Coupled Circuits

Dr. Mohamed Refky Amin
Electronics and Electrical Communications Engineering Department (EECE)
Cairo University
1st.year.circuits@gmail.com
http://scholar.cu.edu.eg/refky/

Outline of This Lecture

- Introduction
- Mutual inductances
- Coupling Coefficient
- Solving the Magnetic-Coupled Circuits
  - Direct Solution Method
  - T-network method

Coupled Circuits

Introduction

Coupled circuits are those in which energy is transferred between them through electric (and/or magnetic) fields varying with time.

When two loops affect each other through the magnetic field generated by one of them, they are said to be magnetically coupled.

The transformer is an electrical device designed on the basis of the concept of magnetic coupling. It uses magnetically coupled coils to transfer energy from one circuit to another.

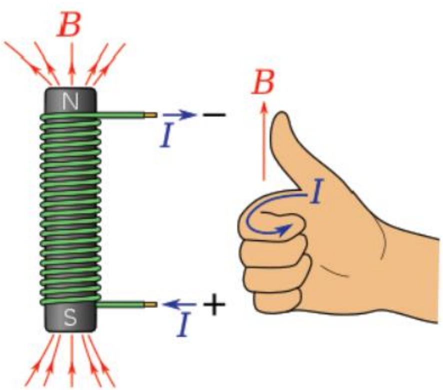
Transformer are used in power systems for stepping up or stepping down AC voltages or currents.

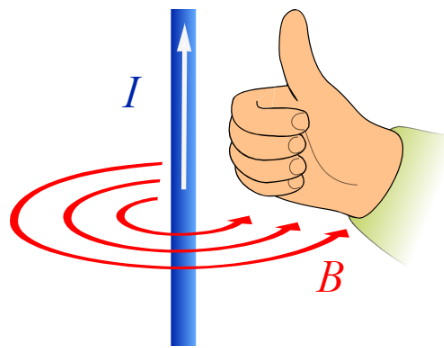
Coupled Circuits

Right-Hand Rule

The right-hand rule is used to get the direction of the magnetic field made by a varying current.

The right thumb and the curled fingers are used to determine the direction of the magnetic field.

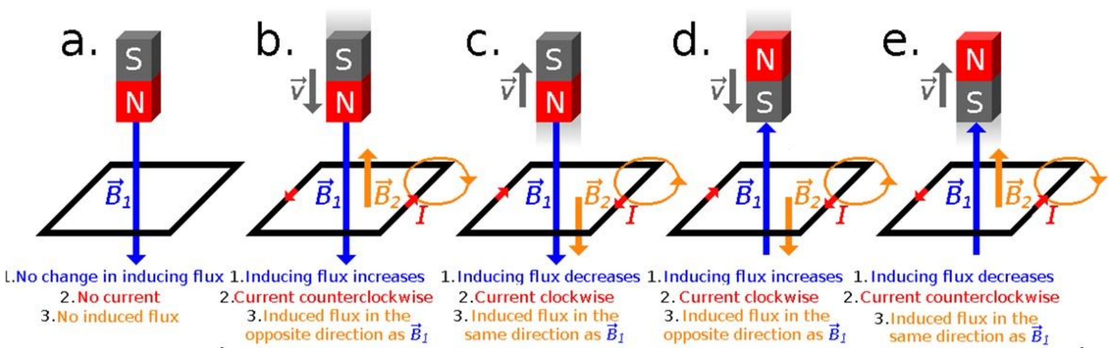




Coupled Circuits

Lenz's law

Lenz's law states that the direction of current induced in a conductor by a changing magnetic field is such that the magnetic field created by the induced current opposes changes in the initial magnetic field.



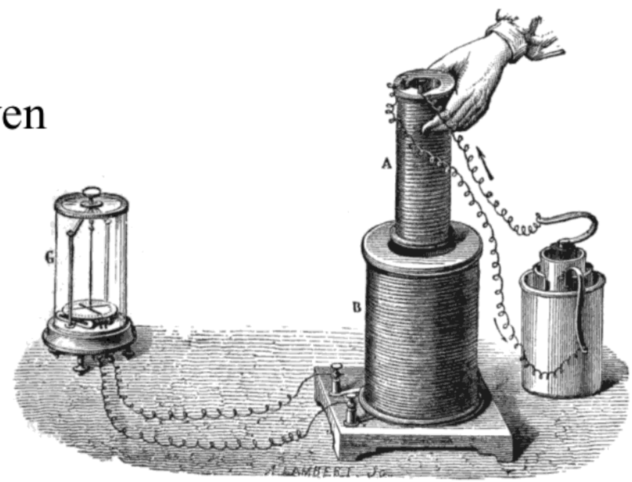
Coupled Circuits

Faraday's law

Faraday's law is a law of electromagnetism predicting how a magnetic field will interact with an electric circuit to produce an induced voltage.

A varying magnetic field $\phi(t)$ generates an induced voltage given by

$$V_{ind} = \frac{d\phi(t)}{dt}$$

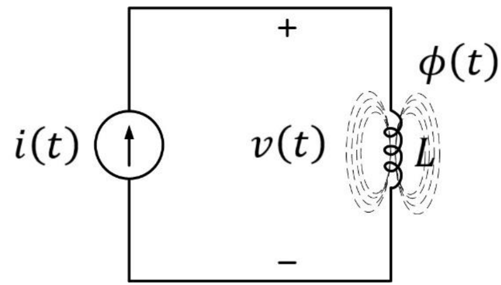


Coupled Circuits

Self Inductance

For a coil L with N turns derived by an AC supply $i(t)$

From Faraday's low



$$v = N\frac{d\phi}{dt} = \frac{d\phi_{total}}{dt}$$

$$= N\frac{d\phi}{di} \times \frac{di}{dt}$$

$$= L\frac{di}{dt} \qquad \rightarrow \qquad L = N\frac{d\phi}{di}$$

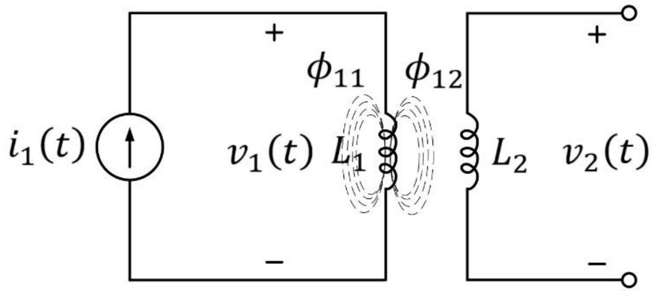
L is called self-inductance. The unit of L is Henry (H).

Coupled Circuits

Mutual inductances

For a magnetic-coupled coils L_1 (N_1 turns) and L_2 (N_2 turns)

$$\phi_1 = \phi_{11} + \phi_{12}$$



ϕ_{11} links only L_1, ϕ_{12} links L_1 and L_2.

$$v_1 = N_1 \frac{d\phi_1}{dt} = L_1 \frac{di_1}{dt}$$

$$v_2 = N_2 \frac{d\phi_{12}}{dt} = N_2 \frac{d\phi_{12}}{di_1} \times \frac{di_1}{dt} = M \frac{di_1}{dt}$$

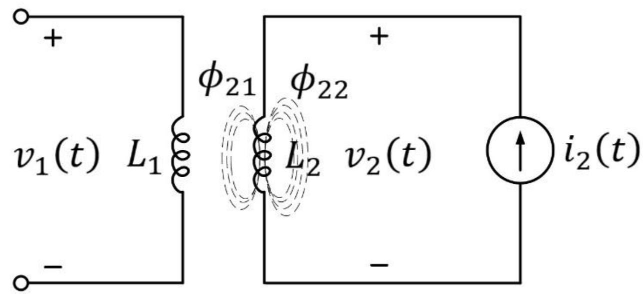
M is called mutual inductance. The unit of M is Henry (H).

Coupled Circuits

Mutual inductances

For a magnetic-coupled coils L_1 (N_1 turns) and L_2 (N_2 turns)

$$\phi_2 = \phi_{22} + \phi_{21}$$



ϕ_{22} links only L_1, ϕ_{21} links L_1 and L_2.

$$v_2 = N_2 \frac{d\phi_2}{dt} = L_2 \frac{di_2}{dt}$$

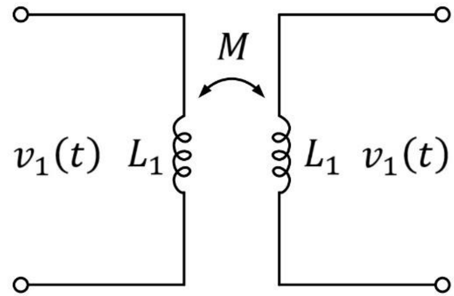
$$v_1 = N_1 \frac{d\phi_{21}}{dt} = N_1 \frac{d\phi_{21}}{di_2} \times \frac{di_2}{dt} = M \frac{di_2}{dt}$$

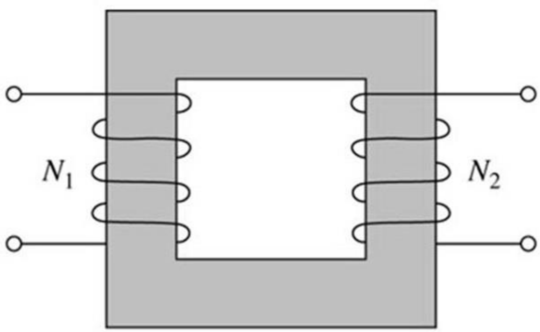
$$M = N_1 \frac{d\phi_{21}}{di_2} = N_2 \frac{d\phi_{12}}{di_1}$$

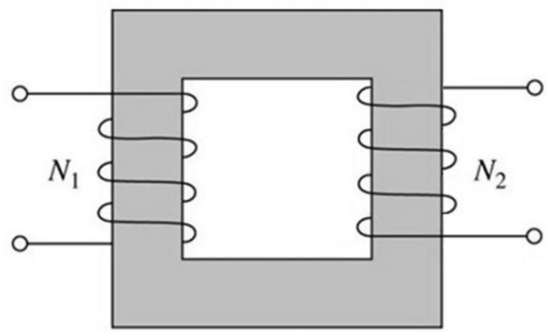
Coupled Circuits

Mutual inductances

Although M is always a positive quantity, the induced voltage may be positive or negative.





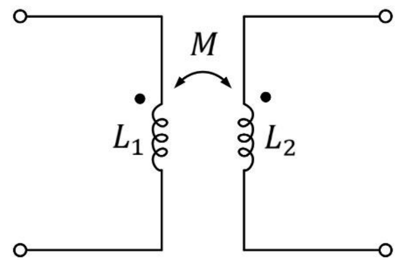


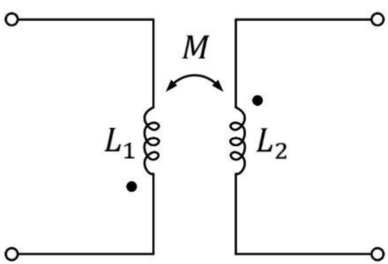
Coupled Circuits

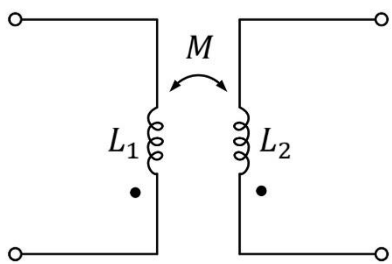
Mutual inductances

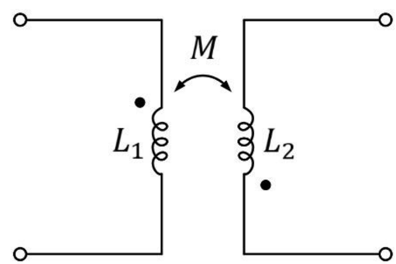
Dot Rule

Since it is inconvenient to show the construction details of coils on a circuit schematic, Dot convention is used instead.







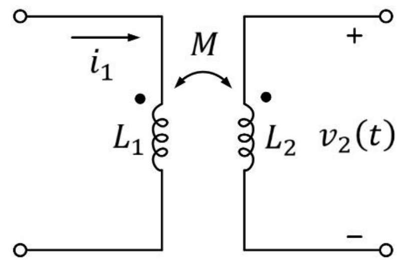


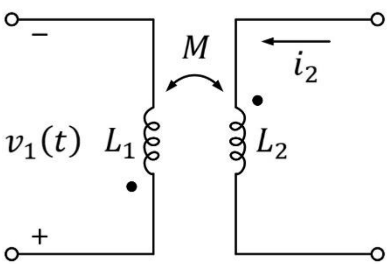
Coupled Circuits

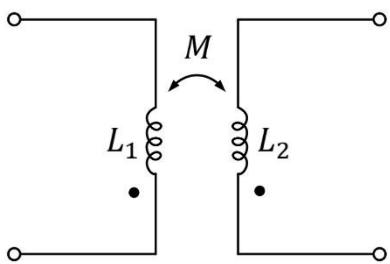
Mutual inductances

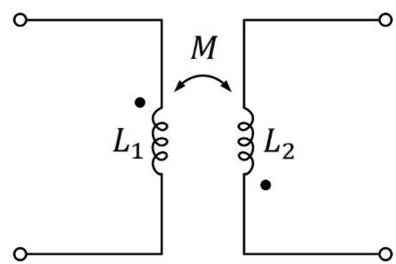
Dot Rule

If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.







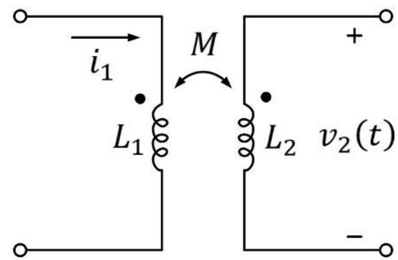


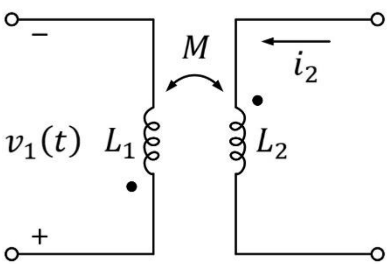
Coupled Circuits

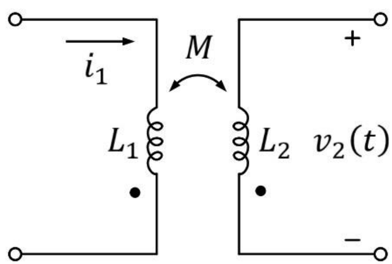
Mutual inductances

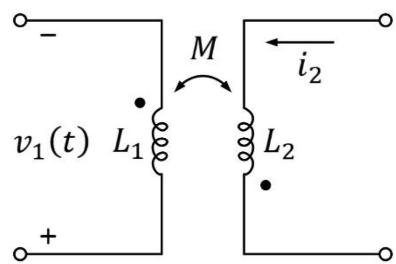
Dot Rule

If a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.









Coupled Circuits

Coupling Coefficient

Coupling coefficient (k) is the fraction of the magnetic flux produced by the current in one coil that links with the other coil (the ratio between the useful part of flux and the total flux).

$$\phi_1 = \phi_{11} + \phi_{12}$$

$$k = \frac{\phi_{12}}{\phi_1}$$

$$\phi_2 = \phi_{22} + \phi_{21}$$

$$k = \frac{\phi_{21}}{\phi_2}$$

$$1 \leq k \leq 0$$

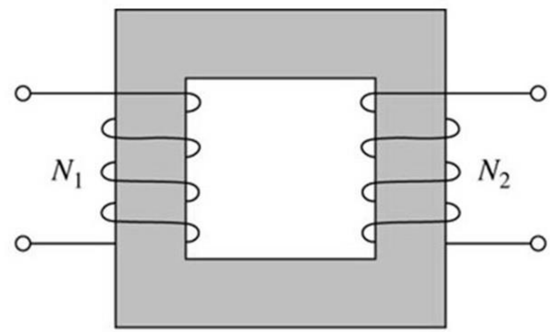
$k = 1$ means perfect coupling while $k = 0$ means no coupling.

Coupled Circuits

Coupling Coefficient

Metallic cores are usually used to make most of the varying flux travel to the coils (increasing k).

k also can be increased by warping the two coils one over the another.



Coupled Circuits

Coupling Coefficient

$$M = N_2 \frac{d\phi_{12}}{di_1}, \qquad M = N_1 \frac{d\phi_{21}}{di_2}$$

Multiply the two equations

$$M^2 = \left(N_2 \frac{d\phi_{12}}{di_1}\right)\left(N_1 \frac{d\phi_{21}}{di_2}\right)$$

Substitute $\phi_{12} = k\phi_1$ and $\phi_{21} = k\phi_2$

$$M^2 = \left(N_2 k \frac{d\phi_1}{di_1}\right)\left(N_1 k \frac{d\phi_2}{di_2}\right)$$

Coupled Circuits

Coupling Coefficient

$$M^2 = k^2 \left(N_1 \frac{d\phi_1}{di_1}\right)\left(N_2 \frac{d\phi_2}{di_2}\right)$$

$$M^2 = k^2 L_1 L_2 \quad \rightarrow \quad M = k\sqrt{L_1 L_2}$$

By multiply the equation with ω

$$\omega M = k\sqrt{\omega L_1 \times \omega L_2} \quad \rightarrow \quad X_M = k\sqrt{X_{L1} X_{L2}}$$

Because $1 \leq k \leq 0$,

$$M \leq \sqrt{L_1 L_2}, \qquad X_M \leq \sqrt{X_{L1} X_{L2}}$$

Coupled Circuits

Coupling Coefficient

$$M = N_2 \frac{d\phi_{12}}{di_1}, \qquad M = k\sqrt{L_1 L_2}$$

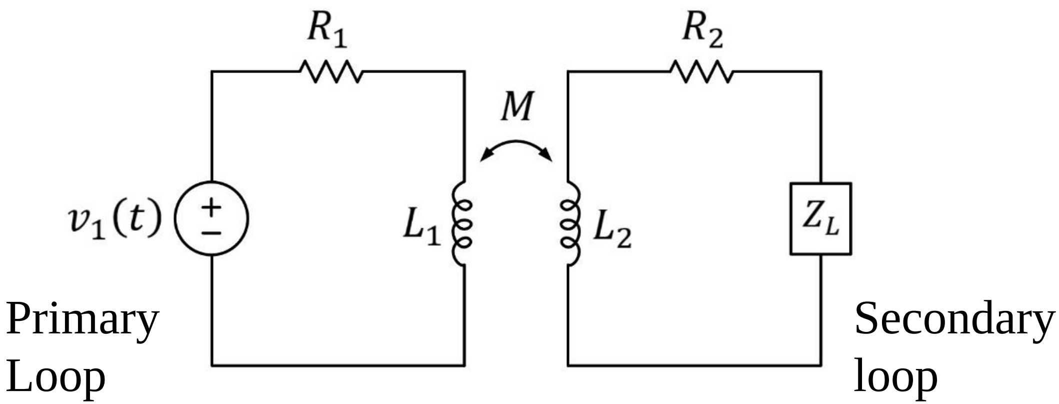
$$M = N_2 k \frac{d\phi_1}{di_1} \qquad \left(\times \frac{N_1}{N_1}\right)$$

$$M = \left(\frac{N_2}{N_1}\right)\left(\frac{M}{\sqrt{L_1 L_2}}\right)\left(N_1 \frac{d\phi_1}{di_1}\right)$$

$$\frac{N_1}{N_2} = \sqrt{\frac{L_1}{L_2}} \qquad\qquad \frac{N_1}{N_2} = a$$ is called the turns ratio

Coupled Circuits

Magnetic-Coupled Circuits

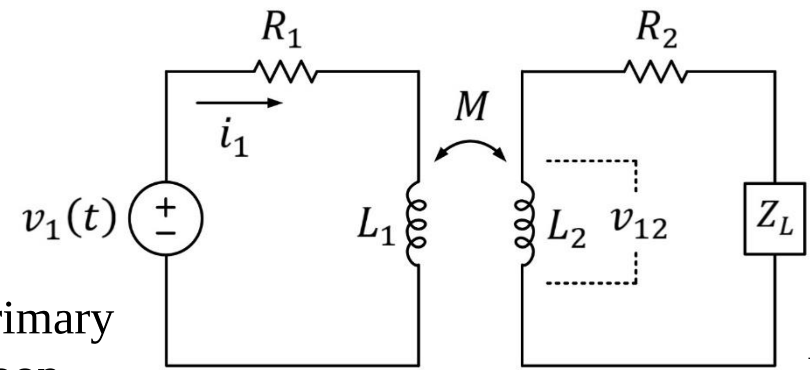


Here, the circuit has 2 coils. L_1 is connected to an AC voltage source $v_1(t)$ and while L_2 is connected to an impedance Z_L.

R_1 and R_2 represents the losses in L_1 and L_2, respectively.

Coupled Circuits

Magnetic-Coupled Circuits



Primary Loop

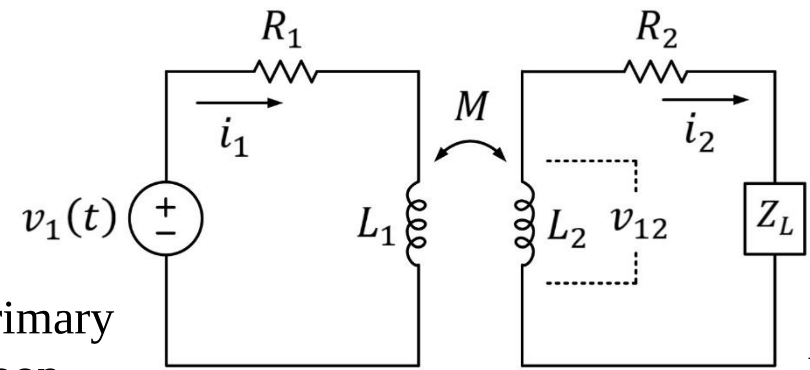
Secondary loop

When a current $i_1(t)$ passes through L_1, $\phi_1(t)$ is generated.

$\phi_{12}(t)$ travels to the L_2 causing the generation of induced voltage $v_{12}(t)$.

Coupled Circuits

Magnetic-Coupled Circuits



Primary Loop

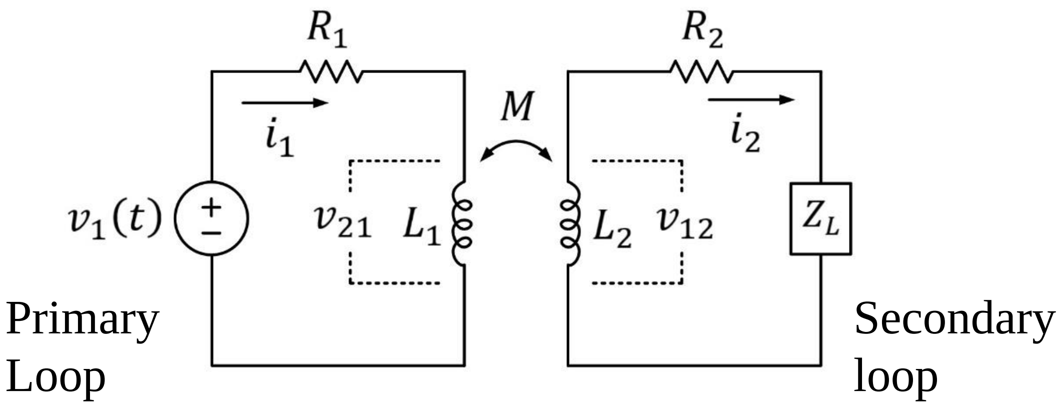
Secondary loop

$v_{12}(t)$ generates an induced current $i_2(t)$ passing through L_2.

When a current $i_2(t)$ passes through L_2, $\phi_2(t)$ is generated.

Coupled Circuits

Magnetic-Coupled Circuits

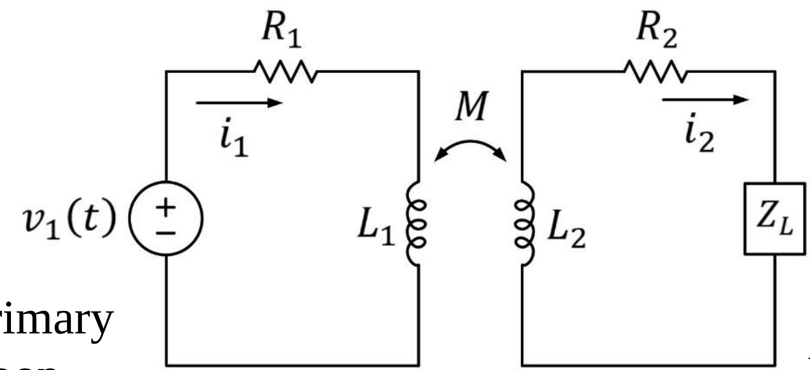


$\phi_{21}(t)$ moves to L_1 through the iron core causing the generation of induced voltage $v_{21}(t)$.

The magnetic coupling is bilateral.

Coupled Circuits

Solving the Magnetic-Coupled Circuits



Primary Loop

Secondary loop

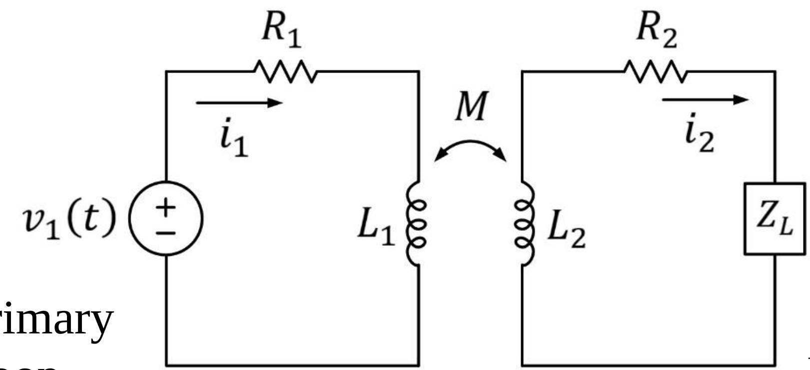
Writing KVL for the two loops in the time domain

Primary Loop $$v_1(t) = i_1(t)R_1 + L_1\frac{di_1(t)}{dt} \pm M\frac{di_2(t)}{dt}$$

Secondary loop $$0 = i_2(t)(Z_L + R_2) + L_2\frac{di_2(t)}{dt} \pm M\frac{di_1(t)}{dt}$$

Coupled Circuits

Solving the Magnetic-Coupled Circuits



Primary Loop

Secondary loop

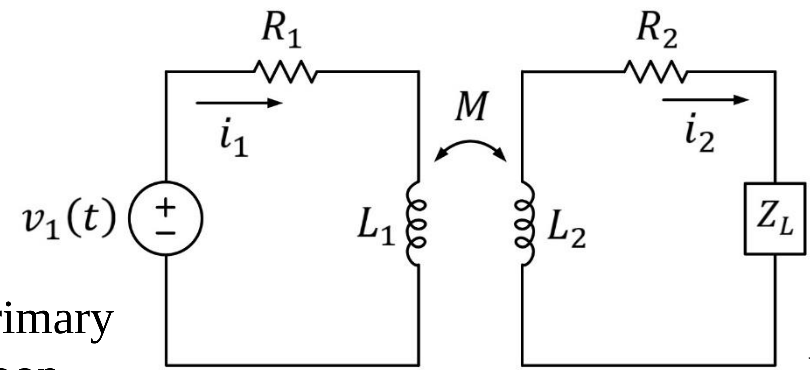
Solving these equations in the time domain is very difficult

Primary Loop
$$v_1(t) = i_1(t)R_1 + L_1 \frac{di_1(t)}{dt} \pm M \frac{di_2(t)}{dt}$$

Secondary loop
$$0 = i_2(t)(Z_L + R_2) + L_2 \frac{di_2(t)}{dt} \pm M \frac{di_1(t)}{dt}$$

Coupled Circuits

Solving the Magnetic-Coupled Circuits



Primary Loop

Secondary loop

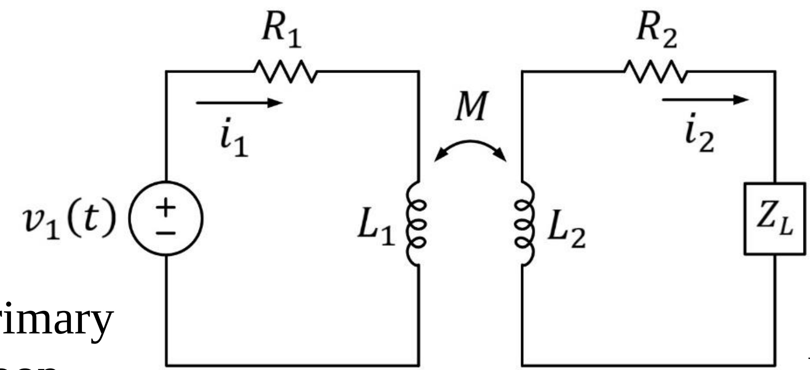
Writing KVL for the two loops in frequency domain (phasor domain)

Primary Loop $$V_1 = I_1(R_1 + j\omega L_1) \pm j\omega M I_2$$

Secondary loop $$0 = I_2(Z_L + R_2 + j\omega L_2) \pm j\omega M I_1$$

Coupled Circuits

Solving the Magnetic-Coupled Circuits



Primary Loop

Secondary loop

ωM is called mutual reactance and $j\omega M$ is called mutual impedance.

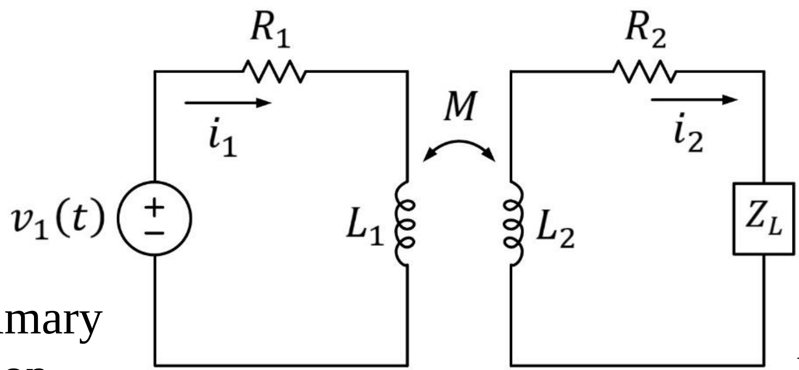
Primary Loop $\quad V_1 = I_1(R_1 + j\omega L_1) \pm j\omega M I_2$

Secondary loop $\quad 0 = I_2(Z_L + R_2 + j\omega L_2) \pm j\omega M I_1$

Coupled Circuits

Solving the Magnetic-Coupled Circuits

Dot Rule



Primary Loop

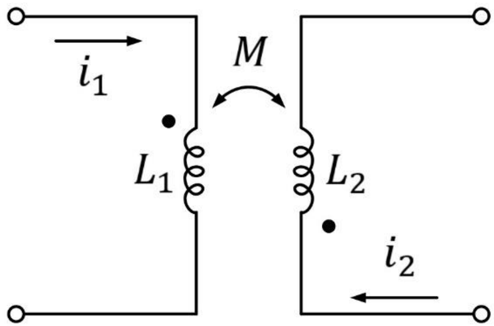
Secondary loop

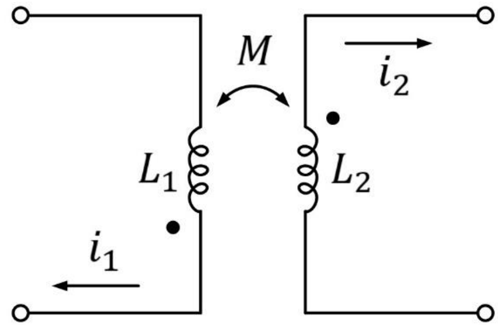
After assigning I_1, and I_2, if I_1, and I_2 are both entering or leaving the doting terminals, the sign of M-term is the same as the L-term in KVL equations. Otherwise, the M-term has opposite sign.

Coupled Circuits

Solving the Magnetic-Coupled Circuits

Dot Rule





The sign of M-term is the same as the L-term

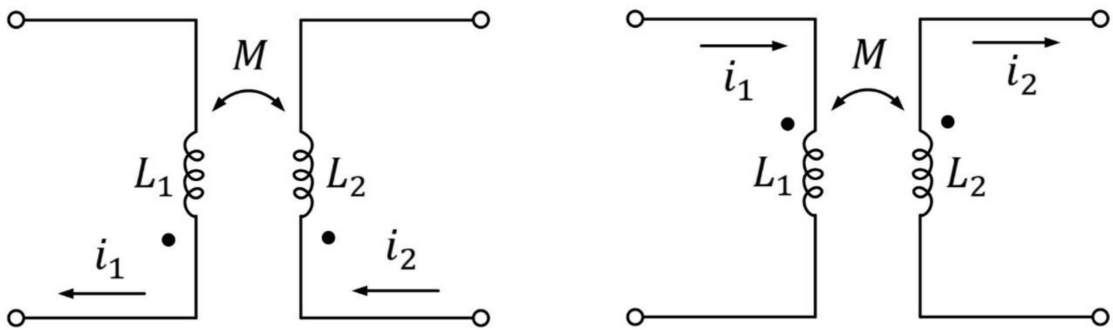
Primary Loop $\quad V_1 = I_1(R_1 + j\omega L_1) + j\omega M I_2$

Secondary Loop $\quad 0 = I_2(Z_L + R_2 + j\omega L_2) + j\omega M I_1$

Coupled Circuits

Solving the Magnetic-Coupled Circuits

Dot Rule



The sign of M-term is opposite to that of the L-term

Primary Loop $\quad V_1 = I_1(R_1 + j\omega L_1) - j\omega M I_2$

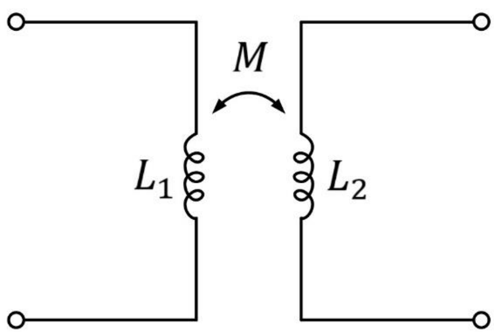
Secondary Loop $\quad 0 = I_2(Z_L + R_2 + j\omega L_2) - j\omega M I_1$

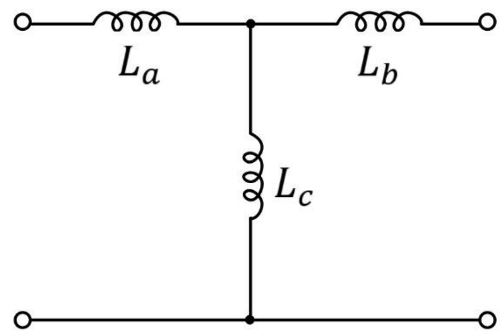
Coupled Circuits

Solving the Magnetic-Coupled Circuits

T-Network

T-Network is an equivalent circuit to the mutual coupled circuits that enable us to use several circuit solving techniques (Simplification, source transformation, etc.).





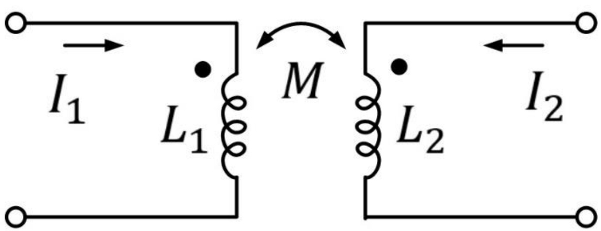
Coupled Circuits

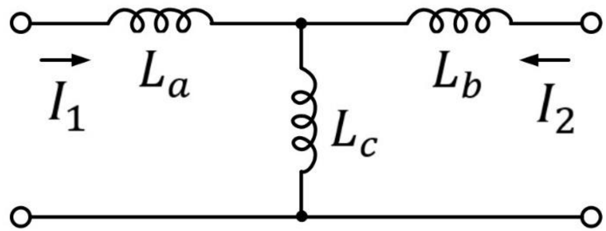
Solving the Magnetic-Coupled Circuits

T-Network

For the case where L_1 and L_2 have similar dots, while the two currents are entering (or leaving) the dots

Loop 1	$V_1 = j\omega L_1 I_1 + j\omega M I_2$	$V_1 = j\omega(L_a + L_c)I_1 + j\omega L_C I_2$
Loop 2	$V_2 = j\omega L_2 I_2 + j\omega M I_1$	$V_2 = j\omega(L_b + L_c)I_2 + j\omega L_C I_1$





Coupled Circuits

Solving the Magnetic-Coupled Circuits

T-Network

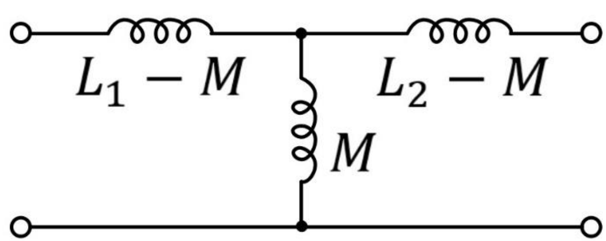
For the case where L_1 and L_2 have similar dots, while the two currents are entering (or leaving) the dots

Loop 1	$V_1 = j\omega L_1 I_1 + j\omega M I_2$	$V_1 = j\omega(L_a + L_c)I_1 + j\omega L_C I_2$
Loop 2	$V_2 = j\omega L_2 I_2 + j\omega M I_1$	$V_2 = j\omega(L_b + L_c)I_2 + j\omega L_C I_1$

$$L_c = M$$

$$L_a = L_1 - M$$

$$L_b = L_2 - M$$



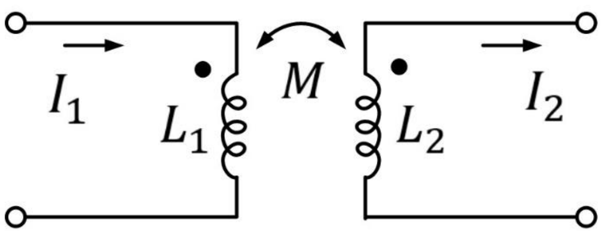
Coupled Circuits

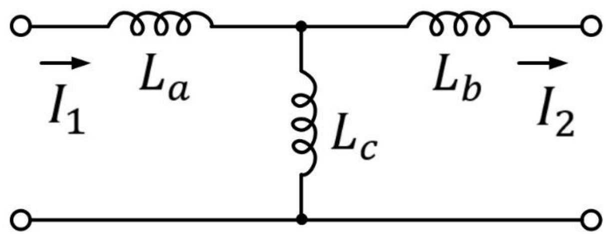
Solving the Magnetic-Coupled Circuits

T-Network

For the case where L_1 and L_2 have similar dots, while one current enters a dot and the other current leaves the other dot.

Loop 1	$V_1 = j\omega L_1 I_1 - j\omega M I_2$	$V_1 = j\omega(L_a + L_c)I_1 - j\omega L_C I_2$
Loop 2	$V_2 = j\omega L_2 I_2 - j\omega M I_1$	$V_2 = j\omega(L_b + L_c)I_2 - j\omega L_C I_1$





Coupled Circuits

Solving the Magnetic-Coupled Circuits

T-Network

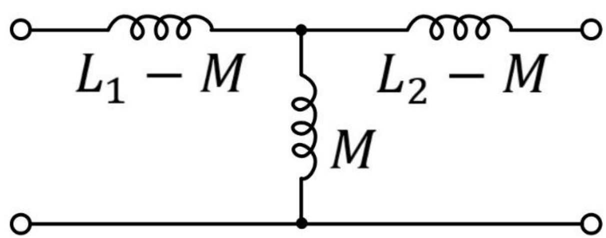
For the case where L_1 and L_2 have similar dots, while one current enters a dot and the other current leaves the other dot.

Loop 1	$V_1 = j\omega L_1 I_1 - j\omega M I_2$	$V_1 = j\omega(L_a + L_c)I_1 - j\omega L_C I_2$
Loop 2	$V_2 = j\omega L_2 I_2 - j\omega M I_1$	$V_2 = j\omega(L_b + L_c)I_2 - j\omega L_C I_1$

$$L_c = M$$

$$L_a = L_1 - M$$

$$L_b = L_2 - M$$



Coupled Circuits

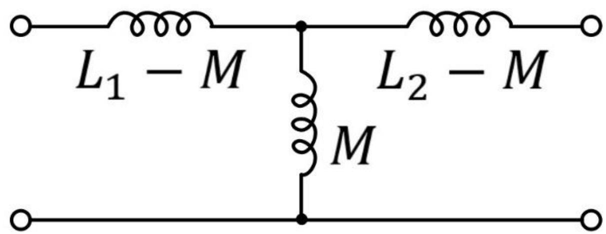
Solving the Magnetic-Coupled Circuits

T-Network

For the case where L_1 and L_2 have similar dots, while one current enters a dot and the other current leaves the other dot.

Loop 1	$V_1 = j\omega L_1 I_1 - j\omega M I_2$	$V_1 = j\omega(L_a + L_c)I_1 - j\omega L_C I_2$
Loop 2	$V_2 = j\omega L_2 I_2 - j\omega M I_1$	$V_2 = j\omega(L_b + L_c)I_2 - j\omega L_C I_1$

The T-network is the same as the one for the case when the two currents were entering (or leaving) the dots.



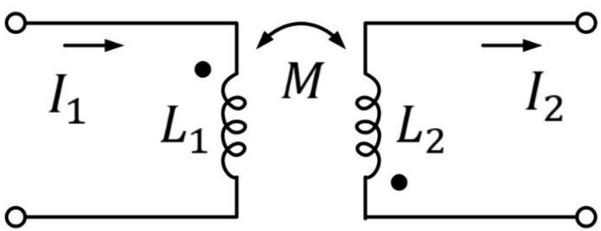
Coupled Circuits

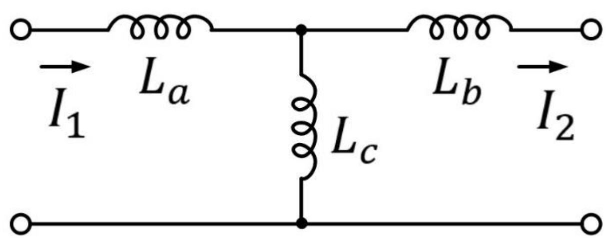
Solving the Magnetic-Coupled Circuits

T-Network

For the case where L_1 and L_2 have different dots, while the two currents are entering (or leaving) the dots

Loop 1	$V_1 = j\omega L_1 I_1 + j\omega M I_2$	$V_1 = j\omega(L_a + L_c)I_1 - j\omega L_C I_2$
Loop 2	$V_2 = j\omega L_2 I_2 + j\omega M I_1$	$V_2 = j\omega(L_b + L_c)I_2 - j\omega L_C I_1$





Coupled Circuits

Solving the Magnetic-Coupled Circuits

T-Network

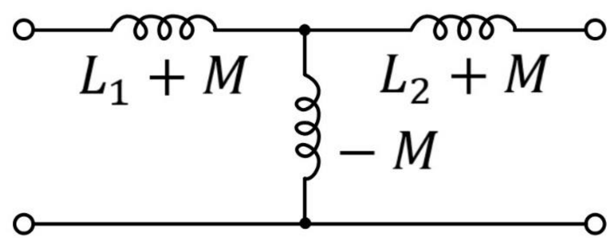
For the case where L_1 and L_2 have different dots, while the two currents are entering (or leaving) the dots

Loop 1	$V_1 = j\omega L_1 I_1 + j\omega M I_2$	$V_1 = j\omega(L_a + L_c)I_1 - j\omega L_C I_2$
Loop 2	$V_2 = j\omega L_2 I_2 + j\omega M I_1$	$V_2 = j\omega(L_b + L_c)I_2 - j\omega L_C I_1$

$$L_c = -M$$

$$L_a = L_1 + M$$

$$L_b = L_2 + M$$



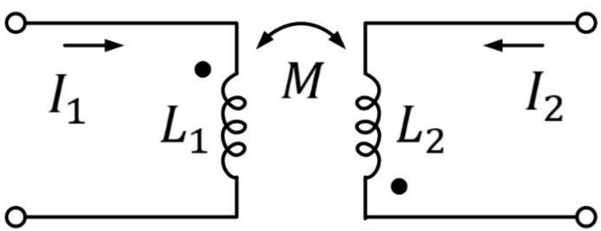
Coupled Circuits

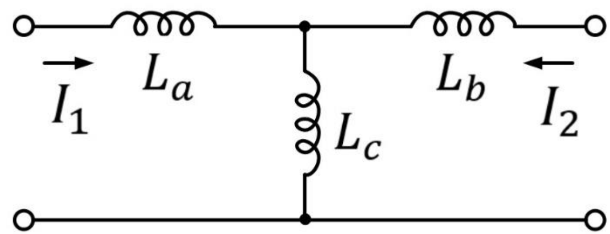
<u>Solving the Magnetic-Coupled Circuits</u>

<u>T-Network</u>

For the case where L_1 and L_2 have different dots, while one current enters a dot and the other current leaves the other dot.

Loop 1	$V_1 = j\omega L_1 I_1 - j\omega M I_2$	$V_1 = j\omega(L_a + L_c)I_1 + j\omega L_C I_2$
Loop 2	$V_2 = j\omega L_2 I_2 - j\omega M I_1$	$V_2 = j\omega(L_b + L_c)I_2 + j\omega L_C I_1$





Coupled Circuits

Solving the Magnetic-Coupled Circuits

T-Network

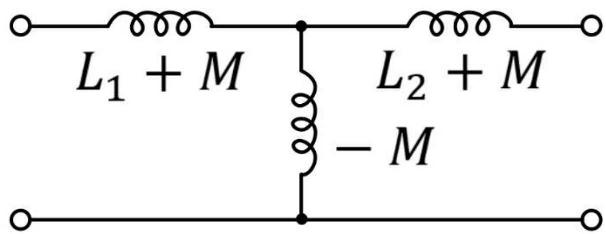
For the case where L_1 and L_2 have different dots, while one current enters a dot and the other current leaves the other dot.

Loop 1	$V_1 = j\omega L_1 I_1 - j\omega M I_2$	$V_1 = j\omega(L_a + L_c)I_1 + j\omega L_C I_2$
Loop 2	$V_2 = j\omega L_2 I_2 - j\omega M I_1$	$V_2 = j\omega(L_b + L_c)I_2 + j\omega L_C I_1$

$$L_c = -M$$

$$L_a = L_1 + M$$

$$L_b = L_2 + M$$



Coupled Circuits

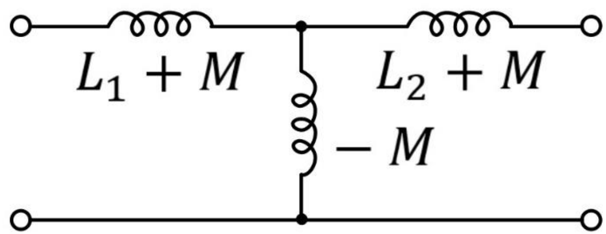
Solving the Magnetic-Coupled Circuits

T-Network

For the case where L_1 and L_2 have different dots, while one current enters a dot and the other current leaves the other dot.

Loop 1	$V_1 = j\omega L_1 I_1 - j\omega M I_2$	$V_1 = j\omega(L_a + L_c)I_1 + j\omega L_C I_2$
Loop 2	$V_2 = j\omega L_2 I_2 - j\omega M I_1$	$V_2 = j\omega(L_b + L_c)I_2 + j\omega L_C I_1$

The T-network is the same as the one for the case when the two currents were entering (or leaving) the dots.

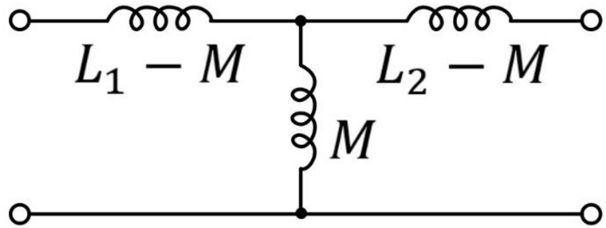


Coupled Circuits

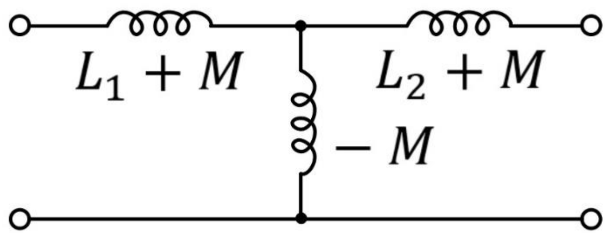
Solving the Magnetic-Coupled Circuits

T-Network

L_1 and L_2 have similar dots



L_1 and L_2 have different dots



Coupled Circuits

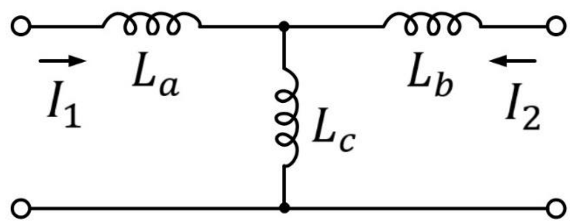
Solving the Magnetic-Coupled Circuits

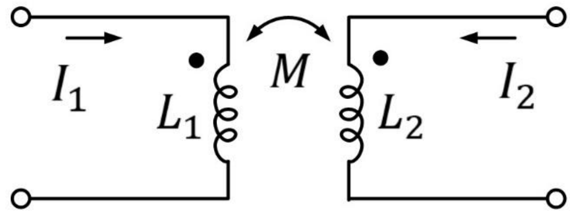
T-Network

The direct solution method and the T-network method are equivalent.

T-network method enable us to use several circuit solving techniques (loop analysis, simplification, source transformation, etc.).

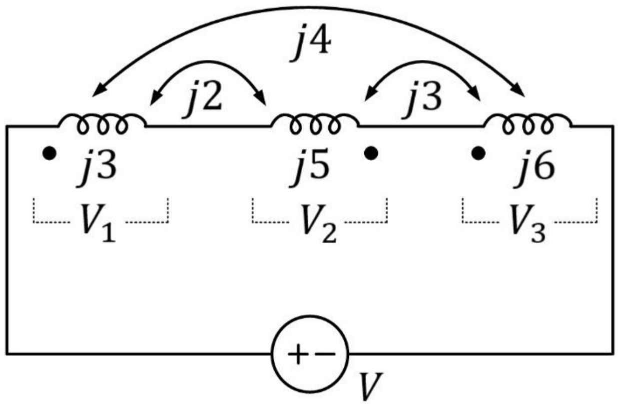
The direct solution method is preferred if the circuit has more than 2 coils.





Coupled Circuits

Example (1)



For the 3 coupled-coils circuit shown above, find X_{eq} and V_2.