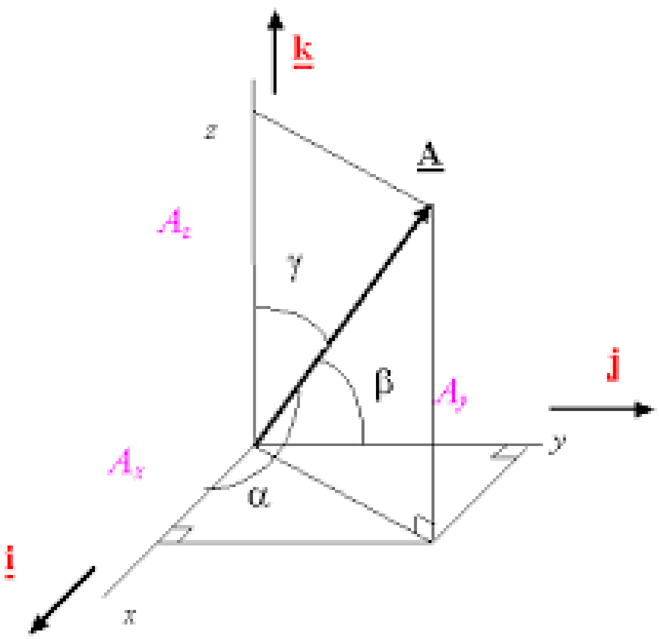
Vector Algebra and Calculus 1

In the framework of the undergraduate course:
"Applied Mathematics"



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Vector Algebra

Scalars

A physical quantity which is completely described by a single real number is called a **scalar**. Physically, it is something which has a magnitude, and is completely described by this magnitude. Examples are **temperature, density** and **mass**. In the following, lowercase (usually Greek) letters, e.g. α, β, γ, will be used to represent scalars.

Vectors

The concept of the **vector** is used to describe physical quantities which have both a magnitude and a direction associated with them. Examples are **force**, **velocity**, **displacement** and **acceleration**.

Geometrically, a vector is represented by an arrow; the arrow defines the direction of the vector and the magnitude of the vector is represented by the length of the arrow, Fig.

Analytically, vectors will be represented by lowercase bold-face Latin letters, e.g. **a**, **r**, **q**.

The **magnitude** (or **length**) of a vector is denoted by $|\mathbf{a}|$ or a. It is a scalar and must be non-negative. Any vector whose length is 1 is called a **unit vector**; unit vectors will usually be denoted by **e**.

(a) (b)

Figure : (a) a vector; (b) addition of vectors

Vector Algebra

The operations of addition, subtraction and multiplication familiar in the algebra of numbers (or scalars) can be extended to an algebra of vectors.

The following definitions and properties fundamentally *define* the vector:

1. Sum of Vectors:

   The addition of vectors **a** and **b** is a vector **c** formed by placing the initial point of **b** on the terminal point of **a** and then joining the initial point of **a** to the terminal point of **b**. The sum is written $\mathbf{c} = \mathbf{a} + \mathbf{b}$. This definition is called the parallelogram law for vector addition because, in a geometrical interpretation of vector addition, **c** is the diagonal of a parallelogram formed by the two vectors **a** and **b**, Fig. 1.1.1b. The following properties hold for vector addition:

   $$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \qquad \text{… commutative law}$$

   $$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} \qquad \text{… associative law}$$

2. The Negative Vector:

For each vector **a** there exists a **negative vector**. This vector has direction opposite to that of vector **a** but has the same magnitude; it is denoted by $-\mathbf{a}$. A geometrical interpretation of the negative vector is shown in Fig. 1.1.2a.

3. Subtraction of Vectors and the Zero Vector:

The **subtraction** of two vectors **a** and **b** is defined by $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$, Fig. 1.1.2b. If $\mathbf{a}=\mathbf{b}$ then $\mathbf{a}-\mathbf{b}$ is defined as the **zero vector** (or **null vector**) and is represented by the symbol **o**. It has zero magnitude and unspecified direction. A **proper vector** is any vector other than the null vector. Thus the following properties hold:

$$\mathbf{a}+\mathbf{o}=\mathbf{a}$$

$$\mathbf{a}+(-\mathbf{a})=\mathbf{o}$$

4. Scalar Multiplication:

The product of a vector **a** by a scalar α is a vector $\alpha\mathbf{a}$ with magnitude $|\alpha|$ times the magnitude of **a** and with direction the same as or opposite to that of **a**, according as α is positive or negative. If $\alpha = 0$, $\alpha\mathbf{a}$ is the null vector. The following properties hold for scalar multiplication:

$(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$ … distributive law, over addition of scalars

$\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$ … distributive law, over addition of vectors

$\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a}$ … associative law for scalar multiplication

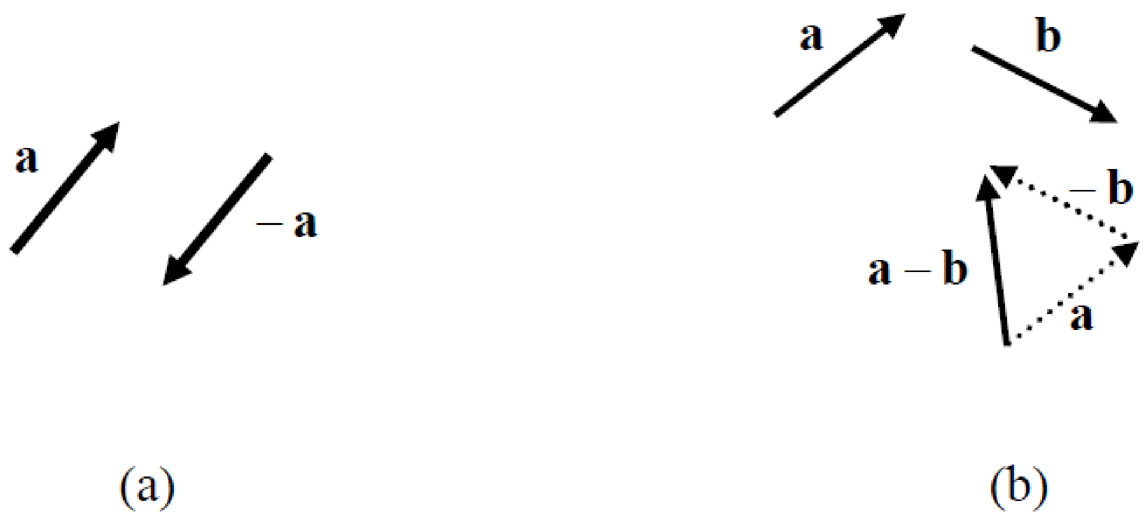


Figure 1.1.2: (a) negative of a vector; (b) subtraction of vectors

Note that when two vectors **a** and **b** are equal, they have the same direction and magnitude, regardless of the position of their initial points. Thus **a** = **b** in Fig. 1.1.3. A particular position in space is not assigned here to a vector – it just has a magnitude and a direction. Such vectors are called **free**, to distinguish them from certain special vectors to which a particular position in space is actually assigned.

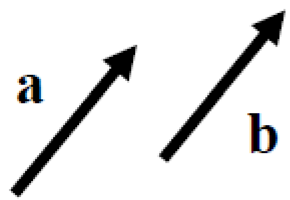


Figure 1.1.3: equal vectors

The vector as something with "magnitude and direction" and defined by the above rules is an element of one case of the mathematical structure, the **vector space**. The vector space is discussed in the next section, §1.2.

The Dot Product

The **dot product** of two vectors **a** and **b** (also called the **scalar product**) is denoted by $\mathbf{a} \cdot \mathbf{b}$. It is a scalar defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta . \qquad (1.1.1)$$

θ here is the angle between the vectors when their initial points coincide and is restricted to the range $0 \le \theta \le \pi$, Fig. 1.1.4.

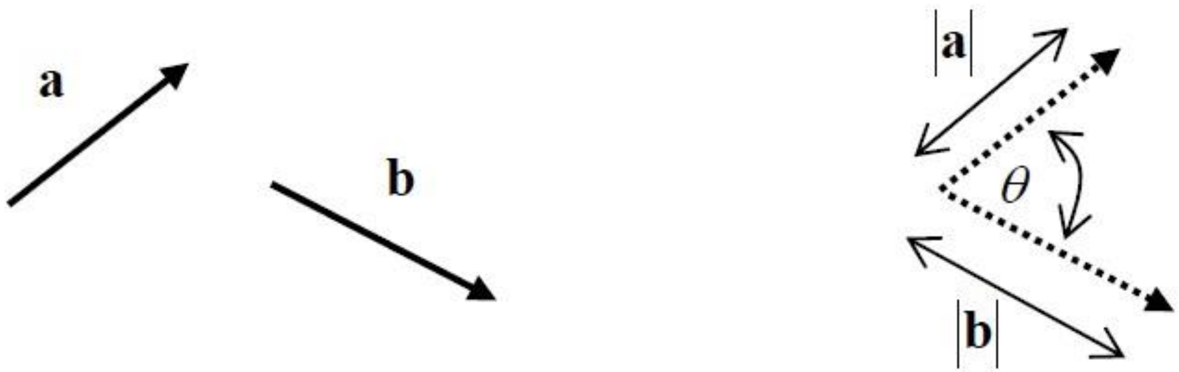
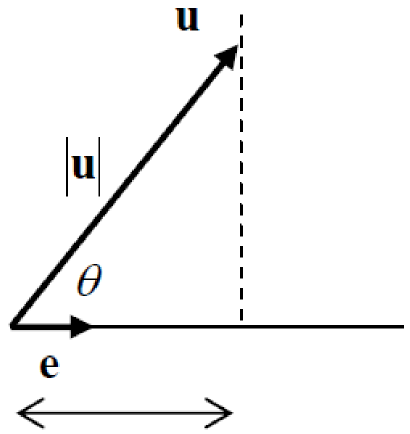


Figure 1.1.4: the dot product

An important property of the dot product is that if for two (proper) vectors **a** and **b**, the relation $\mathbf{a} \cdot \mathbf{b} = 0$, then **a** and **b** are perpendicular. The two vectors are said to be **orthogonal**. Also, $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}|\cos(0)$, so that the length of a vector is $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$.

Another important property is that the **projection** of a vector **u** along the direction of a unit vector **e** is given by $\mathbf{u} \cdot \mathbf{e}$. This can be interpreted geometrically as in Fig. 1.1.5.



$$\mathbf{u}\cdot\mathbf{e} = |\mathbf{u}|\cos\theta$$

Figure 1.1.5: the projection of a vector along the direction of a unit vector

It follows that any vector **u** can be decomposed into a component parallel to a (unit) vector **e** and another component perpendicular to **e**, according to

$$\mathbf{u} = (\mathbf{u}\cdot\mathbf{e})\mathbf{e} + [\mathbf{u} - (\mathbf{u}\cdot\mathbf{e})\mathbf{e}] \tag{1.1.2}$$

The dot product possesses the following properties (which can be proved using the above definition) {▲Problem 6}:

(1) $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ (commutative)
(2) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$ (distributive)
(3) $\alpha(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(\alpha\mathbf{b})$
(4) $\mathbf{a}\cdot\mathbf{a} \ge 0$; and $\mathbf{a}\cdot\mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{o}$

The Cross Product

The **cross product** of two vectors **a** and **b** (also called the **vector product**) is denoted by $\mathbf{a} \times \mathbf{b}$. It is a vector with magnitude

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta \qquad (1.1.3)$$

with θ defined as for the dot product. It can be seen from the figure that the magnitude of $\mathbf{a} \times \mathbf{b}$ is equivalent to the area of the parallelogram determined by the two vectors **a** and **b**.

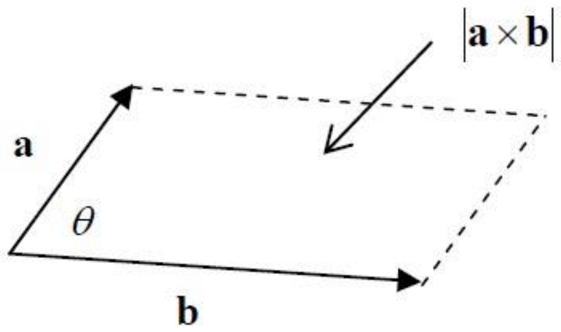


Figure 1.1.6: the magnitude of the cross product

The direction of this new vector is perpendicular to both **a** and **b**. Whether $\mathbf{a} \times \mathbf{b}$ points "up" or "down" is determined from the fact that the three vectors **a**, **b** and $\mathbf{a} \times \mathbf{b}$ form a **right handed system**. This means that if the thumb of the right hand is pointed in the direction of $\mathbf{a}\times\mathbf{b}$, and the open hand is directed in the direction of **a**, then the curling of the fingers of the right hand so that it closes should move the fingers through the angle θ, $0 \le \theta \le \pi$, bringing them to **b**. Some examples are shown in Fig. 1.1.7.

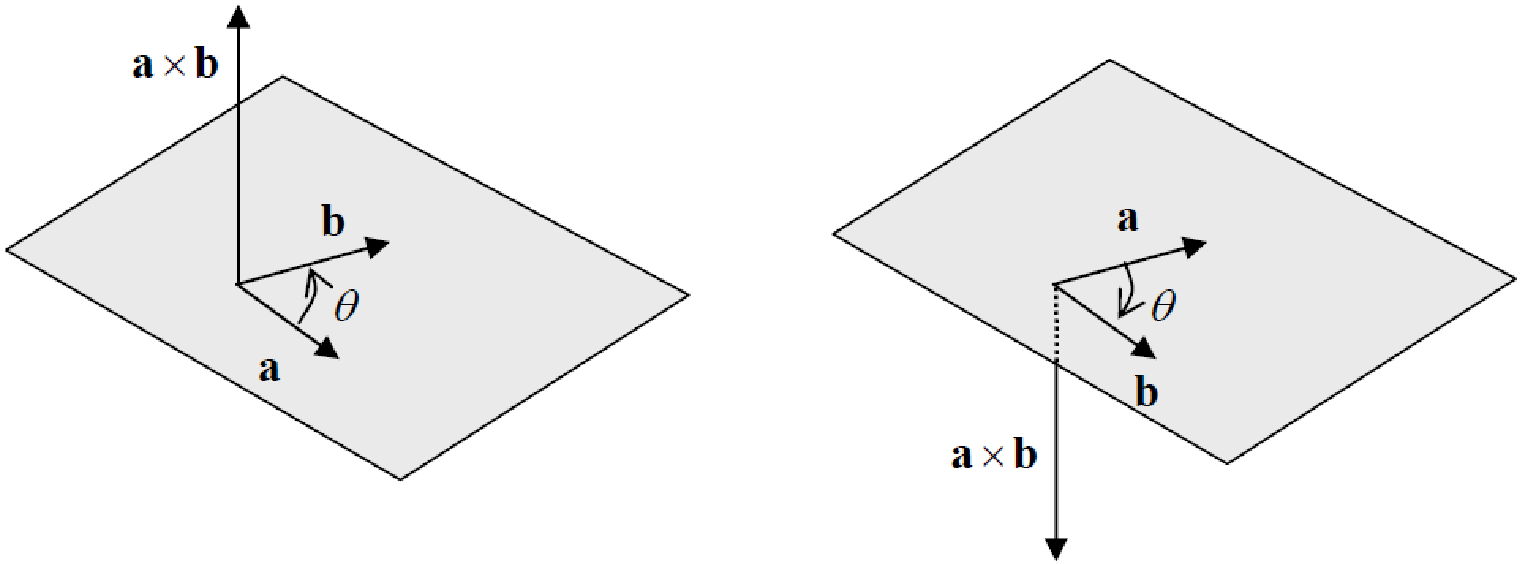


Figure 1.1.7: examples of the cross product

The cross product possesses the following properties (which can be proved using the above definition):

(1) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ (*not* commutative)

(2) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ (distributive)

(3) $\alpha(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(\alpha\mathbf{b})$

(4) $\mathbf{a}\times\mathbf{b} = \mathbf{o}$ if and only if **a** and **b** $(\neq \mathbf{o})$ are parallel ("linearly dependent")

The Triple Scalar Product

The **triple scalar product**, or **box product**, of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is defined by

$$\boxed{(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}=(\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}} \quad \textbf{Triple Scalar Product} \qquad (1.1.4)$$

Its importance lies in the fact that, if the three vectors form a right-handed triad, then the volume V of a parallelepiped spanned by the three vectors is equal to the box product.

To see this, let $\mathbf{e}$ be a unit vector in the direction of $\mathbf{u}\times\mathbf{v}$, Fig. 1.1.8. Then the projection of $\mathbf{w}$ on $\mathbf{u}\times\mathbf{v}$ is $h=\mathbf{w}\cdot\mathbf{e}$, and

$$\begin{aligned}\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) &= \mathbf{w}\cdot\left(|\mathbf{u}\times\mathbf{v}|\mathbf{e}\right)\\ &= |\mathbf{u}\times\mathbf{v}|h \\ &= V\end{aligned} \qquad (1.1.5)$$

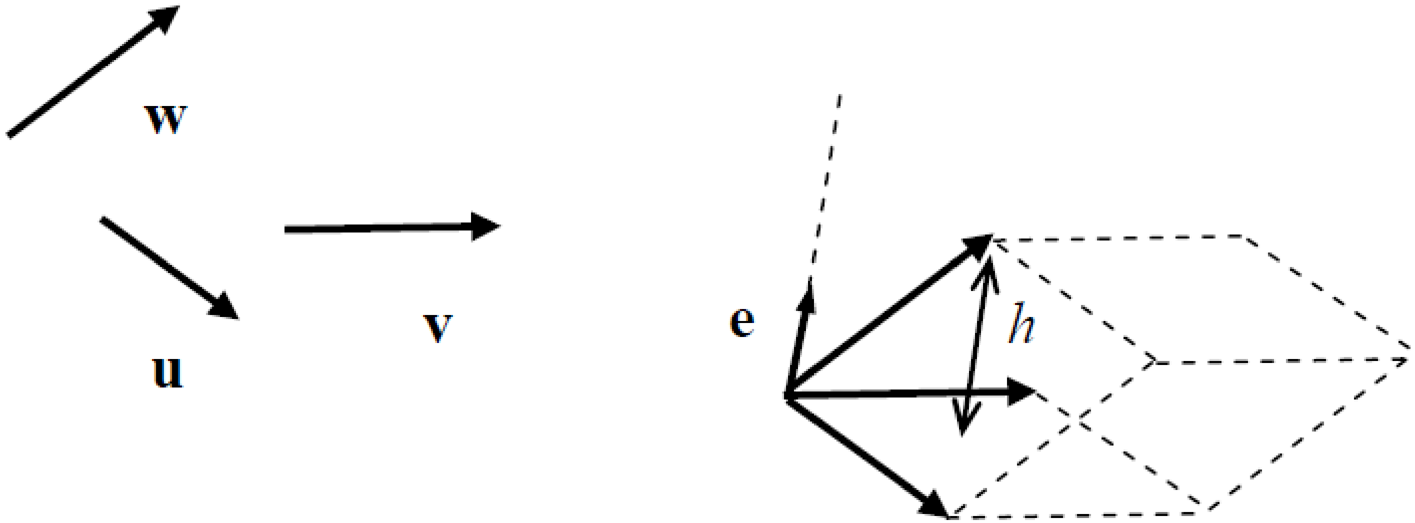


Figure 1.1.8: the triple scalar product

Note: if the three vectors do not form a right handed triad, then the triple scalar product yields the negative of the volume. For example, using the vectors above, $(\mathbf{w} \times \mathbf{v}) \cdot \mathbf{u} = -V$.

Vectors and Points

Vectors are objects which have magnitude and direction, but they do not have any specific location in space. On the other hand, a **point** has a certain position in space, and the only characteristic that distinguishes one point from another is its position. Points cannot be "added" together like vectors. On the other hand, a vector **v** can be added to a point **p** to give a new point **q**, $\mathbf{q} = \mathbf{v} + \mathbf{p}$, Fig. 1.1.9. Similarly, the "difference" between two points gives a vector, $\mathbf{q} - \mathbf{p} = \mathbf{v}$. Note that the notion of point as defined here is slightly different to the familiar point in space with axes and origin – the concept of origin is not necessary for these points and their simple operations with vectors.

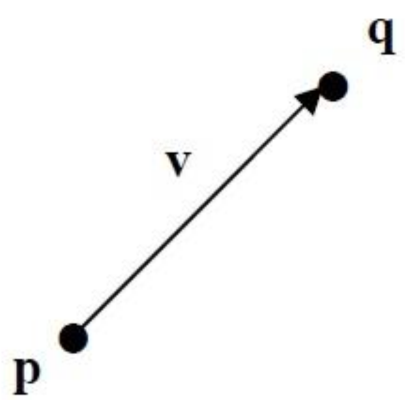
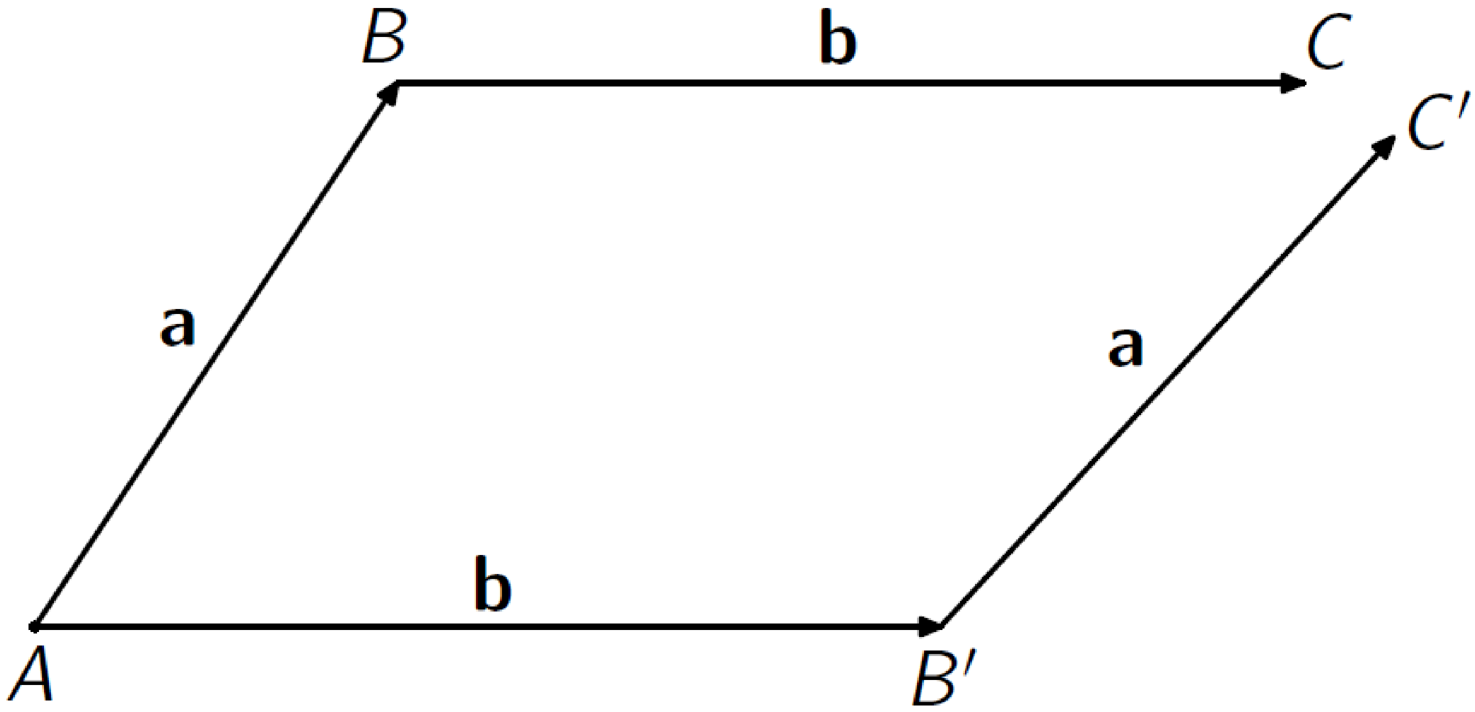


Figure 1.1.9: adding vectors to points

Parallelogram rule

Let $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$, $\overrightarrow{AB'} = \mathbf{b}$, and $\overrightarrow{B'C'} = \mathbf{a}$.
Then $\mathbf{a} + \mathbf{b} = \overrightarrow{AC}$, $\mathbf{b} + \mathbf{a} = \overrightarrow{AC'}$.



Wrong picture!

Parallelogram rule

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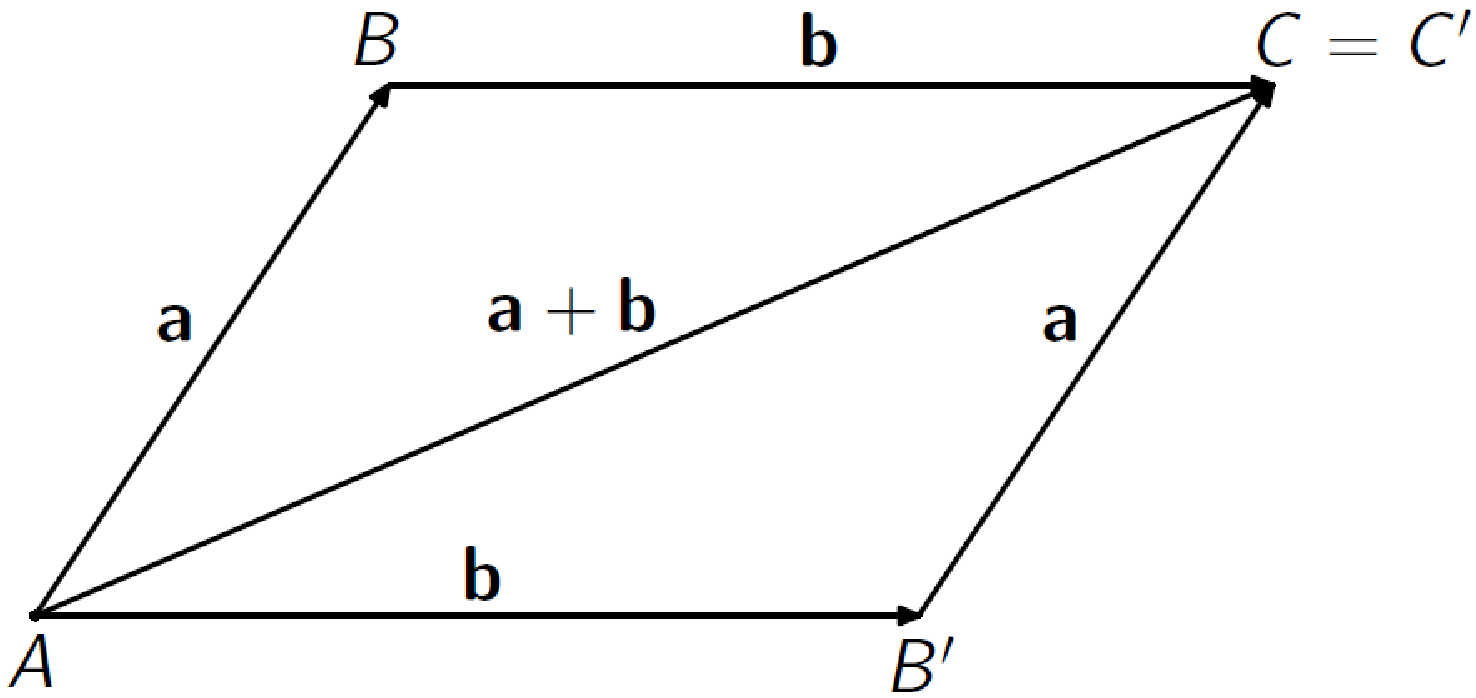


Right picture!

Provided we use an orthogonal coordinate system, the magnitude of a 3-vector is

$$a = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

and of an n-vector

$$a = |\mathbf{a}| = \sqrt{\sum_i a_i^2}$$

Summary

Key Point

A vector has both magnitude and direction, and both these properties must be given in order to specify it. A quantity with magnitude but no direction is called a scalar.

Key Point

The length of a vector $\overline{AB}$ is written as

$$AB \text{ or } |\overline{AB}|,$$

and the length of a vector $\mathbf{a}$ is written as

$$a \text{ (in ordinary type, or without the bar) or as } |\mathbf{a}|.$$

If two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, we write

$$\mathbf{a}//\mathbf{b}$$

Key Point

We can add two vectors $\mathbf{a}$ and $\mathbf{b}$ by making $\mathbf{b}$ start where $\mathbf{a}$ finishes, and completing the triangle. Alternatively, we can make $\mathbf{a}$ and $\mathbf{b}$ start at the same place, and take the diagonal of the parallelogram.

Key Point

$\mathbf{a} - \mathbf{b}$ means $\mathbf{a} + (-\mathbf{b})$

Key Point

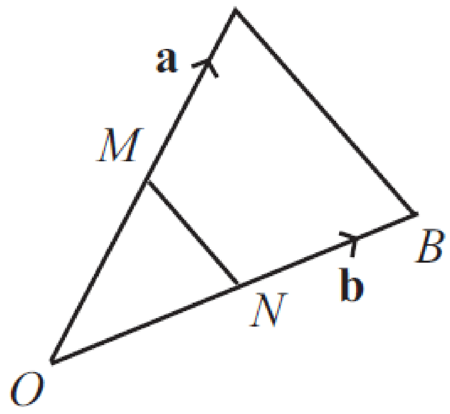
A vector $n\mathbf{a}$ is in the same direction as the vector $\mathbf{a}$, but n times as long.

Examples

Example 1

There is a useful theorem in geometry called the *mid-point theorem*. In this theorem, we take two points A and B, defined with respect to an origin O. Let us write $\mathbf{a}$ for the position vector of A, and $\mathbf{b}$ for the position vector of B. We can join A and B with a line, to give a triangle.

Now take the mid-point M of the line OA, and the mid-point N of the line OB, and join M to N with a line. Can we say anything about the relationship between the line MN and the line AB?



We can answer this very easily with vectors. We can write the vector for the line segment $\overline{AB}$ as $\overline{AO} + \overline{OB}$. Now $\overline{AO}$ is the reverse of the vector $\mathbf{a}$, so it is $-\mathbf{a}$. And $\overline{OB}$ is the same as the vector $\mathbf{b}$. Therefore

$$\begin{aligned} \overline{AB} &= \overline{AO} + \overline{OB} \\ &= (-\mathbf{a}) + \mathbf{b} \\ &= \mathbf{b} - \mathbf{a}. \end{aligned}$$

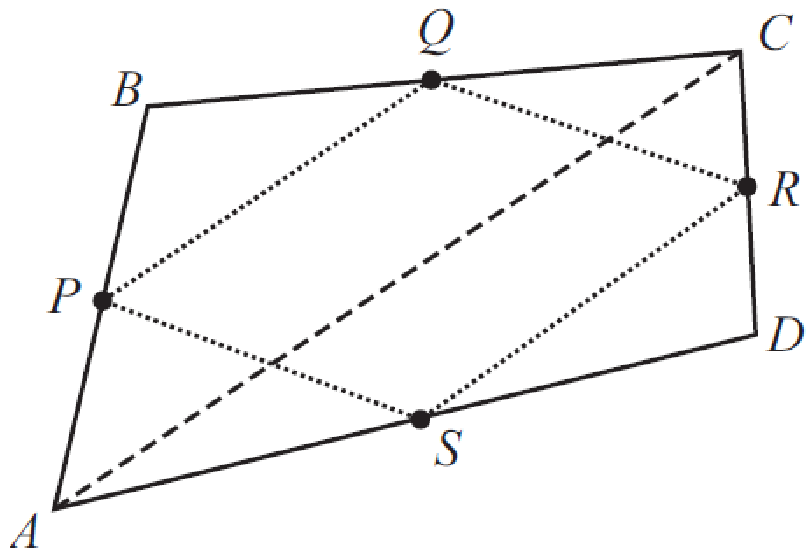
What about $\overline{MN}$? Following the same reasoning, this is $\overline{MO}+\overline{ON}$. But what is $\overline{MO}$? This is a vector half the length of $\overline{AO}$, and in the same direction, so it must be $\frac{1}{2}(-\mathbf{a})$. In the same way, $\overline{ON}$ is in the same direction as $\overline{OB}$, but is half the length, so it must be $\frac{1}{2}\mathbf{b}$. Therefore

$$\begin{aligned}\overline{MN} &= \overline{MO}+\overline{ON}\\ &= \tfrac{1}{2}(-\mathbf{a})+\tfrac{1}{2}\mathbf{b}\\ &= \tfrac{1}{2}(\mathbf{b}-\mathbf{a}).\end{aligned}$$

Now we can compare $\overline{AB}$ and $\overline{MN}$. From our calculation, we can see that $\overline{MN}$ is $\frac{1}{2}\overline{AB}$. So, as this is a vector equation, it tells us two things. First, it tells us about magnitude, so that $MN = \frac{1}{2}AB$. Also, it tells us that MN and AB must be in the same direction, so that $MN//AB$. This is called the mid-point theorem for a triangle. It states that if you join the mid-points of two sides of a triangle then the resulting line is equal to half of the third side of the triangle, and is parallel to it.

Example 2

We can apply the mid-point theorem to a quadrilateral, or indeed to any four points in space, to give an interesting geometrical result. We shall call the four points A, B, C and D. We shall also give labels to the mid-points of the four sides: we shall call the mid-points P, Q, R and S. Now let us join the four mid-points, to make a new shape $PQRS$. What kind of shape is this?



We can identify the shape by joining the points A and C.

If we apply the mid-point theorem to triangle ABC, we see that

$$\overline{PQ} = \tfrac{1}{2}\overline{AC}.$$

But if we apply the mid-point theorem to the triangle ADC, we also see that

$$\overline{RS} = \tfrac{1}{2}\overline{AC}.$$

If we combine these two equations, we then obtain

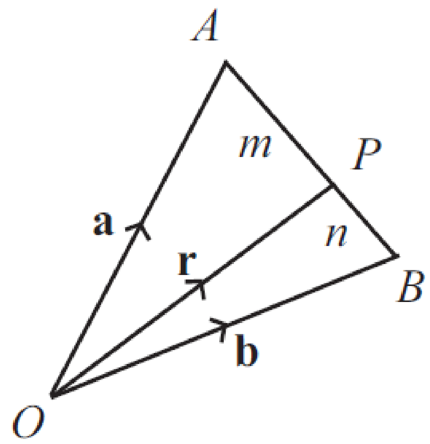
$$\overline{PQ} = \overline{RS}.$$

Now this is a vector equation, and so it tells us two things. First, it tells us that the length of PQ is the same as the length of RS. And secondly, it tells us that the direction of PQ is the same as the direction of RS, so that PQ and RS are parallel. But having two parallel sides of equal length is a property which defines a parallelogram, and so the shape $PQRS$ must be a parallelogram.

Example 3

We shall now use vectors to prove one more theorem.

Take two points A and B, having position vectors $\mathbf{a}$, $\mathbf{b}$ with respect to an origin O. Draw the line AB, and take a point P on that line which divides it in the ratio of m to n. What is the position vector of P with respect to O?.



We can use the same method that we used before. We know that

$$\overline{OP} = \overline{OA} + \overline{AP}, \tag{1}$$

and we also know that $\overline{OA} = \mathbf{a}$. But what is $\overline{AP}$?

Now $\overline{AP}$ is in the same direction as $\overline{AB}$, and their lengths are in the ratio of m to $m + n$. So

$$\overline{AP} = \frac{m}{m+n}\overline{AB}. \tag{2}$$

We also know that

$$\begin{aligned} \overline{AB} &= \overline{AO} + \overline{OB} \\ &= \mathbf{b} - \mathbf{a}. \end{aligned}$$

Now we can put these three statements together, replacing $\overline{AP}$ in equation (1) by using equation (2), and replacing $\overline{AB}$ in equation (2) by using the equation (3), so that everything will be written in terms of $\mathbf{a}$ and $\mathbf{b}$. This gives us

$$\overline{OP} = \mathbf{a} + \frac{m}{m+n}(\mathbf{b} - \mathbf{a}).$$

Putting all this over a common denominator then gives

$$\overline{OP} = \frac{(m+n)\mathbf{a} + m(\mathbf{b} - \mathbf{a})}{m+n}.$$

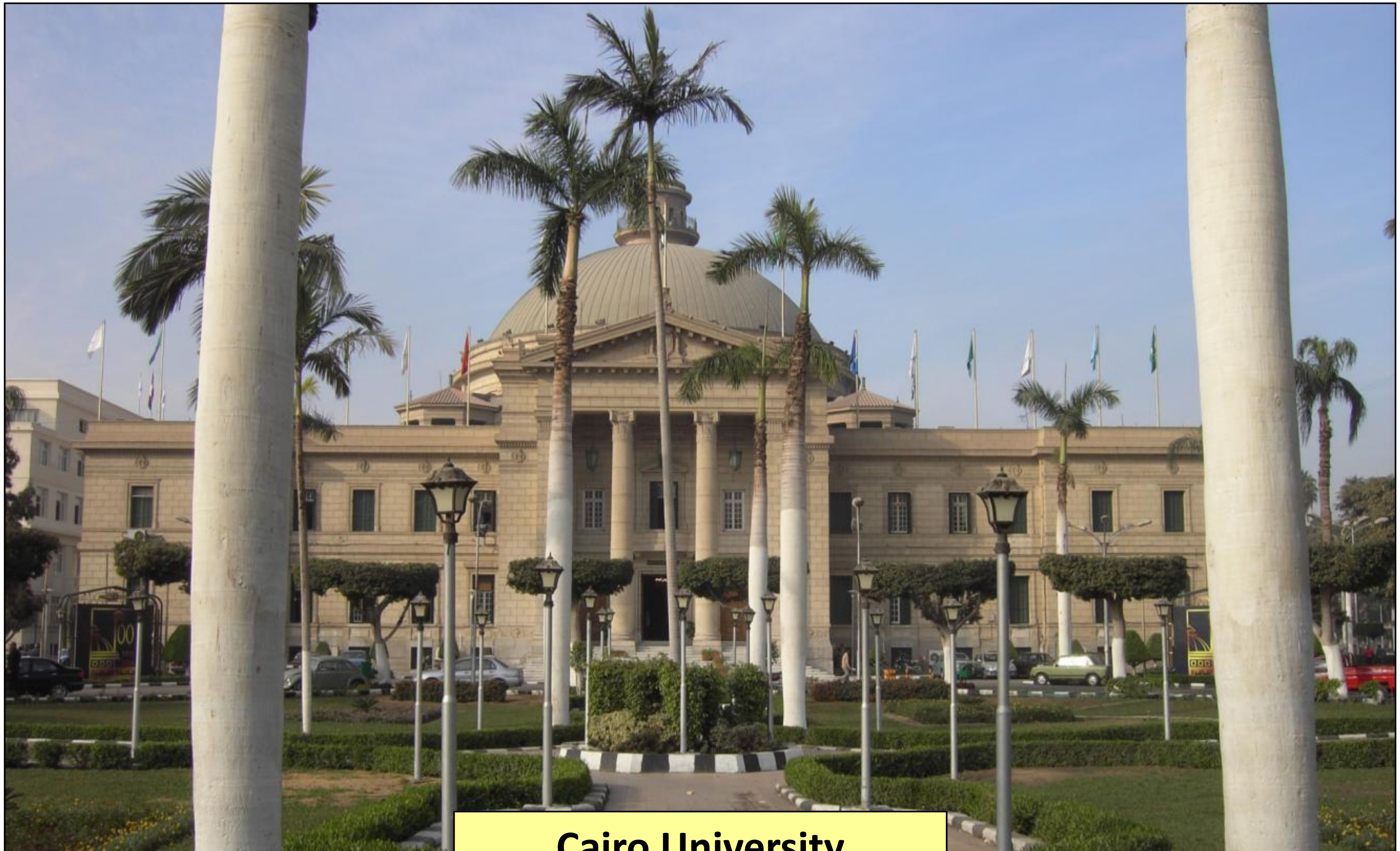
If we expand the brackets, the term $m\mathbf{a}$ will cancel with the term $m(-\mathbf{a})$, and so finally we have

$$\overline{OP} = \frac{n\mathbf{a} + m\mathbf{b}}{m+n}.$$

This formula gives us a way of calculating the position vector of the point P. For instance, if m and n were both 1 then P would be the mid-point of AB. The position vector of the midpoint would be $(\mathbf{a} + \mathbf{b})/2$. As another example, if $m = 2$ and $n = 1$, so that P was two-thirds of the way along the line, then the position vector of P would be $(\mathbf{a} + 2\mathbf{b})/3$.

Exercises and Solutions

Thank You!



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