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On multiple soliton similariton-pair solutions, conservation laws via multiplier and stability analysis for the Whitham–Broer–Kaup equations in weakly dispersive media

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In this article, the generalized unified method (GUM) is used for finding multiwave solutions of the coupled Whitham-Broer-Kaup (WBK) equation with variable coefficients. Which describes the propagation of of shallow water waves. Here, we study the effects of the indirect nonlinear interaction of one-, two- and three-solitonic similaritons on the behavior of propagation of waves, in quasi-periodic distributed system. This study can unable us to control the dynamics of type soliton (soliton, anti-soliton) similaritons waves in dispersive waveguides. To give more physical insight to the obtained solutions, they are shown graphically. Their different structures are depicted by taking appropriate arbitrary functions. Further, with the suitable parameters, the indirect nonlinear interaction between two and three-soliton waves are shown weal, in the sense that their amplitude does not blow up. Moreover, because of the importance of conservation laws Cls and stability analysis SA in the investigation of integrability, internal properties, existence, and uniqueness of a differential equation, we compute the Cls via multiplier technique and stability analysis via the concept of linear stability analysis for the WBK equations using the constant coefficients.

KEYWORDS

conservation laws, similariton, stability analysis, the generalized unified method (GUM), Whitham–Broer–Kaup WBK equation

1 | INTRODUCTION

In recent years, studies on nonlinear evolution equations (NLEE) are devoted to investigate various phenomena in nonlinear science. Focusing attention on different branches, such as fluid mechanics, plasma physics, chemistry, biology, solid-state materials, etc.[1-5] To this issue , various methods were suggested in the literature in the last few decades. With relevance to multiwave solutions.[6-13]

On the other hand, when the inhomogeneities of media are taken into account, that is the NLEE are considered with space variable coefficients. This case represents more realistic models than those which are considered with constant coefficients.[14-17] The soliton wave propagation in the above-mode has been called "non-autonomous soliton." Hence, N-soliton nonautonomous soliton interactions in various play a definitive role in the formation of the structure of wave and also the propagation direction with a phase shift in dispersive media.[18-24]

The dispersive long (surface) water waves with time-dependent coefficient are given by the equations

$$u_t + u u_x + v_x - \beta(t) u_{xx} = 0\,,$$
$$v_t + (uv)_x + \alpha(t) u_{xxx} - \beta(t) v_{xx} = 0\,, \tag{1}$$

where $u(x, t)$ is the field of horizontal velocity, $v(x, t)$ is the height waves, $\alpha(t)$ and $\beta(t)$ are the dispersion coefficients varying. If $\alpha(t)$ and $\beta(t)$ are real constant, then the Equation 1 represents the WBK equation with constant coefficients, which had been suggested by Whitham,[25] Broer,[26] and Kaup.[27] The cases of Equation 1 are classified as follows: when $\alpha(t)$= 1 and $\beta(t) = 0$, the Equation 1 is reduced to the propagation of long waves in the shallow water.[28,29]

Motivated by the above mentioned works, in this paper, the solutions of Equation 1 are constructed from which the integrability of Equation 1 can be verified, see also Abdel-Gawad and Abdel-Gawad and Tantawy[30-34] of Equation 1. Explicit multi(two and three) soliton solutions of the Equation 1 are obtained. In the study of stability of homogeneous solutions, influence of the group-velocity dispersion equation is shown. The effects of $\beta(t)$ on the propagation and interaction of the nonautonomous waves. Geometric structures are discussed via graphical presentation in the case of weakly dispersive medium, where the wave-amplitude does not grow with time.

Furthermore, because of the importance of Cls and SA in the investigation of integrability, internal properties, existence, and uniqueness of a differential equation, several authors have studied the Cls and SA for various nonlinear differential equations.[35-40] Thus, we compute the Cls via multiplier technique and SA via the concept of linear stability analysis for the governing equation.

This paper is organized as follows. In Section 2, the generalized unified method (GUM) are classified. Section 3, is devoted to considered multisoliton solutions over using the numerical results to illustrate the dynamical of waves. In Section 4, we present the Cls via multiplier and SA via the concept of linear stability analysis. The conclusion is addressed in Section 5.

2 | THE GUM METHOD

We will briefly present the main steps of the GUM method that will be applied to the nonlinear Equation 1, as in the following steps:

Step 1. Let us have the general evolution equation

$$F_k(x,\, t,\, u_{i_1}, u_{i_2},\, \dots u_{i_1 t}, u_{i_2 t} \dots) = 0,\; k,\, i_j, = 1, 2 \dots s, \tag{2}$$

where F is polynomial in their argument and x and t are missing Equation 3 that has traveling wave solutions (TWS) (or here is called semiself similar). In this case, Equation 2 reduces to
and take the wave transformation

$$u_i(x, t) = U_i(\xi),\; \xi = \kappa x + \int_0^t \omega(\tau)\, d\tau, \tag{3}$$

$$G_k\left(U_{i_1}, U_{i_2},\, \dots U'_{i_1}, U'_{i_2},\, \dots U''_{i_1}, U''_{i_2} \dots\right) = 0,\; U' = \frac{dU_i}{d\xi},\; i_j = 1, 2,\, \dots s. \tag{4}$$

Step 2. We suppose that Equation 4 has forms of a polynomial or a rational solution with ordinary differential equation, which can be take as

(i) Polynomial function solutions with ordinary equation is

$$u(\xi) = \sum_{i=0}^{n} a_i\, \varphi^i(\xi),\; (\varphi'(\xi))^p = \sum_{i=0}^{k} c_j\, \varphi^i(\xi),\; p = 1, 2, \tag{5}$$

where n is found by using the balance condition from highest order derivative term and the highest order nonlinear term of Equation 1, while k is determined from the consistency condition. Because the solitary wave is an exact balance between nonlinearity and dispersion, it was long assumed that rather special initial conditions were necessary to make one.

(ii) Rational function solutions.

The rational function solution can be written as follows:

$$u(\xi)=\frac{\sum_{i=0}^{n}a_i\varphi^i(\xi)}{\sum_{i=0}^{r}q_i\varphi^i(\xi)},\ (\varphi'(\xi))^p=\sum_{i=0}^{k}c_j\varphi^i(\xi),\ p=1,2, \tag{6}$$

where a_i, q_i, and c_j are unknown parameters.

Step 3. Two and three wave solutions are can be obtained by accounting for two and three ordinary differential equations.

The rational function solutions (two-, three-nonautonomous, or similariton-pair solitons) take the following expression:

$$u(\varphi_1,\varphi_2)=\frac{a_0+\sum_{i=1}^{2}a_i\varphi_i+a_3\varphi_1\varphi_2}{q_0+\sum_{i=1}^{2}q_i\varphi_i+q_3\varphi_1\varphi_2}, \tag{7}$$

$$u(\varphi_1,\varphi_2,\varphi_3)=\frac{a_0+\sum_{i=1}^{3}a_i\varphi_i+a_4\varphi_1\varphi_2+a_5\varphi_2\varphi_3+a_6\varphi_2\varphi_3+a_7\varphi_1\varphi_2\varphi_3}{q0+\sum_{i=1}^{3}a_q\varphi_i+q_4\varphi_1\varphi_2+q_5\varphi_2\varphi_3+q_6\varphi_2\varphi_3+q_7\varphi_1\varphi_2\varphi_3}.$$

$$\varphi_i'(\xi_j)=c_i\varphi_j(\xi_j)+c_{j0},\ i=j=1,2,3.$$

Step 4. We substitute Equations 5 to 7 into (4) and solving the ordinary equations. As a result of this substitution with (Mathematica, Matlab, or other programming), we get a polynomial of φ_i.

We calculate all the coefficients of same power of φ_i to 0. The conditions of variable coefficients and the explicit solutions of Equation 4 solution are determination.

If possible, we may conclude with the uniform formula of N-nonautonomous solutions for any $N \geq 1$. Here, we mention that ξ_j and t are independent variables.

Step 5. Ensure that the solutions are satisfies the Equation 4.

It is worth notching that there $\varphi_i, i = 1,2,3$ are exponential function method (when $k = p = 1$).[10] This the present method is the same as the Exp-function expansion method. Otherwise includes, it is generalized mentioned method when $k > 1, orp > 1$.

3 | MULTIPLE SOLITON SIMILARITON-PAIRS

In this section, we give the detailed description of the semiself-similar solutions of Equation 1 including one, two-, and three-soliton similariton pairs.

Here, we use the transformation $v(x,t) = u_x(x,t)$ and substituting by this condition into Equation 1, we find that, the first of Equation 1 (or in the second in [1] and integration with neglecting the constant of integration with taken $\alpha = 1$), is equivalent to the variant Burger equation

$$u_t+uu_x+\gamma(t)u_{xx}=0. \tag{8}$$

By bearing in mind that $\gamma(t) = 1 - \beta(t)$.

3.1 | One-soliton

By considering the wave transformations $u(x,t) = u(\xi_1,t)$, and $\xi_1=\kappa_1x+\int_0^t\omega_1(\tau)d\tau$, we change the Equation 8 to the following equation:

$$\omega(t)u_{\xi_1}+\kappa_1uu_{\xi_1}+\kappa_1^2\gamma(t)u_{\xi_1\xi_1}=0, \tag{9}$$

when $p = 1$. By taking $n = r$ (when $k = 1$) and by using Equation 6, the solution of Equation 9 has the form

$$u(\xi_1,t)=\frac{a_1\varphi(\xi_1)+a_0}{q_1\varphi(\xi_1)+q_0},$$
$$\varphi_{\xi_1}=c_0-c_1\varphi(\xi_1), \tag{10}$$

by substituting from Equation 10 into 9, we find that the solutions are

$$
\begin{aligned}
u(x,t) &= \frac{1}{q_0}\left[a_0 + \frac{2k_1 q_1 \gamma(t)\sigma_1\left(c_1 e^{\xi_1 c_1} - c_{10}\right)}{c_1\left(q_1 e^{\xi_1 c_1} + q_0\right) - c_{10} q_1}\right] \\
v(x,t) &= u_x(x,t), \\
a_1 &= \frac{q_1\left(a_0 + 2k_1\gamma(t)\left(\sigma_1\right)\right)}{q_0}, \ \omega_1(t)=k_1\left(k_1 c_1 \gamma(t) - \frac{a_1}{q_1}\right) \\
\sigma_1 &= c_1 q_0 - c_{10} q_1,
\end{aligned} \tag{11}
$$

where q_i and $c_j, i, j = 0, 1$ are arbitrary parameters.

3.2 | Two-soliton

In this case, the obtained analytical solutions for two-soliton solutions, we write transformation $u(x,t) = u(\xi_1, \xi_2)$, $\xi_j = \kappa_j x + \int_0^t \omega_j(\tau)d\tau$, and $j = 1, 2$. Substituting Equation 7_1 into Equation 8 and using Mathematica, then equating to 0, each coefficient of the same order power of the ordinary functions φ_i yields a set of equations. Suppose that Equation 7_1 admits a solution of the form

$$
\begin{aligned}
u(\xi_1,\xi_2) &= P_2(\varphi_{1,2})/Q_2(\varphi_{1,2}), \\
P_2(\varphi_{1,2}) &= e^{\xi_2 c_2} c_2\left(-2k_1 k_2 c_1 \gamma(t)\sigma_{12}\right. \\
&\quad + \left(c_1\left(q_3(t)e^{\xi_1 c_1} + q_2\right) - c_{10} q_3\right) \\
k_2^2 c_2 \gamma(t) - \omega_2(t)\big) &- c_1 \sigma_{21} e^{\xi_1 c_1}\left(k_2^2 c_2 \gamma(t) + \omega_2\right) \\
Q_2(\varphi_{1,2}) &= k_2\left(c_2 e^{\xi_2 c_2}\left(c_1\left(q_3(t)e^{\xi_1 c_1} + q_2\right) - c_{10} q_3\right)\right. \\
&\quad \left. c_1 \sigma_{21} + e^{\xi_1 c_1}\right),
\end{aligned} \tag{12}
$$

where

$$
\begin{aligned}
\omega_1(t) &= k_1\left(k_1 c_1 \gamma(t) - k_2 c_2 \gamma(t) + \frac{\omega_2(t)}{k_2}\right), \\
\sigma_{12} &= c_1 q_2 - c_{10} q_3, \ \sigma_{21} = c_2 q_1 - c_{20} q_3,
\end{aligned} \tag{13}
$$

and q_i, c_j, and $c_{0j}, i, = 0, 1, 2, j = 1, 2.$ are arbitrary parameters.

3.3 | Three-soliton

By the substitution of Equation 7_2 into Equation 8, three-soliton solutions can be obtained with the iterative algorithm of the GUM.

According to the solution of Equation 8 with auxiliary equations is

$$
\begin{aligned}
u(x,t) &= P_3(\varphi_{1,2,3})/Q_3(\varphi_{1,2,3}), \ v = u_x \\
P_3(\varphi_{1,2,3}) &= c_1 c_2\left(c_3 q_4 - c_{30} q_7\right)\left(-e^{\xi_1 c_1 + \xi_2 c_2}\right) \\
\kappa_3^2 c_3 \gamma(t) &+ \omega_3(t) - e^{\xi_3 c_3} c_3\left(c_2 e^{\xi_2 c_2}\right. \\
\left(c_1 q_5 - c_{10} q_7\right) &\left(\kappa_3 \gamma\left(2\kappa_1 c_1 - \kappa_3 c_3\right) + \omega_3(t)\right) \\
&+ e^{\xi_1 c_1} c_1\left(c_2 q_7\left(-e^{\xi_2 c_2}\right)\left(\kappa_3^2 c_3 \gamma(t) - \omega_3(t)\right)\right. \\
&\quad + \left(c_2 q_6 - c_{20} q_7\right)\left(\kappa_3 \gamma(t)\left(2\kappa_2 c_2 - \kappa_3 c_3\right) + \omega_3(t)\right)))) \\
Q_3(\varphi_{1,2,3}) &= \kappa_3\left(c_1 c_2\left(c_3 q_4(t) - c_{30} q_7\right) e^{\xi_1 c_1 + \xi_2 c_2}\right. \\
&\quad + e^{\xi_3 c_3} c_3\left(c_2\left(c_1 q_5 - c_{10} q_7\right) e^{\xi_2 c_2}\right. \\
&\quad \left.\left. c_1 + e^{\xi_1 c_1}\left(c_2\left(q_7(t)e^{\xi_2 c_2} + q_6\right) - c_{20} q_7\right)\right)\right),
\end{aligned} \tag{14}
$$

where

$$
\omega_1(t)=k_1\left(k_1 c_1 \gamma(t) - k_2 c_2 \gamma(t) + \frac{\omega_2(t)}{k_2}\right), \ \omega_2(t)=k_2^2 c_2 \gamma(t), \tag{15}
$$

and we take $q_i = 1, \ i = 4, 6, 7$, but a_7 and $\kappa_i, i = 1, 2, 3$ are arbitrary parameters.

We have stated the observable variable-coefficient effect $\gamma(t)$ effect on multisoliton waves controlling and collsion. In Figure 1A describes the similariton pairs of to kind waves via two and three-soliton and two, three-antisoliton. Here, weak collision is shown, and soliton amplitude is proportional to the diffusion coefficient $\gamma(t)$. Moreover, Figure 1A shows tunneling in the periodic two-soliton waves. We remark that amplitude does not grow with time. This agree with the weak dispersive. Figure 1B shows periodic lattice wave with tunnelling.

Although solitons in Figure 1A,B hold larger wave amplitude than those of Figure 1C,D. They both can propagate for long distances under $\gamma(t)$ is periodic and pulses waves.

Also the dynamical characteristics of two kinds of soliton including the amplitude, width, and phase have the same values for along-distance distribution.

The Figures 2A and 2B show multigraded index and periodic zigzag propagation for the three- soliton collisions at $v(x, t)$. The Figures 3A, 3B, and 3C, the propagations of two-soliton incoming in upper layer and two-antisoliton outgoing in

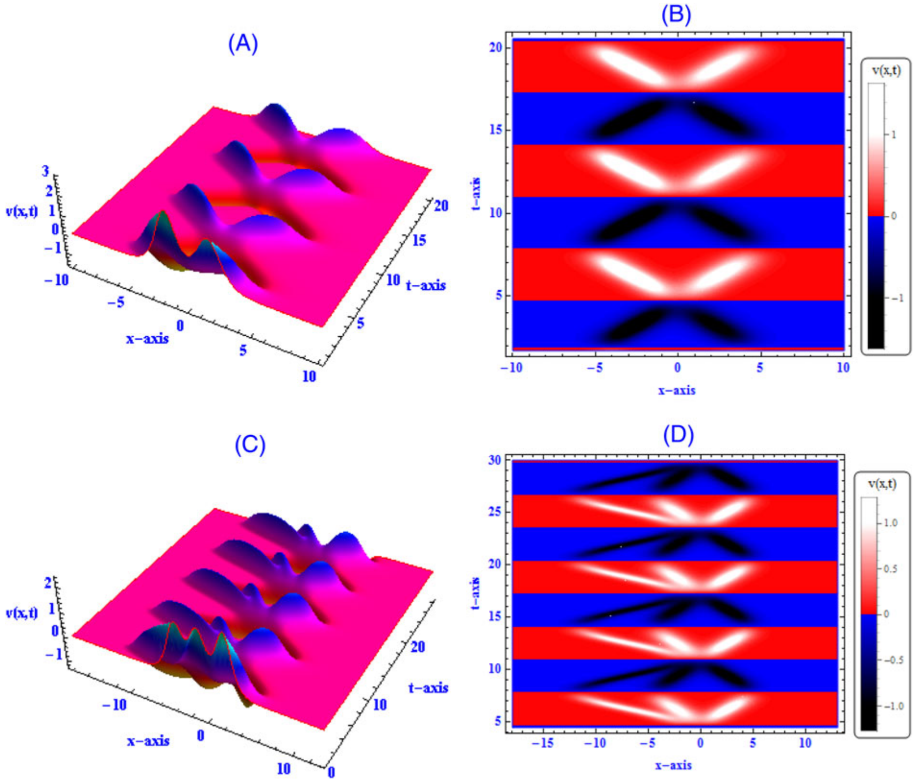

FIGURE 1 A, b, c, and d, show 3D and control plot for the solutions of Equations 12 and 15 are depicted against x and t. Parameters are chosen as A, B, $c_1(t) = 2$, $c_2(t) = 4$, $c_{10}(t) = 0.2$, $c_{20}(t) = -0.1$, $\kappa_1 = 0.9$, $\kappa_2 = -0.4$, $\gamma(t) = \cos(t) + sech(t)$, and C, D, $c_1(t)=10$, $c_2(t)=5$, $c_3(t)=4$, $c_{10}(t)=0.2$, $c_{20}(t)=0.4$, $c_{30}(t)=0.3$, $\kappa_1=0.3$, $\kappa_2=0.3$, $\kappa_3= -0.4$ and $\gamma(t)=\cos(t)+\text{sech}(t)$. In this figures, $q_i(t) = 1$, $i = 4, 5, 6, 7$ [Colour figure can be viewed at wileyonlinelibrary.com]

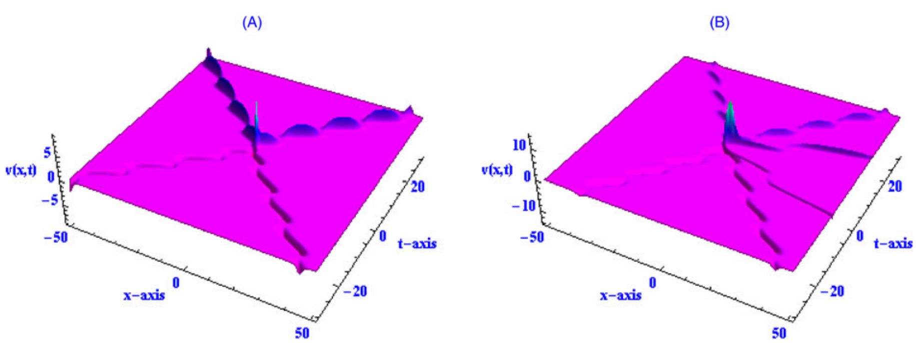

FIGURE 2 A, and B, are the solutions of Equations 12 and 15 are depicted against x and t, where the free parameters are the same values of Figure 1 except$\gamma(t) = \text{sn}(t, 0.5) + \tanh(t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

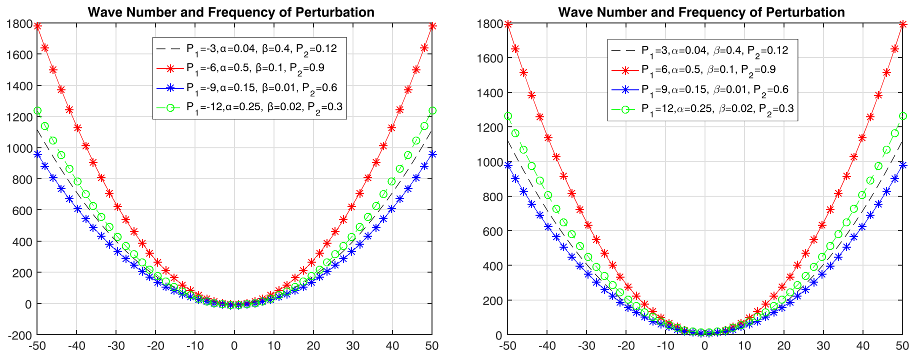

FIGURE 3 Frequency of the perturbation against the wave number with different parameter values [Colour figure can be viewed at wileyonlinelibrary.com]

the lower layer. Here, no periodic waves are propagating along the $t - axis$.

Moreover, Figure 2A shows the effects of indirect interaction propagation of three on the periodic soliton waves, while Figure 2B shows multiperiodic lattice with tunneling. We have stated the observable variable-coefficient effect($\gamma(t)$) on multisoliton waves controlling and collusion. In Figures 1 and 2 describe the similariton pairs of to kind waves via two and three-soliton and two, three-antisoliton. Here, weak collision is shown, and soliton amplitude is proportional to the diffusion coefficient $\gamma(t)$

4 | CONSERVATION LAWS FOR EQUATION (1) BY MULTIPLIER

Consider N independent variable such as $X = (X_1, X_2, \dots, X_N)$ and $U = (U^1, U^2, \dots, U^M)$, be M dependent variables. Take into consideration, also a system of R PDEs with $K^{th}-$order as[35]

$$Q_\gamma[U] = Q_\gamma(X, U, U_{(1)}, U_{(2)}, \dots, U_{(K)}), \quad \gamma = 1, 2, \dots, R, \tag{16}$$

with $U_{(1)} = \{U^\gamma_{(i)}\}, U_{(2)} = \{U^\gamma_{(ij)}\}, \{U^\gamma_{(i)}\} = \frac{\partial U^\gamma_i}{\partial X_i}, \{U^\gamma_{(ij)}\} = \frac{\partial^2 U^\gamma}{\partial X_i \partial X_j}, \dots$, assume that $u = (u^2, u^2, \dots, u^n)$ represents an arbitrary functions of the independent variables X and depicts partial derivatives $\frac{\partial}{\partial x_i}$ by subscripts i. ie, $u^\sigma_i = \frac{\partial u^\sigma}{\partial X_i}, u^\sigma_{ij} = \frac{\partial^2 u^\sigma}{\partial X_i X_j}$, etc.

1. $$D_i = \frac{\partial}{\partial X_i} + U^\gamma_i \frac{\partial}{\partial U^\gamma_i} + U^\gamma_{ij} \frac{\partial}{\partial U^\gamma_i} + U^\gamma_{ijk} \frac{\partial}{\partial U^\gamma_{jk}} + \dots, \tag{17}$$

   where $i, j, K, \dots = 1, 2, \dots, M$.
2. The multipliers of Equation 16 are a function $\{\Xi^\gamma[u]\}$ such that

   $$\Xi^\gamma[u] Q_\gamma[u] = D_i T^i[u], \tag{18}$$

   for some functions $T^i[u]$. If $u^\sigma = u^\sigma(X)$ is a solution for Equation 16, then Equation 18 gives the Cls[36]

   $$D_i T^i[u] = 0, \tag{19}$$

   for Equation 16 and for every i, $T^i[u]$ is a flux.
3. The well-known Euler generators possessing the differential function u^j and the derivatives $u^i_j, u^j_{i_1 i_2} \dots$ are generators stated by
4. $$E^j_u = \frac{\partial}{\partial u^j} - D_i \frac{\partial}{\partial u^j_i} + \dots + (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u^j_{i_1 \dots i_s}}, \tag{20}$$

for every $j = 1, 2, \ldots, M\{\Xi^{\gamma}[u]\}$ generates a set of multipliers for a Cls of Equation 16 provided that each Euler generator Equation 20 eradicates the left side of Equation 18

$$E_u^j(\Xi^{\gamma}[u]Q_{\gamma}[u]) \equiv 0, \quad j = 1, \ldots, N, \tag{21}$$

for arbitrary u, u_i, u_{ij}...etc.

Suppose that α and β are an arbitrary constant in Equation 1, from determining Equation 21 for multipliers, we attain the first-order multipliers $\Xi^1(x, t, u, v, u_x, v_x, u_t, v_t)$, $\Xi^2(x, t, u, v, u_x, v_x, u_t, v_t)$ for the governing equation presented by

$$\begin{aligned} &\Xi^1 = 0, \\ &\Xi^2 = c_1\alpha u_{xx} - c_1\beta v_x + c_1 uv + c_2\beta u_x + \frac{1}{2}c_2u^2 + c_2v, \end{aligned} \tag{22}$$

where c_1 is an arbitrary constant. Therefore, the multipliers for the nontrivial local Cls involving the cases isolated by free constants can be obtained as

$$\begin{aligned} &\Xi^1 = 0, \\ &\Xi^2 = 1. \end{aligned} \tag{23}$$

Subsequently, we obtain the following fluxes:

$$\begin{aligned} &T^t = 0, \\ &T^x = \alpha u_{xx} - \beta v_x + uv. \end{aligned} \tag{24}$$

When

$$\begin{aligned} &\Xi^1 = 1, \\ &\Xi^2 = 0, \end{aligned} \tag{25}$$

we obtain the following fluxes

$$\begin{aligned} &T^t = 0, \\ &T^x = \beta u_x + \frac{1}{2}u^2 v. \end{aligned} \tag{26}$$

The presented conservation laws could be constructed from pairs of symmetries and adjourn symmetries using the general theory presented in Ma.[37,38] This result will be investigated in the future studies.

4.1 | Stability analysis to Equation 1

In this subsection, the concept of linear stability analysis[39-44] will be applied to investigate the stability analysis for the governing equation. Suppose that α and β are an arbitrary constant in Equation 1, then by considering the perturbed solution of the form

$$\begin{aligned} u(x,t) &= P_1 + \epsilon w(x,t) \\ v(x,t) &= P_2 + \tau r(x,t), \end{aligned} \tag{27}$$

it is easy to see that any constants P_1 and P_2 are a steady state solution of Equation 1. Inserting Equation 27 into Equation 1, one gets

$$\begin{aligned} \epsilon w_t + \tau r_x + \epsilon P_1 w_x + \epsilon^2 w w_x + \epsilon\beta w_{xx} &= 0, \\ \tau r_t + \tau P_1 r_x + \epsilon\tau w r_x + \epsilon\tau r w_x + \epsilon P_2 w_x - \beta\tau r_{xx} + \alpha\epsilon w_{xxx} &= 0, \end{aligned} \tag{28}$$

linearizing (28) in ϵ and τ give

$$\begin{aligned} \epsilon w_t + \tau r_x + \epsilon P_1 w_x + \epsilon\beta w_{xx} &= 0, \\ \tau r_t + \tau P_1 r_x + \epsilon P_2 w_x - \beta\tau r_{xx} + \alpha\epsilon w_{xxx} &= 0. \end{aligned} \tag{29}$$

Suppose that Equation 29 has a solution of the form

$$\begin{aligned} w(x,t) &= \alpha_1 e^{(ikx + i\sigma t)} \\ r(x,t) &= \alpha_2 e^{(ikx + i\sigma t)}, \end{aligned} \tag{30}$$

where k is the normalized wave number, substituting Equation 30 into Equation 29 yields

$$\begin{aligned} ik^2\epsilon\beta\alpha_1 + \epsilon\sigma\alpha_1 + k\epsilon P_1\alpha_1 + k\tau\alpha_2 = 0, \\ -ik^3\alpha\epsilon\alpha_1 + ik\epsilon P_2\alpha_1 + k^2\beta\tau\alpha_2 + i\tau\sigma\alpha_2 + ik\tau P_1\alpha_2 = 0. \end{aligned} \tag{31}$$

Collecting terms with α_1, α_2 gives

$$\begin{pmatrix} \epsilon(ik^2\beta + \sigma + kP_1) & k\tau \\ -ik\epsilon(k^2\alpha - P_2) & \tau(k^2\beta + i\sigma + ikP_1) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{32}$$

and taking the determinant of the above yields

$$ik^4\alpha\epsilon\tau + ik^4\epsilon\beta^2\tau + i\epsilon\tau\sigma^2 + 2ik\epsilon\tau\sigma P_1 + ik^2\epsilon\tau P_1^2 - ik^2\epsilon\tau P_2 = 0. \tag{33}$$

Solving for σ yields

$$\sigma(k) = -kP_1 \pm i\sqrt{k^4\alpha + k^4\beta^2 + k^2P_2}. \tag{34}$$

The relations for the dispersion in Equation 34 will be investigated. The sign of the real part (Re) of σ suggests either the solution will become bigger or vanish in a given period of time. When the sign of Re for $\sigma(k)$ is negative for all k values, then any superposition of solutions of the form $e^{(i\sigma t + ikx)}$ will come to vanished. In other words, if the Re is positive for some values of k, then with time some components of a superposition will become bigger rapidly. The former case is called stable, whereas the latter is unstable. If the maximum of the Re is exactly 0, the situation is called marginally stable. It is more difficult to assess the long-term behavior in this case. Thus, from Equation 34, one can observe that the Re is always negative for $k > 0$, which implies that the dispersion relation is stable. If $k < 0$, the Re will be positive, hence in the case the dispersion is unstable. When $k = 0$, the Re will be 0, which suggests that the dispersion is marginally stable in this case.

5 | CONCLUSIONS

With the help of the GUM and symbolic computation, we have investigated the possibility of supporting multisolitons in a variable-coefficient for Equation 1 and given the graphical analysis. Figures have been plotted for us to analyze the propagation and collusion of several kinds of rational soliton solutions for long distance distrbution through the choice of free parameters.

Further, some main features of solutions have been shown. These obtained solutions can be used to describe the possible construct technicality for fluid, oceanic, and long wave phenomenon in shallow water. Moreover, because of the importance of Cls and SA in the investigation of integrability, internal properties, existence, and uniqueness of a differential equation, we computed the Cls via multiplier technique and stability analysis via the concept of linear stability analysis for the Equation 1 when α and β are constants.

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