0$. Equivalently, the last identity becomes

$$\sum_{i=0}^{s_1} h_i(a_{r_0}(x,t), d_{r_2}(x,t), b_{r_1}(x,t), c_{r_3}(x,t),\ldots)\phi^i \equiv 0,$$
$$r_0 = 0,\ldots,n, r_1,\ r_3 = 0,\ldots,k, \quad \text{and} \quad r_2 = 0,\ldots,r. \tag{11}$$

In (11), by equating the coefficients of $\phi^i, i = 0,1,\ldots,s_1$ to zero, we get a set of (s_1+1) equations that determine the functions a_i, b_i, c_i, and d_i. We mention that these later functions count $n+2k+r+4$. As in Sect. 2.1, by assuming that the higher nonlinear term is $u^j u_x$ in Eq. (2), then the balancing condition is

$$\begin{cases} nj + n + k - 1 + r = n - m + mk + mr + r(j-(m+1)), \\ \qquad\qquad m+1 < j \\ nj + r((m+1)-j) = n - m + mk + mr = s_1, \\ \qquad\qquad m+1 > j \end{cases} \tag{12}$$

Now by solving (12) for n, we find that it depends on m, j, r and k and, in the two cases in the above. They give rise to the same equation for $n - r$. Hereafter, we distinguish between the two cases mentioned in (12). From the last results and when $j < m+1$, the number of the determining equations, balances the number of unknowns, is over-determined or is under-determined when the difference, namely $(n - m + m(k+2) + rm) - (n + 2k + r + 4)$ is 0, >0, or <0 respectively. But when $j > m+1$ this difference is $(n - m + m(k+2) + rm + r(j-(m+1))) - (n + 2k + r + 4)$. From these last conditions, we may determine the consistency condition that will be identified in the following Lemma.

Lemma 2.2. *For solitary wave-rational solutions of Eq. (2) to exist it is necessary that*

(i) $(m-1)(k-1)/j (:= n - r)$ is an integer

(ii) $r(m-1)+(k-1)m-3 \leq m, j < m+1$ *or* $r(j-2)+(k-1)m-k-2 \leq 2$, $j > m+1$, *in the case when Eq. (2) passes the Painleve' test. Otherwise* $r(m-1)+(k-1)m-k-2 \leq 2, j < m+1$ *or* $r(j-2)+(k-1)m-k-2 \leq 2, j > m+1$.

For details see Ref. 20.

3. Exact Solutions of KdV Equation with Space Dependent Coefficients

Here, we extend the unified method to the variable coefficient KdV equation

$$u_t + f(x)u_{xxx} + g(x)uu_x = 0, t > 0, x > 0, \quad (13)$$

where f and g are arbitrary functions in x. For $x < 0$, the solutions of Eq. (13) hold if we replace x by $|x|$ and assuming that $f(-x) = -f(x)$ and $g(-x) = -g(x)$. We mention that Eq. (13) describes the propagation of waves in a medium with space dependent dispersion and conviction. In fact, partial differential equations with variable coefficients may be of practical interests as the behavior of solutions depends significantly on the dispersion and conviction coefficients. Notably when one of these coefficients is highly greater than the other one's. Some exact solutions were obtained in Refs. 21 and 12 when the coefficients in Eq. (13) are time dependent, namely $f(t)$ and $g(t)$. In these works, solutions were obtained only when $f(t) = cg(t)$, where c is a constant [or $f(t)$ and $g(t)$ are linearly dependent].

3.1 The polynomial function solutions

In what follows we shall derive a polynomial solution of equation Eq. (13).

In Lemma 2.1, the consistency condition holds when $k = 2, 3$ but it does not hold when $k \geq 4$. So that, only the cases $k = 2, 3$ will be considered.

I. When $k = 2$, $n = 2$, by substituting into (5), (6), and (13), we get six principle equations. We mention that calculations are carried out by using MATHEMATICA where standard functions in symbolic computations are only needed.

The steps of computations are as follows;

Step 1. Solving the principle equations, where five of them are solved explicitly to

$$a_2(x,t) = -12h(x)c_2(x,t)^2,$$
$$a_1(x,t) = -\frac{12}{5}(5c_1(x,t)c_2(x,t)h(x) + c_2(x,t)h'(x) + 5h(x)c_{2x}(x,t)), \quad (14)$$

together with explicit equations for $b_j(x,t)$, $j = 0, 1, 2$ (they are too lengthy to written here) where $h(x) = f(x)/g(x)$ and $r(x) = 1/g(x)$. It remains only one unsolved equation of the principle ones.

Step 2. We consider the compatibility equations that result from $\phi_{xt} = \phi_{tx}$ and they are given formally by

$$b_0(x,t)c_1(x,t) - b_1(x,t)c_0(x,t) + c_{0t}(x,t) - b_{0x}(x,t) = 0,$$
$$2b_0(x,t)c_2(x,t) - 2b_2(x,t)c_0(x,t) + c_{1t}(x,t) - b_{1x}(x,t) = 0,$$
$$-b_2(x,t)c_1(x,t) + b_1(x,t)c_2(x,t) + c_{2t}(x,t) - b_{2x}(x,t) = 0. \quad (15)$$

To simplify the computations, we make the transformations

$$c_{2x}(x,t) = p(x,t)c_2(x,t), c_1(x,t) = -p(x,t) + C_1(x,t),$$
$$c_0(x,t) = \frac{-2C_{1x}(x,t) + C_1^2(x,t) + 4C_0(x,t)}{4c_2(x,t)}, \quad (16)$$

where $C_0(x,t)$, $C_1(x,t)$ are arbitrary functions. To evaluate $a_0(x,t)$ the following steps are used.

i. Solve the last equation in (15) for a_{0x}
ii. Eliminate a_{0xx}, a_{0xxx}
iii. Substitute in the middle equation in (15) to get $a_0(x,t)$
iv. Calculate a_{0x} from the last step and identify it by a_{0x} from step (i), we get an equation in C_0, $C_{0x}, \ldots$.

As the computations are too lengthy in the general case, we consider a power law functions $h(x) = h_0x^n, r(x) = r_0x^m$. In the original variable $g(x) = r_0^{-1}x^{-m}, f(x) = h_0/r_0x^{n-m}$.

v. Solve the equation that result from (iv) in C_{0x}
vi. Substitute into the first equation in (15) and solve for C_{0t}. Thus (15) solved completely.
vii. Calculate C_{0tx} from (vi) and balance it with C_{0xt} from (v), we get the following algebraic equations

$$30 + 5m^2 + m(25-9n) - 22n + 4n^2 = 0, \quad (17)$$

or

$$-187500m + 31250m^2 + 125000m^3 + 31250m^4 - 150000n + 893750mn + 84375m^2n - 362500m^3n - 90625n^4 + 695000n^2 - 1605000mn^2 - 477500m^2n^2 + 325000m^3n^2 + 81250m^4n^2 - 1274000n^3 + 1338500mn^3 + 581750m^2n^3 - 87500m^3n^3 - 21875m^4n^3 + 1178000n^4 - 511500mn^4 - 255750m^2n^4 - 578000n^5 + 71750mn^5 + 35875m^2n^5 + 143000n^6 - 14000n^7 + 5(5-n)\sqrt{W(m,n)} = 0, \quad (18)$$

where

$$W(m,n) = (2-n)(5m^2 + 4(-1+n)n + m(-5+9n))^2 \times ((-450000m^5(5-8n+3n^2) + 25m^4 \times (-209750 + 499175n - 270640n^2 + 24307n^3) + 10m^3(-788750 + 1335125n + 328025 \times n^2 - 939245n^3 + 237213n^4) + m^2(9381250 - 53065625n + 94436500n^2 - 68148950n^3 - 1899953n^5) + 4(n-1)^2(1406250 + 1621875n - 6959000n^2) + 5762450n^3 - 1866010n^4 + 214003n^5) - 4m(-5718750 + 21213125n - 27985375n^2 + 14726550n^3 + 1320440n^4 - 1178459n^5 + 263349n^6)). \quad (19)$$

The solution of Eq. (17) leads to $n = m + 3$ or $n = 5(2+m)/4$.

In what follows we find the solution of Eq. (13):

Case (1): when $n = m + 3$, Eq. (18) leads to $m = 2, 16/3, -1$.

First when $m = 2$, we get $h(x) = h_0x^5$, $r(x) = r_0x^2$ and in the original variable $f(x) = (h_0/r_0)x^3$, $g(x) = 1/r_0x^2$, by solving the first auxiliary equation in (6) [we get $\phi(x,t)$] and substituting into second auxiliary equation in (6) to find the arbitrary time dependent function of integration, we get $C_{1t}(x,t) = 0$, or $C_1(x,t) = C_1(x)$ and so, we get $\phi(x,t)$ as

$$\phi(x,t) = \frac{2 + (x - e^{\frac{10h_0 t}{r_0}})C_1(x)}{2(e^{\frac{10h_0 t}{r_0}} - x)c_2(x,t)}. \qquad (20)$$

By substituting from (20), into (5) we get a solution of (13) as

$$u(x,t) = -\frac{2h_0 x^3 (x^2 + e^{\frac{20h_0 t}{r_0}} + 4x e^{\frac{10h_0 t}{r_0}})}{(-e^{\frac{10h_0 t}{r_0}} + x)^2}. \qquad (21)$$

It is worth noticing that one can verify that the solution given by (21) satisfies (13).

When $m = 16/13$ in a way similar to the above, we get $h(x) = h_0 x^{\frac{55}{13}}$, $r(x) = r_0 x^{\frac{16}{13}}$ and in the original variable $f(x) = (h_0/r_0)x^3$, $g(x) = 1/(r_0 x^{\frac{16}{13}})$, also the auxiliary equation (6) solve to

$$\phi(x,t) = -\frac{22e^{\frac{2310h_0 t}{2197 r_0}} + 195x^{\frac{2}{13}} + 26e^{\frac{2310h_0 t}{2197 r_0}} x C_1(x,t) + 169x^{\frac{15}{13}} C_1(x,t)}{52e^{\frac{2310h_0 t}{2197 r_0}} x c_2(x,t) + 338x^{\frac{15}{13}} c_2(x,t)}, \qquad (22)$$

and we get the solution of Eq. (13) as

$$u(x,t) = -\frac{48h_0 x^{\frac{33}{13}}}{(2e^{\frac{2310h_0 t}{2197 r_0}} + 13x^{\frac{2}{13}})^2}. \qquad (23)$$

When $m = -1$, we get the following results

$$\phi(x,t) = -\frac{6e^{\frac{3h_0}{5r_0}t} + 40x^{\frac{3}{5}} + 15e^{\frac{3h_0}{5r_0}t} x C_1(x,t) + 25x^{\frac{8}{5}} C_1(x,t)}{30e^{\frac{3h_0}{5r_0}t} + 50x^{\frac{8}{5}} C_1(x,t)}, \qquad (24)$$

$$u(x,t) = \frac{h_0(261e^{\frac{6h_0}{5r_0}t} + 870e^{\frac{3h_0}{5r_0}t} x^{\frac{3}{5}} - 1975x^{\frac{6}{5}})}{25(3e^{\frac{3h_0}{5r_0}t} + 5x^{\frac{3}{5}})^2} \qquad (25)$$

Case (2): when $n = 5(2+m)/4$, Eq. (18) leads to $m = 3/2$ and we get the following results

$$\phi(x,t) = -\frac{\left(1 - \frac{28h_0 t}{9r_0}\right)(2 + 3xC_1(x,t) + 12x^{\frac{1}{3}} + 9x^{\frac{4}{3}} C_1(x,t)}{6\left(1 - \frac{28h_0 t}{9r_0}\right) x c_2(x,t) + 18x^{\frac{4}{3}} c_2(x,t)}, \qquad (26)$$

$$u(x,t) = -\frac{972h_0 r_0^2 x^2}{(28h_0 t - 9r_0(1 + 3x^{\frac{1}{3}}))^2}. \qquad (27)$$

Again, the solutions (23) or (25) or (26) verify Eq. (13).

4. Conclusions

In this paper, we suggested the extended unified method for finding exact solutions to evolution equations with variable coefficients. A wide class of exact solutions to KdV equation with space dependent coefficients is obtained. The method and the solutions that were obtained here are completely new and we can use this method to find exact solutions of coupled evolution equations. But in this case we think that parallel computations should be used.

*hamdyig@yahoo.com
†Corresponding author. nasser.elazab@yahoo.com
‡mofatzi@yahoo.com

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