Maxwell's equations: Time-Harmonic

$$\nabla \times \underline{E}_h = -j\omega \underline{B}_h \qquad (1)$$

$$\nabla . \underline{D}_h = \rho_h \qquad (2)$$

$$\nabla \times \underline{H}_h = \underline{J}_h + j\omega \underline{D}_h \qquad (3)$$

$$\nabla . \underline{B}_h = zero \qquad (4)$$

Fields in Good conductors at Low freq

($\varepsilon = \varepsilon_o \varepsilon_r$, $\mu = \mu_o \mu_r$, $\sigma >>$ zero, $\rho =$ zero) + low ω

$$\nabla \times \underline{H}_h = \sigma \underline{E}_h + j\omega\varepsilon \underline{E}_h$$

$$\frac{\sigma}{\omega\varepsilon} >> 1$$

$$\nabla \times \underline{H}_h = \sigma \underline{E}_h$$

$$\nabla \times \underline{H}_h = \underline{J}_h$$

Displacement current is neglected

Fields in Good conductors at Low freq

($\varepsilon = \varepsilon_o \varepsilon_r$, $\mu = \mu_o \mu_r$, $\sigma >>$ zero, $\rho =$ zero) + low ω

$$\nabla^2 \underline{H}_h - j\omega\mu\sigma \underline{H}_h + \omega^2 \mu\varepsilon \underline{H}_h = zero$$

$$\Rightarrow \nabla^2 \underline{H}_h - j\omega\mu\sigma \underline{H}_h = zero$$

Magnetic Diffusion Eq.

$$\nabla^2 \underline{H}_h + \gamma^2 \underline{H}_h = zero$$

$$\gamma = \frac{(1-j)}{\delta}$$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

$$\underline{J}_h = \nabla \times \underline{H}_h$$

$$\underline{J}_h = \sigma \underline{E}_h$$

Solving Magnetic Diffusion Eq.

$$\nabla^2 \underline{H}_h + \gamma^2 \underline{H}_h = zero$$

$$\gamma = \frac{(1-j)}{\delta}$$

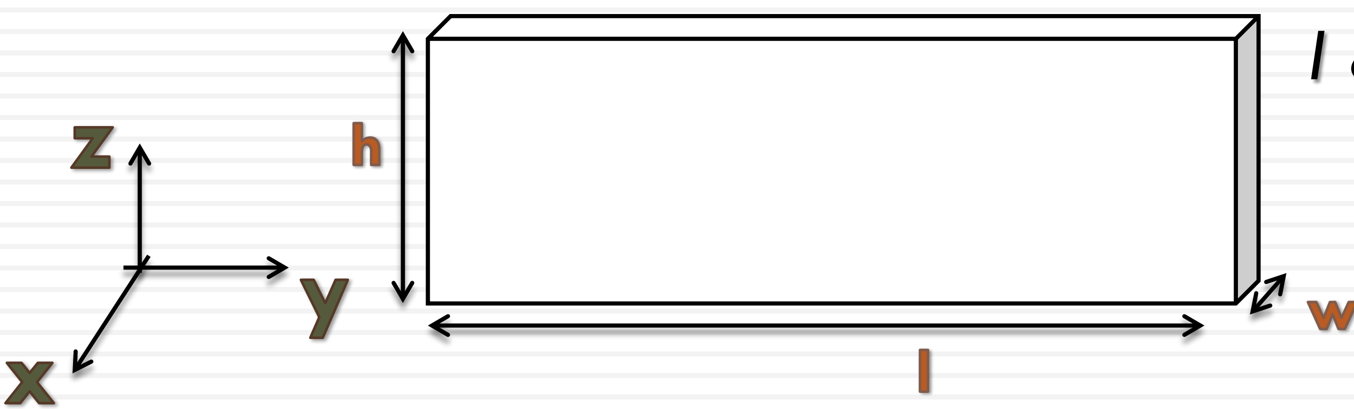
$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

Solution of magnetic diffusion equation:

1. Consider simplifying the equation to make H <u>function in only one variable</u>. (i.e. consider some dimensions infinite)

$$\underline{H}_h(\underline{r}) = \underline{H}_h(u)$$

u = x or y or z



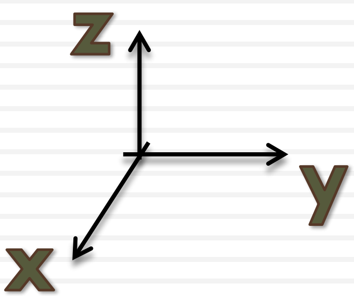
l & h>> w

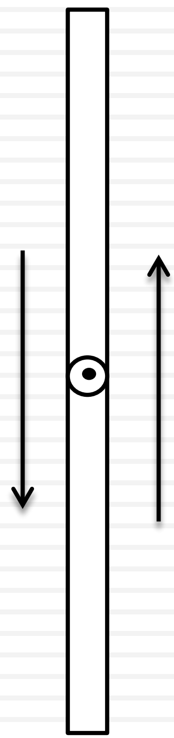
Solving Magnetic Diffusion Eq.

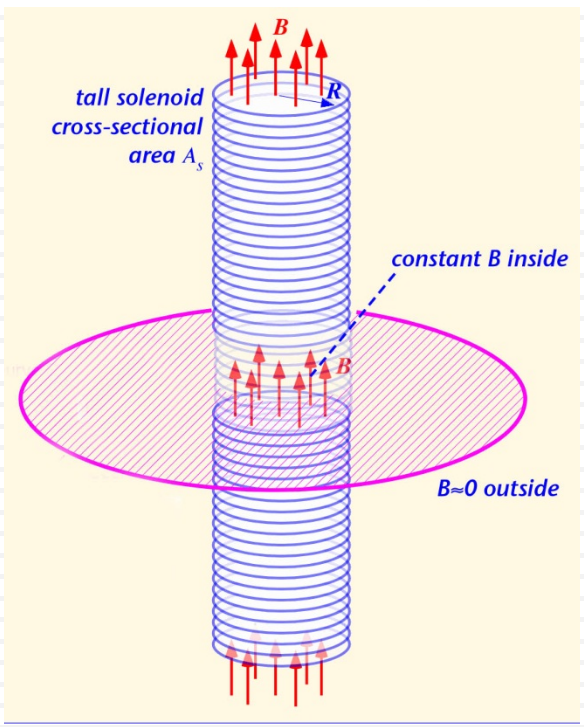
2. Find the direction of $\underline{H}$. (using RHR)

$$\underline{H}_h(u) = H_v(u)\underline{u}_v$$

$\underline{u}_v = \underline{u}_x$ or $\underline{u}_y$ or $\underline{u}_z$







Solving Magnetic Diffusion Eq.

3. Solve the O.D.E:

$$\frac{\partial^2 H_v(u)}{\partial u^2} + \gamma^2 H_v(u) = zero$$

$$\Rightarrow M^2 + \gamma^2 = zero \quad \Rightarrow M = \pm j\gamma$$

$$\Rightarrow M = \pm j\frac{(1-j)}{\delta} \quad \Rightarrow M = \pm\frac{(1+j)}{\delta} \qquad \beta = \frac{(1+j)}{\delta}$$

$$\Rightarrow M = \pm\beta$$

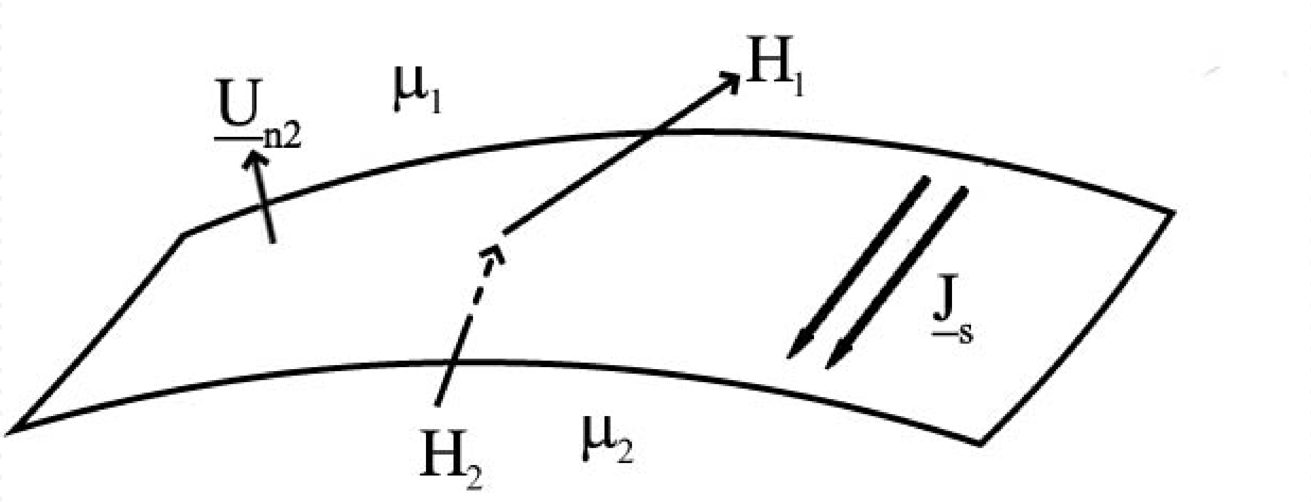
$$\Rightarrow H_v(u) = A_1 e^{\beta u} + C_1 e^{-\beta u}$$

Solving Magnetic Diffusion Eq.

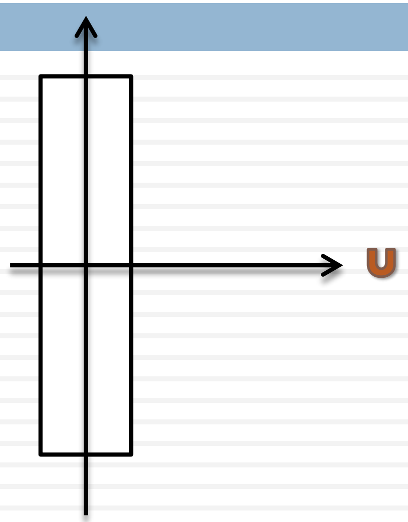
4. Get the constants (A_1 & C_1) from the boundary conditions:

$$H_{1t} - H_{2t} = J_s$$

$$B_{1n} = B_{2n}$$



Solving Magnetic Diffusion Eq.



Special Cases:

1. Even symmetry (i.e. $H(u) = H(-u)$)

$$A_1 = C_1$$

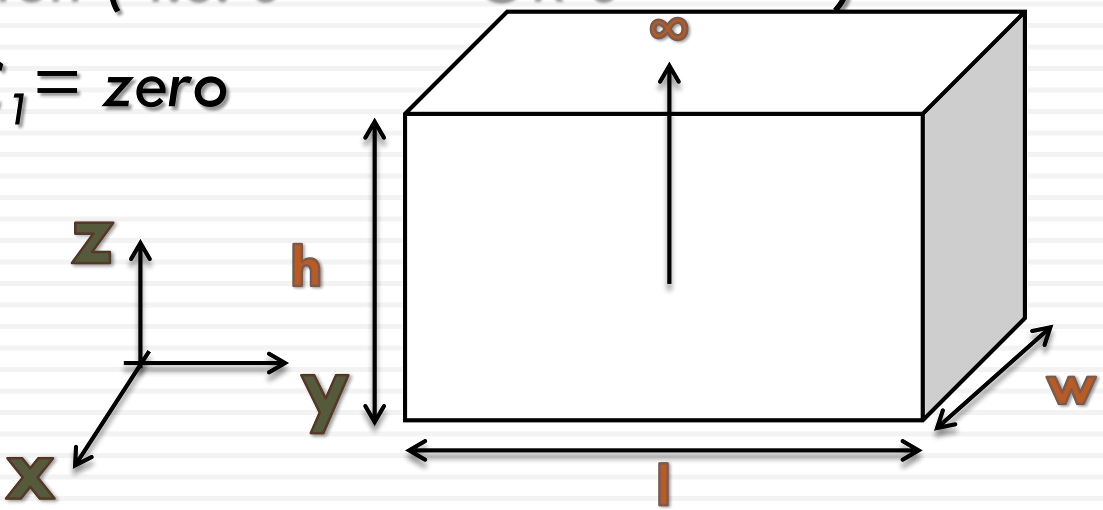
2. Odd symmetry (i.e. $H(u) = -H(-u)$)

$$A_1 = -C_1$$

3. Semi-infinite extension (i.e. $u \rightarrow \infty$ OR $u \rightarrow -\infty$)

$$A_1 = \text{zero OR } C_1 = \text{zero}$$

- w & $l >> h$
- h large $\rightarrow + \infty$



Eddy Currents & Skin Effect

h large $\longrightarrow +\infty$

$$\underline{H} = H_o \cos(\omega t)\underline{u}_x$$

$$\underline{H}(t) = \mathrm{Re}[\underline{H}_h\, e^{j(\omega t)}]$$

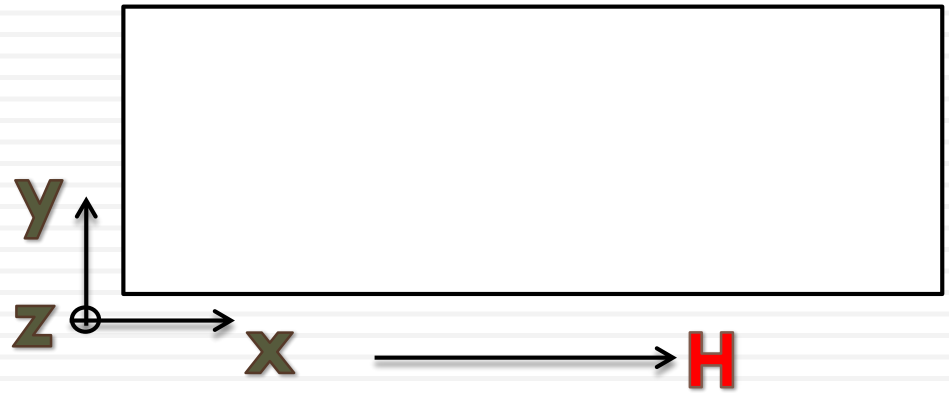
$$\underline{H}_h = H_o \underline{u}_x$$

Eddy Currents & Skin Effect

$$\nabla^2 \underline{H}_h + \gamma^2 \underline{H}_h = zero$$

$$\gamma = \frac{(1-j)}{\delta}$$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$



1. Assumptions:

- *infinitely extended in x and z.*
- *semi infinite slab ($y \rightarrow +\infty$).*

$$\underline{H}_h(\underline{r}) = \underline{H}_h(y)$$

2. Direction of H:

$$\underline{H}_h = H_x(y)\underline{u}_x$$

Eddy Currents & Skin Effect

3. Solve the O.D.E

$$\frac{\partial^2 H_x(\mathrm{y})}{\partial y^2} + \gamma^2 H_x(\mathrm{y}) = zero$$

$$H_x(\mathrm{y}) = A_1 e^{\beta y} + C_1 e^{-\beta y}$$

$$\beta = \frac{(1+j)}{\delta}$$

Since $y \rightarrow +\infty$ then A_1 zero

$$H_x(\mathrm{y}) = C_1 e^{-\beta y}$$

4. Get the constant from B.C: Boundary between air and conduction material at y = zero

$$H_{air.x}(0) = H_o$$

$$H_{cond.x}(0) = C_1$$

Eddy Currents & Skin Effect

$$H_{1t} - H_{2t} = \cancel{J_s}$$

$$H_{1t} = H_{2t}$$

$$H_o = C_1$$

$$\Rightarrow \quad H_x(\mathrm{y}) = H_o e^{-\beta y}$$

$$\underline{H}_h = H_o e^{-\beta y} \underline{u}_x$$

$$\underline{J}_h = \nabla \times \underline{H}_h = \begin{vmatrix} \underline{u}_x & \underline{u}_y & \underline{u}_z \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ H_x & 0 & 0 \end{vmatrix} = -\frac{\partial}{\partial y} H_x \underline{u}_z$$

$$\Rightarrow \quad J_z(\mathrm{y}) = \beta H_o e^{-\beta y}$$

Eddy Currents & Skin Effect

$$J_z(y) = \beta H_o e^{-\beta y}$$

$$\underline{J}_h = \beta H_o e^{-\beta y} \underline{u}_z$$

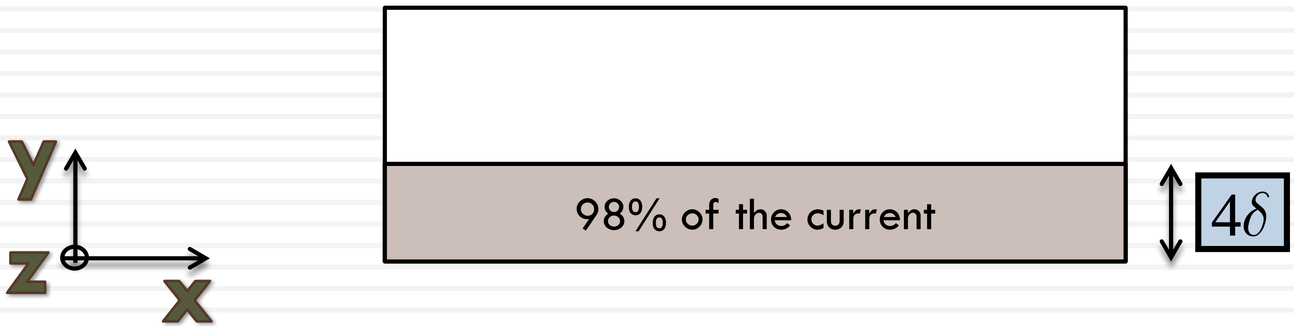
$$\underline{J}(\underline{r},t) = \mathrm{Re}[\beta H_o e^{-\beta y} e^{j\omega t}]\underline{u}_z$$

$$\beta = \frac{(1+j)}{\delta} = \frac{\sqrt{2}}{\delta} e^{j\frac{\pi}{4}}$$

$$\underline{J}(\underline{r},t) = \mathrm{Re}[\frac{\sqrt{2}}{\delta} H_o e^{-\frac{y}{\delta}} e^{j\frac{\pi}{4}} e^{-j\frac{y}{\delta}} e^{j\omega t}]\underline{u}_z$$

$$\underline{J}(\underline{r},t) = \frac{\sqrt{2}}{\delta} H_o e^{-\frac{y}{\delta}} \mathrm{Re}[e^{j(\frac{\pi}{4}-\frac{y}{\delta}+\omega t)}]\underline{u}_z$$

Eddy Currents & Skin Effect



$$\underline{J}(\underline{r},t)=\frac{\sqrt{2}}{\delta}H_o e^{-\frac{y}{\delta}}\cos(\omega t+\phi)\underline{u}_z$$

$$\phi=\frac{\pi}{4}-\frac{y}{\delta}$$

Skin depth

$$\delta=\sqrt{\frac{2}{\omega\mu\sigma}}$$

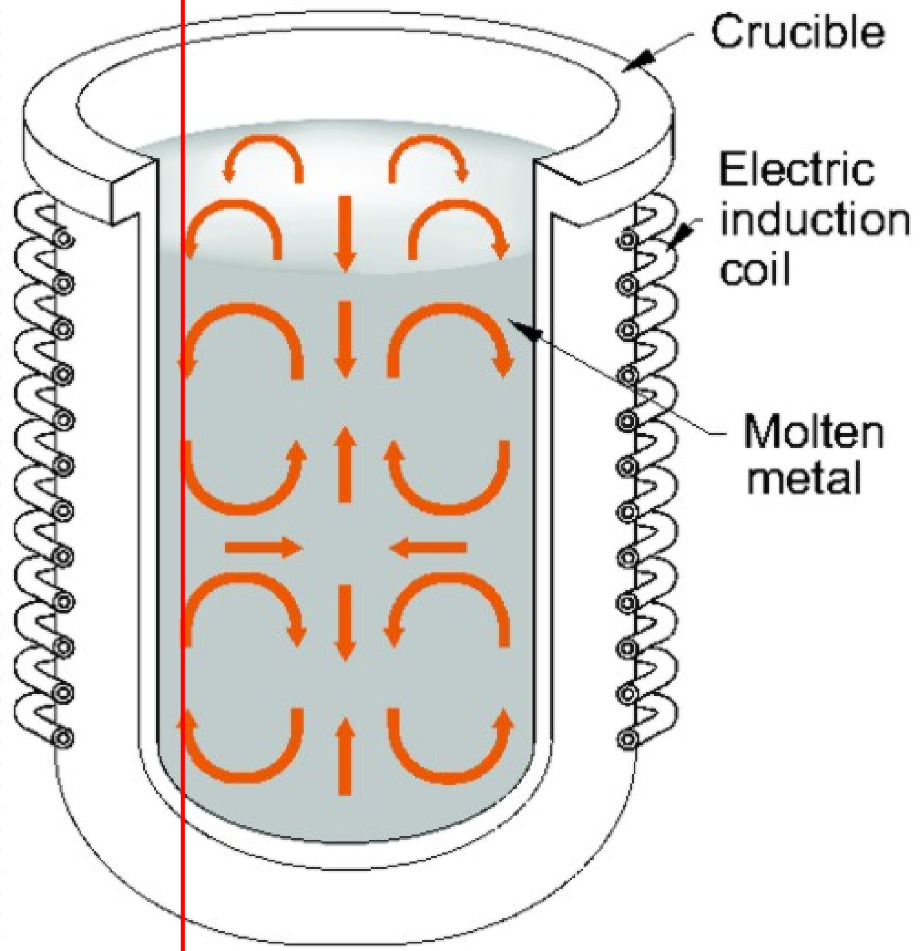
At power frequency: Copper $\delta \approx 10mm$

Aluminum $\delta \approx 12mm$

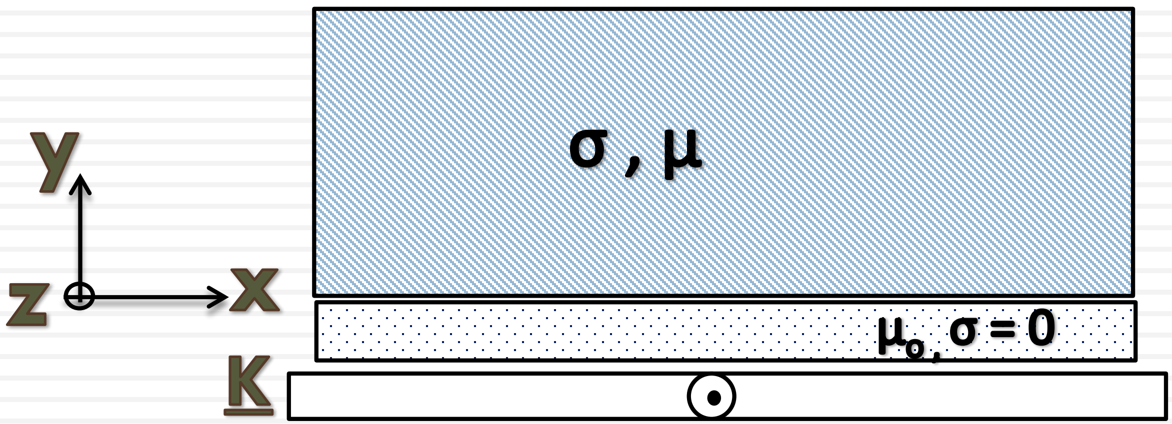
Applications

1. ***Induction Heating.***
2. ***Losses in transformer core.***
3. ***Bus bars. (skin effect)***
4. ***Magnetic Shielding.***
5. ***Magnetic Braking.***

1. Induction heating



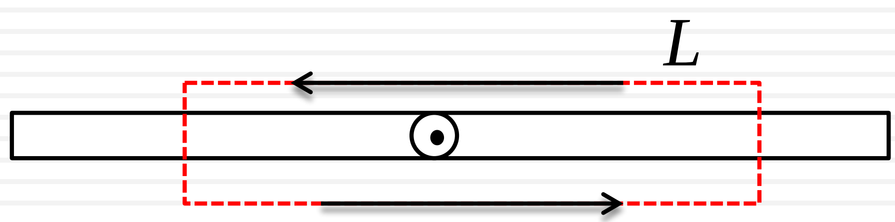
1. Induction heating



$$\underline{K} = K_o \cos(\omega t)\underline{u}_z$$

$$\underline{K}(t) = \mathrm{Re}[\underline{K}_h \, e^{j(\omega t)}]$$

$$\underline{K}_h = K_o \underline{u}_z$$



$$\oint_C \underline{H}.\underline{dl} = I_{en}$$

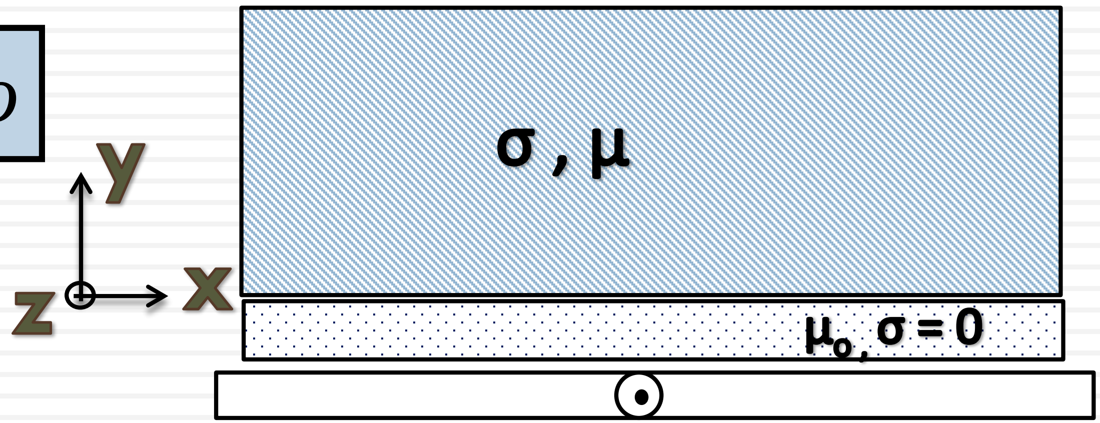
$$H*L + H*L = K_o * L$$

$$H_h = \frac{K_o}{2}$$

1. Induction heating

$$\nabla^2 \underline{H}_h + \gamma^2 \underline{H}_h = zero$$

$$\gamma = \frac{(1-j)}{\delta} \qquad \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$



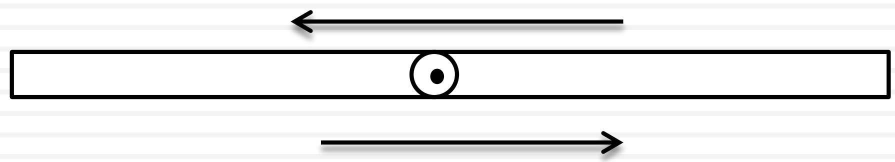
1. Assumptions:

- *infinitely extended in x and z.*
- *semi infinite slab (y → + ∞).*

$$\underline{H}_h(\underline{r}) = \underline{H}_h(y)$$

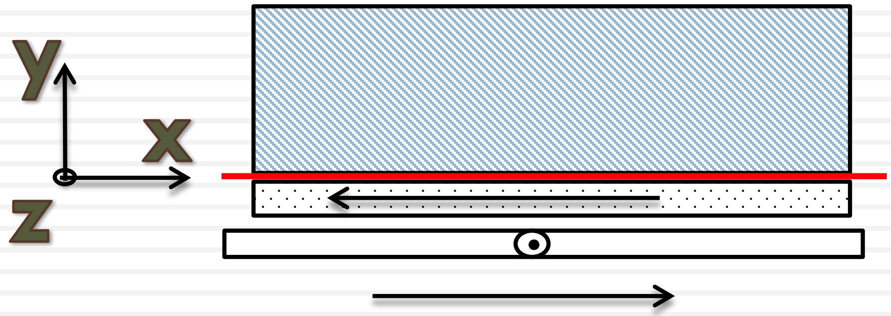
2. Direction of H: *using RHR*

$$\underline{H}_h = H_x(y)\underline{u}_x$$



1. Induction heating

3. Solve the O.D.E



$$\frac{\partial^2 H_x(y)}{\partial y^2} + \gamma^2 H_x(y) = zero$$

$$H_x(y) = A_1 e^{\beta y} + C_1 e^{-\beta y}$$

$$\beta = \frac{(1+j)}{\delta}$$

Since $y \rightarrow +\infty$ *then* A_1 *zero*

$$\Rightarrow \quad H_x(y) = C_1 e^{-\beta y}$$

4. Get the constant from B.C: *Boundary between insulator and conduction material at y = zero*

$$\Rightarrow \quad H_{ins.x}(0) = -\frac{K_o}{2}$$

$$H_{cond.x}(0) = C_1$$

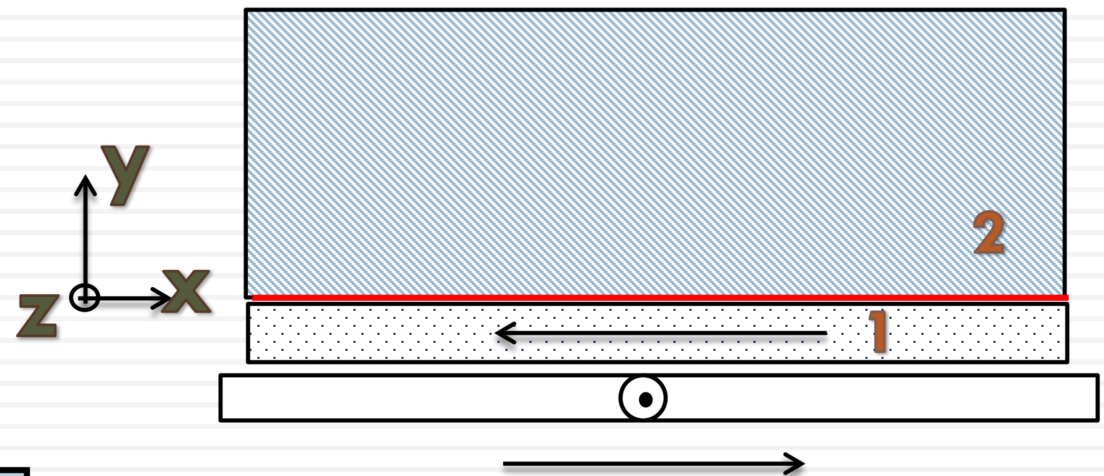
1. Induction heating

$$H_{1t} - H_{2t} = \cancel{J_s}$$

$$H_{1t} = H_{2t}$$

$$-\frac{K_o}{2} = C_1$$

$$H_x(\mathrm{y}) = -\frac{K_o}{2} e^{-\beta y}$$



$$\underline{H}_h = -\frac{K_o}{2} e^{-\beta y} \underline{u}_x$$

$$\underline{J}_h = \nabla \times \underline{H}_h = \begin{vmatrix} \underline{u}_x & \underline{u}_y & \underline{u}_z \\ 0 & \frac{\partial}{\partial y} & 0 \\ H_x & 0 & 0 \end{vmatrix} = -\frac{\partial}{\partial y} H_x \underline{u}_z$$

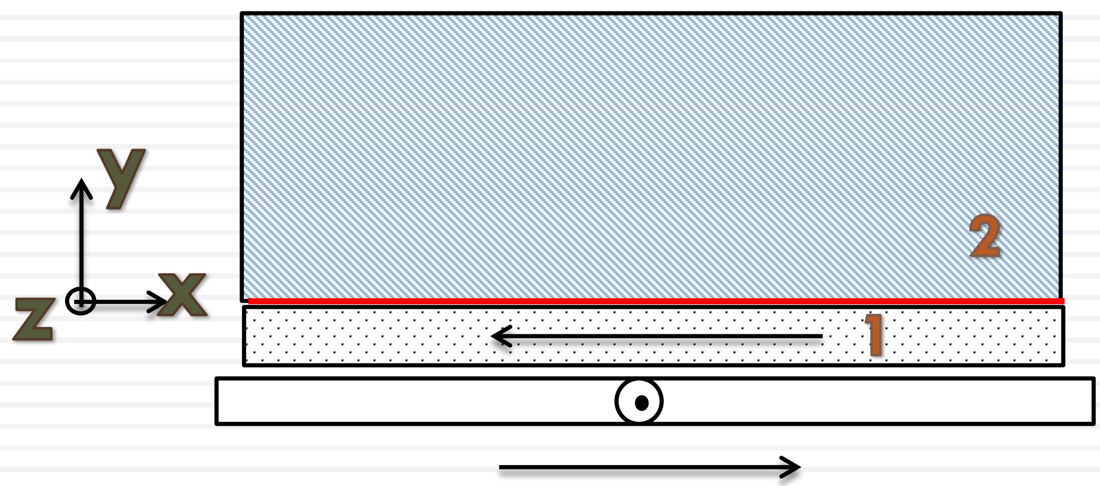
$$J_z(\mathrm{y}) = -\beta \frac{K_o}{2} e^{-\beta y}$$

1. Induction heating

$$\underline{H}_h = -\frac{K_o}{2} e^{-\beta y} \underline{u}_x$$

$$\underline{J}_h = -\beta \frac{K_o}{2} e^{-\beta y} \underline{u}_z$$

$$\underline{E}_h = -\beta \frac{K_o}{2\sigma} e^{-\beta y} \underline{u}_z$$



$$\underline{S}_h = \frac{1}{2} \underline{E}_h \times \underline{H}_h^*$$

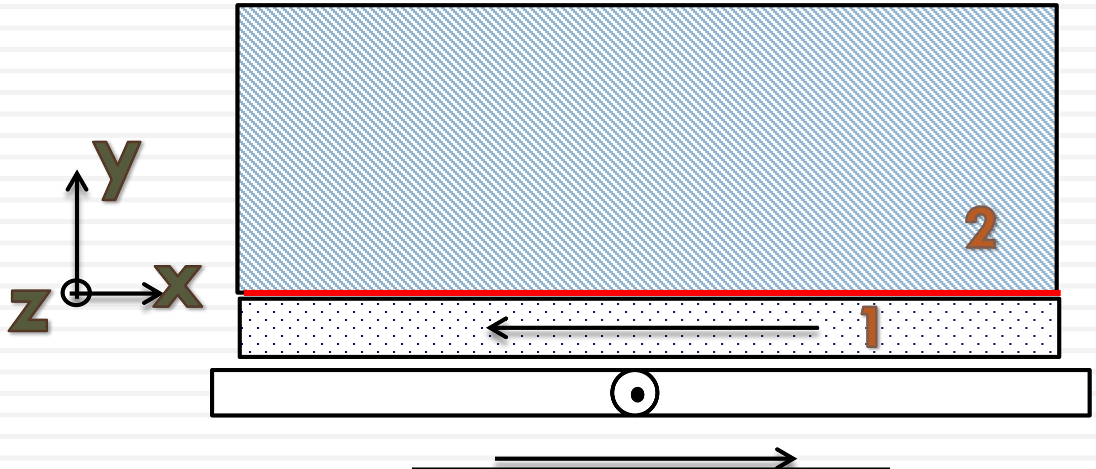
$$-\oint_S \underline{S}_h . \underline{dS} = P + jQ$$

$$\underline{S}_{y=0} = \frac{1}{2}\left(-\frac{\beta}{\sigma}\frac{K_o}{2}\underline{u}_z\right) \times \left(-\frac{K_o}{2}\underline{u}_x\right)$$

$$\underline{S}_{y=0} = \frac{1}{8}\frac{\beta K_o^2}{\sigma}(\underline{u}_y)$$

1. Induction heating

$$-\oint_S \underline{S}_h . \underline{dS} = P + jQ$$



$$\underline{S}_{y=0} = \frac{1}{8}\frac{\beta K_o^2}{\sigma}\underline{u}_y$$

$$\underline{dS} = dxdz(-\underline{u}_y)$$

$$-\oint_S \underline{S}_h . \underline{dS} = -\int_{x=0}^{x=1}\int_{z=0}^{z=1} -\frac{1}{8}\frac{\beta K_o^2}{\sigma}dxdz = \frac{1}{8}\frac{\beta K_o^2}{\sigma}$$

$$\beta = \frac{(1+j)}{\delta}$$

$$P + jQ = \frac{1}{8}\frac{(1+j)}{\delta}\frac{K_o^2}{\sigma}$$

$$P = \frac{1}{8}\frac{K_o^2}{\sigma\delta}$$

$$Q = \frac{1}{8}\frac{K_o^2}{\sigma\delta}$$