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Real Analysis
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Chapter 0

The Real Number System

1. Introduction

We begin our study of real analysis by constructing $\mathbb{R}$ and proving some of its basic properties, assuming the existence of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$, and their familiar properties. Our purpose here is to show that we do not need any ad hoc axioms about the structure of $\mathbb{R}$, provided we believe that $\mathbb{Q}$ is a sufficiently reasonable starting point.

A layman's definition of a real number is probably "a decimal expression of the form $\pm n_0 . \alpha_1\alpha_2\cdots$, where $n_0 \in \mathbb{N}$ and $\alpha_i \in \{0,\dots,9\}$". This approach can be pursued rigorously; see for example [5]. We shall instead construct $\mathbb{R}$ by using sequences of rationals. This will define $\mathbb{R}$ as a certain *completion* of $\mathbb{Q}$. This approach has the advantage of generality, as it can be used to complete an arbitrary metric space in place of $\mathbb{Q}$. Later on, we prove in Section 4 that each real number has an essentially unique decimal expansion.

Our exposition certainly seems incomplete : we will construct $\mathbb{R}$ from $\mathbb{Q}$, but how can we construct $\mathbb{Q}$? We will not delve into this question here. We reassure the reader that $\mathbb{Q}$ can be constructed from $\mathbb{Z}$, that $\mathbb{Z}$ can be constructed from $\mathbb{N}$, and that $\mathbb{N}$ can be constructed from almost nothing : the empty set and binary operations. These considerations go somehow beyond the scope of mathematical analysis, so we refer the interested reader to [3] for details.

Sections 2 and 3 are based on [1], see also [2]. Section 4 is based on [4]

2. Construction of $\mathbb{R}$

The basic idea behind the following construction is that if $x \in \mathbb{R}$, then $x_n := \frac{[10^n x]}{10^n}$ is a sequence of rationals converging to x. The problem now is that x_n is defined in terms of x, so we cannot construct $\mathbb{R}$ from $\mathbb{Q}$ in this way. To avoid this problem, we shall use Cauchy sequences of rationals.

Definition 2.1. A sequence $x : \mathbb{N} \to \mathbb{Q}$ is called a *Cauchy sequence* of rationals if for each $a \in \mathbb{Q}_+^*$ there is an $N \in \mathbb{N}$ such that $|x_m - x_n| < a$ for all $m, n \geq N$. The set of all Cauchy sequences of rationals is denoted by $\mathcal{C}$.

Definition 2.2. A sequence $x : \mathbb{N} \to \mathbb{Q}$ is called a *zero sequence* of rationals if for each $a \in \mathbb{Q}_+^*$ there is an $N \in \mathbb{N}$ such that $|x_n| < a$ for all $n \geq N$. The set of all zero sequences of rationals is denoted by $\mathcal{Z}$.

Clearly, $\mathcal{Z} \subset \mathcal{C}$. As usual, given $x, y \in \mathcal{C}$, we define $x + y$ and xy to be the sequences with nth term $x_n + y_n$ and $x_n y_n$, respectively.

We are now ready to define the set of real numbers :

Definition 2.3. We define $\mathbb{R}$ to be the set of all equivalence classes of rational Cauchy sequences, where $x, x' \in \mathcal{C}$ are equivalent (denoted $x \sim x'$) if $x - x' \in \mathcal{Z}$. Each real number is thus a set $E_x = \{x' \in \mathcal{C} : x' \sim x\}$ for some representative $x \in \mathcal{C}$.

The reason why we consider equivalence classes of rationals is that different Cauchy sequences may converge to the same real number. We now want to define addition and multiplication in $\mathbb{R}$ by $E_x + E_y = E_{x+y}$ and $E_x E_y = E_{xy}$. However, for this definition to make sense, we must make sure that if $E_x = E_{x'}$ and $E_y = E_{y'}$, then $E_{x+y} = E_{x'+y'}$ and $E_{xy} = E_{x'y'}$. In order to verify this property, we first state the following

Lemma 2.4. *Let $x : \mathbb{N} \to \mathbb{Q}$ and $y : \mathbb{N} \to \mathbb{Q}$ be two sequences.*
(1) *If $x, y \in \mathcal{C}$, then $x + y \in \mathcal{C}$ and $xy \in \mathcal{C}$.*
(2) *If $x, y \in \mathcal{Z}$, then $x + y \in \mathcal{Z}$ and $xy \in \mathcal{Z}$.*
(3) *If $x \in \mathcal{Z}$ and $y \in \mathcal{C}$, then $xy \in \mathcal{Z}$.*

The proof is left as an exercise. Note that by (1), $E_{x+y}, E_{xy} \in \mathbb{R}$ whenever $E_x, E_y \in \mathbb{R}$.

Lemma 2.5. *Given $E_x, E_y \in \mathbb{R}$, the operations*

$$E_x + E_y := E_{x+y} \quad \text{and} \quad E_x E_y := E_{xy}$$

are well defined additions and multiplication on $\mathbb{R}$. That is, if $x, x', y, y' \in \mathcal{C}$ satisfy $x \sim x'$ and $y \sim y'$, then $x + y \sim x' + y'$ and $xy \sim x'y'$.

Proof. Suppose $x' = x + p$ and $y' = y + q$ for some $p, q \in \mathcal{Z}$. Then $(x' + y') - (x + y) = p + q \in \mathcal{Z}$ and $(x'y') - (xy) = py + xq + pq \in \mathcal{Z}$ by Lemma 2.4. Thus, $x + y \sim x' + y'$ and $xy \sim x'y'$. □

We now want to show that $\mathbb{R}$ is a field. To check that each nonzero element has an inverse, we first prove the following

Lemma 2.6. *If $x \in \mathcal{C}$ and $x \notin \mathcal{Z}$, then there is a rational $a > 0$ and $M, N \in \mathbb{N}$ such that*
(1) *$|x_n| > a$ for all $n \geq M$.*
(2) *Either $x_n > a$ for all $n \geq N$, or $x_n < -a$ for all $n \geq N$.*

Proof. (1) Assume on the contrary that for each rational $a > 0$ and each $M \in \mathbb{N}$, there is an $n \geq M$ such that $|x_n| \leq a$. Let $b > 0$ be a rational. Since $x \in \mathcal{C}$, we may find $M \in \mathbb{N}$ such that $|x_n - x_m| \leq b/2$ for all $m, n \geq M$. Moreover, by hypothesis we may find $n \geq M$ such that $|x_n| \leq b/2$. Thus, given $m \geq M$, we have $|x_m| \leq |x_n| + |x_n - x_m| \leq b$, i.e. $x \in \mathcal{Z}$, a contradiction.
(2) Choose $N \geq M$ such that $|x_m - x_n| \leq a$ for all $n, m \geq N$ and let $n, m \geq N$. If x_n and x_m have opposite signs, then $|x_n - x_m| = |x_n| + |x_m| > 2a$ by (1), which is impossible. Thus, using (1) again, we have either $x_n > a$ for all $n \geq N$ or $x_n < -a$ for all $n \geq N$. □

Theorem 2.7. *$\mathbb{R}$ is a field.*

Proof. Let $\bar{0}$ and $\bar{1}$ be the constant sequences consisting of all 0s and all 1s. Then $E_{\bar{0}}$ is the additive identity for $\mathbb{R}$, and $E_{\bar{1}}$ is the multiplicative identity for $\mathbb{R}$.

Suppose $E_x \neq E_{\bar{0}}$. Then $x \in \mathcal{C}$ and $x \notin \mathcal{Z}$, so by Lemma 2.6, there is a rational $a > 0$ and $N \in \mathbb{N}$ such that $|x_n| > a$ for all $n > N$. Define $y : \mathbb{N} \to \mathbb{Q}$ by $y_n = 0$ if $n < N$ and $y_n = 1/x_n$ if $n \geq N$. Then given $n, m \geq N$, $|y_n - y_m| = |x_n - x_m|/|x_n x_m| \leq (1/a^2)|x_n - x_m|$. Given a rational $b > 0$, choose $M \geq N$ such that $|x_n - x_m| \leq a^2 b$ for $n, m \geq M$. Then $|y_n - y_m| \leq b$ for $n, m \geq M$, hence $y \in \mathcal{C}$. Moreover, $xy \sim 1$ by definition of y, hence $E_x E_y = E_1$. We thus showed that each nonzero element in $\mathbb{R}$ has a multiplicative inverse.

It is now clear that $(\mathbb{R}, +)$ and $(\mathbb{R}^*, \times)$ are abelian groups. It only remains to prove the distributive law. For this, let $a, b, c \in \mathcal{C}$. Then

$$(a(b + c))_n = a_n(b + c)_n = a_n b_n + a_n c_n = (ab)_n + (ac)_n = ((ab) + (ac))_n$$

by distributivity in $\mathbb{Q}$. Thus, given $E_a, E_b, E_c \in \mathbb{R}$ with $a, b, c \in \mathcal{C}$ we get

$$E_a(E_b + E_c) = E_a E_{b+c} = E_{a(b+c)} = E_{(ab)+(ac)} = E_{ab} + E_{ac} = E_a E_b + E_a E_c\,,$$

which proves the distributive law. □

Definition 2.8. A sequence $x : \mathbb{N} \to \mathbb{Q}$ is said to be *eventually positive* if there is an $N \in \mathbb{N}$ such that $x_n > 0$ for all $n \geq N$.

A real number $E_x \in \mathbb{R}$ is said to be *positive* if each $y \in E_x$ is eventually positive. The set of positive real numbers will be denoted by $P_{\mathbb{R}}$.

Remarks 2.9. (i) $E_{\bar{0}} \notin P_{\mathbb{R}}$, since $(-1/n) \in E_{\bar{0}}$ is not eventually positive.

(ii) Since $E_{(1/n)} = E_{\bar{0}}$ we see that even if the representative x is eventually positive, this does not guarantee that $E_x \in P_{\mathbb{R}}$.

Lemma 2.10. *$E_x \in P_{\mathbb{R}}$ iff there exist $a \in \mathbb{Q}_+^*$ and $N \in \mathbb{N}$ such that $x_n > a$ for all $n \geq N$.*

Proof. Suppose $E_x \in P_{\mathbb{R}}$. If $x \in \mathcal{Z} = E_{\bar{0}}$, then $x \sim \bar{0} \sim (-1/n)$, so $(-1/n) \in E_x$. But $(-1/n)$ is not eventually positive, so $E_x \notin P_{\mathbb{R}}$, a contradiction. Hence, $x \notin \mathcal{Z}$. Using Lemma 2.6 and the fact that x is eventually positive, it follows that there exist $a \in \mathbb{Q}_+^*$ and $N \in \mathbb{N}$ such that $x_n > a$ for all $n \geq N$, as asserted.

Conversely, suppose there exists $a \in \mathbb{Q}_+^*$, $N \in \mathbb{N}$ such that $x_n > a$ for all $n \geq N$. Then x is eventually positive. Moreover, if $y \sim x$, then we may find $K \geq N$ such that $|y_n - x_n| \leq \frac{a}{2}$ for $n \geq K$. Thus, for $n \geq K$ we have $y_n = x_n + (y_n - x_n) \geq a - \frac{a}{2} = \frac{a}{2}$, so y is also eventually positive. Hence, $E_x \in P_{\mathbb{R}}$. □

Theorem 2.11. (1) *For each $E_x \in \mathbb{R}$, exactly one of the following three cases is true : $E_x = E_{\bar{0}}$, or $E_x \in P_{\mathbb{R}}$, or $-E_x \in P_{\mathbb{R}}$.*

(2) *If $E_x, E_y \in P_{\mathbb{R}}$, then $E_x + E_y \in P_{\mathbb{R}}$ and $E_x E_y \in P_{\mathbb{R}}$.*

We say that $\mathbb{R}$ is an *ordered* field.

Proof. Suppose $E_x \neq E_{\bar{0}}$. Then $x \notin \mathcal{Z}$, so by Lemma 2.6, there exists $a \in \mathbb{Q}_+^*$ and $N \in \mathbb{N}$ such that either $x_n > a$ for all $n \geq N$, or $x_n < -a$ for all $n \geq N$. Hence, by Lemma 2.10, either $E_x \in P_{\mathbb{R}}$, or $-E_x = E_{-x} \in P_{\mathbb{R}}$. This proves (1).

For (2), let $E_x, E_y \in P_{\mathbb{R}}$. Then by Lemma 2.6, there are rationals $a, b > 0$ and $N \in \mathbb{N}$ such that $x_n > a$ and $y_n > b$ for all $n \geq N$. Thus, $x_n + y_n > a + b > 0$ and $x_n y_n > ab > 0$ for all $n \geq N$. Hence, by Lemma 2.10, $E_x + E_y = E_{x+y} \in P_{\mathbb{R}}$ and $E_x E_y = E_{xy} \in P_{\mathbb{R}}$. □

Theorem 2.12. *Let $\varphi : \mathbb{Q} \to \mathbb{R}$ be defined by $\varphi(p) = E_{\bar{p}}$, where $\bar{p} \in \mathcal{C}$ is the constant sequence with all terms equal to $p \in \mathbb{Q}$. Then φ is one-to-one and preserves addition, multiplication and order.*

We call φ the *canonical embedding* of $\mathbb{Q}$ in $\mathbb{R}$.

Proof. If $E_{\bar{p}} = E_{\bar{q}}$, then $\bar{p} \sim \bar{q}$ and $\bar{p} - \bar{q} \in \mathcal{Z}$. But a constant sequence is a zero sequence iff the constant is zero. Hence, $p - q = 0$ and φ is one-to-one. Next, note that

$$\varphi(p+q) = E_{\overline{p+q}} = E_{\bar{p}} + E_{\bar{q}} = \varphi(p) + \varphi(q)\,.$$

Similarly, $\varphi(pq) = \varphi(p)\varphi(q)$. Hence, φ preserves addition and multiplication. Finally, let $p \in \mathbb{Q}_+^* =: P_{\mathbb{Q}}$. Then $p > \frac{p}{2}$ for all n, so $E_{\bar{p}} \in P_{\mathbb{R}}$ by Lemma 2.10. Thus, $\varphi(P_{\mathbb{Q}}) \subset P_{\mathbb{R}}$ and φ preserves order. □

We have finally showed that $\mathbb{R}$ is an ordered field that contains an isomorphic copy of $\mathbb{Q}$ as a subfield. From now on, we denote any real number by a single letter $r \in \mathbb{R}$ and ignore the difference between $\mathbb{Q}$ and $\varphi(\mathbb{Q})$. We also write $r > 0$ to indicate that $r \in P_{\mathbb{R}}$.

3. Further properties of $\mathbb{R}$

We shall now prove that $\mathbb{Q}$ is dense in $\mathbb{R}$, that $\mathbb{R}$ satisfies the least upper bound property, and that $\mathbb{R}$ is an archimedean field.

Lemma 3.1. *Let $r > 0$, $r \in \mathbb{R}$. Then there is a $p \in \mathbb{Q}$ such that $0 < p < r$.*

Proof. Let $r = E_x$, where $x \in \mathcal{C}$ is any representative of r. By Lemma 2.10, we may find $a \in \mathbb{Q}_+^*$ and $N \in \mathbb{N}$ such that $x_n > a$ for all $n \geq N$. Hence, $x_n - \frac{a}{2} > \frac{a}{2}$ for all $n \geq N$. Let $p = \frac{a}{2}$. Then by Lemma 2.10, $E_{x-\bar{p}} \in P_{\mathbb{R}}$, i.e. $r - p > 0$. Hence, $r > p > 0$. □

Theorem 3.2. *Let $r, s \in \mathbb{R}$ and $r < s$. Then there is a $q \in \mathbb{Q}$ such that $r < q < s$.*

This shows that $\mathbb{Q}$ is *dense* in $\mathbb{R}$.

Proof. Since $0 < s-r$, by Lemma 3.1, we may find $p \in \mathbb{Q}$ such that $0 < p < s-r$. Assume $r = E_x$ and $s = E_y$. Then $E_{y-x-\bar{p}} = s-r-p > 0$, hence $y-x-\bar{p}$ is eventually positive. Let $M \in \mathbb{N}$ such that $y_m > x_m + p$ for all $m \geq M$. Since $x \in \mathcal{C}$, we may find $N \geq M$ such that $|x_N - x_n| \leq p/4$ for all $n \geq N$. Let $q = x_N + (p/2)$. Then for $n \geq N$,

$$q - x_n = x_N + (p/2) - x_n \geq p/2 - |x_N - x_n| \geq p/4,$$
$$q = x_N + (p/2) = x_n + (x_N - x_n) + (p/2) \leq x_n + (3p/4) < y_n - (p/4)\,.$$

Thus, $q - r = E_{\bar{q}-x} > 0$ and $s - q = E_{y-\bar{q}} > 0$ by Lemma 2.10. Hence, $r < q < s$. □

Remark 3.3. It follows in particular that if $M \in \mathbb{R}$, then $\exists n_1, n_2 \in \mathbb{Z}$ such that $n_1 < M < n_2$. Indeed, $\exists q_1, q_2 \in \mathbb{Q}$ such that $M - 1 < q_1 < M < q_2 < M + 1$, say $q_j = a_j/b_j$, with $a_j, b_j \in \mathbb{Z}$, $j = 1, 2$. Then $q_2 \leq |a_2|$ and $q_1 \geq -|a_1|$, so we may take $n_1 = -|a_1|$ and $n_2 = |a_2|$. This crude property will be refined later on, when we define the integer part of a real number.

We will need the following property of $\mathbb{Z}$:

Theorem 3.4. *Any nonempty subset of $\mathbb{Z}$ which is bounded above admits a greatest element. That is, if $A \subset \mathbb{Z}$ is nonempty and there exists $M \in \mathbb{Z}$ such that $a \leq M$ for all $a \in A$, then there exists $a_0 \in A$ such that $a \leq a_0$ for all $a \in A$.*

Proof. Suppose $A \neq \emptyset$ and $M \in \mathbb{Z}$ satisfies $a \leq M$ $\forall a \in A$. Then $B := \{M - a : a \in A\}$ is a nonempty subset of $\mathbb{N}$. By the well-ordering principle[1], B has a least element, i.e. $\exists b \in B$ with $b \leq M - a$ for all $a \in A$. Since $b \in B$, $b = M - a_0$ for some $a_0 \in A$. Hence, $M - a_0 \leq M - a$ for all $a \in A$, i.e. $a \leq a_0$ for all $a \in A$. □

Definition 3.5. We say that $M \in \mathbb{R}$ is an *upper bound* for $A \subset \mathbb{R}$ if $a \leq M$ for all $a \in A$.

Lemma 3.6. *Let A be a nonempty set of real numbers. Assume A has an upper bound. Then there are two sequences of rationals (p_n) and (q_n) such that*

(1) *p_n is not an upper bound for A and q_n is an upper bound for A,*
(2) *p_n is increasing and q_n is decreasing,*
(3) $q_n - p_n = (1/2)^{n-1}$.

Moreover, (p_n) and (q_n) are equivalent Cauchy sequences.

Proof. Let $M \in \mathbb{R}$ be an upper bound and $a \in A$. By Remark 3.3, we may choose $n_1, n_2 \in \mathbb{Z}$ such that $n_1 < a$ and $M < n_2$. Then n_1 is not an upper bound for A, and any $n \geq n_2$ is an upper bound for A. The set $B = \{k \in \mathbb{Z} : k \text{ is not an upper bound of } A\}$ is thus nonempty (contains n_1) and bounded above (by n_2), hence it contains a greatest element $m \in B$ by Theorem 3.4. Choosing $p_1 = m$ and $q_1 = p_1 + 1$, statements (1) and (3) of the lemma are verified at $n = 1$.

Assume that p_n and q_n have been constructed. Let $s_n = (p_n + q_n)/2$. If s_n is not an upper bound, let $p_{n+1} = s_n$ and $q_{n+1} = q_n$. If s_n is an upper bound, let $p_{n+1} = p_n$ and $q_{n+1} = s_n$. Then the requirements are also satisfied at $n+1$. Hence, the sequences p_n and q_n are defined by the induction principle.

We finally show that (p_n) and (q_n) are equivalent Cauchy sequences. By (2) and (3), if $m \geq n$, then

$$0 \leq p_m - p_n \leq q_m - p_n \leq q_n - p_n = (1/2)^{n-1} \to 0\,,$$

[1] The well-ordering principle says that any non-empty subset of $\mathbb{N}$ has a least element. It is equivalent to mathematical induction. Let us show that induction implies the well-ordering principle as a theorem.

Let E be any nonempty subset of $\mathbb{N}$. Given $n \in \mathbb{N}$, let $P(n)$ be the following property : If $\exists 0 \leq k \leq n$ such that $k \in E$, then E has a least element. $P(0)$ is true: if $0 \in E$, then 0 is a least element for E, since $E \subseteq \mathbb{N}$. Suppose $P(n)$ is true and assume $\exists 0 \leq k \leq n+1$ such that $k \in E$. If $\exists e \in E$ such that $0 \leq e \leq n$, then E admits a least element by $P(n)$. If not, we must have $k = n+1$, and any $e \in E$ satisfies $e > n$. Thus, $k \in E$ is a least element for E and $P(n+1)$ is true. Hence, $P(n)$ is true for all n by induction. Finally, if $E \subseteq \mathbb{N}$ is nonempty, then there exists $n \in E$, hence E admits a least element by $P(n)$.

hence (p_n) is Cauchy. Similarly, (q_n) is Cauchy. Since $q_n - p_n = (1/2)^{n-1} \to 0$, (p_n) and (q_n) are equivalent. □

Theorem 3.7. *Let A be a nonempty set of real numbers. Assume A has an upper bound. Then A has a least upper bound. That is, there exists $r \in \mathbb{R}$ such that r is an upper bound for A, but if $s < r$, then s is not an upper bound for A.*

The real number r above is called the *supremum* of A and is denoted by $r = \sup(A)$.

Proof. Let (p_n) and (q_n) be the sequences in Lemma 3.6. Since they are equivalent Cauchy sequences, they represent the same real $r \in \mathbb{R}$. We show that $r = \sup(A)$.

Suppose r is not an upper bound for A. Then there is $a \in A$ such that $r < a$. By Theorem 3.2, we may find $q \in \mathbb{Q}$ with $r < q < a$. Then $E_{\bar{q}-(q_n)} = q - r > 0$, hence $q - q_n$ is eventually positive. Hence, there is an $N \in \mathbb{N}$ such that $q > q_n$ for all $n \geq N$. Thus, $q_n < q < a$ and q_n is not an upper bound for A. This contradiction shows that r must be an upper bound for A.

Suppose that A has an upper bound $s < r$. Let $p \in \mathbb{Q}$ such that $s < p < r$. Then $E_{(p_n)-\bar{p}} = r - p > 0$, hence $p_n - p$ must be eventually positive. In particular, there is an $n \in \mathbb{N}$ such that $p_n > p > s$. Thus, p_n is an upper bound for A. This contradiction shows that any $s < r$ cannot be an upper bound for A. Thus, $r = \sup(A)$. □

Theorem 3.8. *Given $x \in \mathbb{R}_+^*$ and $y \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $nx \geq y$.*

We refer to this as the *archimedean property* of the reals.

Proof. Suppose on the contrary that $nx < y$ for all $n \in \mathbb{N}$. Then the set $A = \{nx : n \in \mathbb{N}\}$ is a nonempty set of real numbers which has an upper bound y. Hence, it has a least upper bound $r = \sup(A)$. Since $x > 0$, $r - x < r$ and $r - x$ is not an upper bound for A. So there exists $m \in \mathbb{N}$ such that $r - x < mx$, hence $r < (m+1)x$, which is impossible, since r is an upper bound for A. This contradiction proves the theorem. □

Corollary 3.9. *Given $x, y \in \mathbb{R}$ with $x > 1$, there exists $n \in \mathbb{N}$ such that $x^n \geq y$.*

Proof. Let $x = 1 + h$, $h > 0$. Then by the binomial theorem, $(1+h)^n \geq 1 + nh$. By the archimedean property, we may find n such that $nh \geq y - 1$. Hence, $x^n \geq y$. □

Lemma 3.10. *Given $x \in \mathbb{R}$ and $a \in \mathbb{R}_+^*$, there exists a unique $n \in \mathbb{Z}$ such that $na \leq x < (n+1)a$.*

Proof. ♢ *Uniqueness :* If n and n' satisfy the assertion, then

- $n \leq \frac{x}{a} < n' + 1$ implies $n - n' < 1$, i.e. $n - n' \leq 0$,
- $n' \leq \frac{x}{a} < n + 1$ implies $n' - n < 1$, i.e. $n' - n \leq 0$.

Hence, $n = n'$.

♢ *Existence :* By the archimedean property, we may find $n_1, n_2 \in \mathbb{N}$ such that $x \leq an_1$ and $-x \leq an_2$. The set $A = \{k \in \mathbb{Z} : ka \leq x\}$ is thus a nonempty subset of $\mathbb{Z}$ (it contains $-n_2$) and has an upper bound (n_1). Hence by Theorem 3.4, A has a greatest element n such that $na \leq x$. Since $n + 1 \notin A$, we have $x < (n+1)a$. □

The case $a = 1$ in the previous lemma is of special interest :

Definition 3.11. If $x \in \mathbb{R}$, the unique $n \in \mathbb{Z}$ satisfying $n \leq x < n+1$ is called the *integer part* of x, and denoted by $[x]$. The real $x - [x]$ is called the *fractional part* of x, and is denoted by $\{x\}$.

4. Expansion in base b

Euclidean division. *Given $a \in \mathbb{Z}$ and $b \in \mathbb{N}^*$, there exist unique $q, r \in \mathbb{Z}$ such that $a = bq + r$ and $0 \le r < b$. Moreover, if $a \in \mathbb{N}$, then $q \in \mathbb{N}$.*

The integers q and r are called the *quotient* and *remainder* of the Euclidean division of a by b, respectively.

Proof. $\diamondsuit$ *Uniqueness :* Suppose $a = bq_1 + r_1 = bq_2 + r_2$ with $0 \le r_1, r_2 < b$. Then $r_2 - r_1 = b(q_1 - q_2)$, hence b divides $r_2 - r_1$. But $0 \le r_1, r_2 < b$, so $-b < r_2 - r_1 < b$. The only multiple of b in $]-b, b[$ is 0, hence $r_2 = r_1$. Since $r_2 - r_1 = b(q_1 - q_2)$ and $b \neq 0$, we have also have $q_1 = q_2$.

$\diamondsuit$ *Existence :* Let $E = \{k \in \mathbb{Z} : kb \le a\}$. If $a \ge 0$, then $0 \in E$ and a is an upper bound for E (because $b \ge 1$). If $a < 0$, then $a \in E$ and 0 is an upper bound for E. In any case, E is a nonempty subset of $\mathbb{Z}$ which is bounded above, so it has a greatest element q by Theorem 3.4. We thus have $qb \le a < (q+1)b$, and taking $r := a - bq$, we get $0 \le r < b$.

Finally, note that if $a \ge 0$, then as we saw, $0 \in E$, hence $q = \sup(E) \ge 0$. $\square$

Any integer can easily be expressed in base 10. For example, $2586 = 6 + 8 \times 10 + 5 \times 100 + 2 \times 1000$. More generally, we can express any $n \in \mathbb{N}$ in base b, for $b \ge 2$, as the following lemma shows.

Lemma 4.1. *Given $n \in \mathbb{N}$ and $b \ge 2$, there exists a unique $m \in \mathbb{N}$ and unique $a_0, \ldots, a_m \in \{0, 1, \ldots, b-1\}$ such that*

$$n = a_0 + a_1 b + a_2 b^2 + \ldots + a_m b^m, \qquad a_m \neq 0\,.$$

In analogy with the base 10 case, we may denote this by $n = (a_m \ldots a_1 a_0)_b$.

Proof. By Euclidean division, we may find $r_0, q_0 \in \mathbb{N}$ such that $n = bq_0 + r_0$, and for $n \ge 1$, we may find r_n, q_n such that $q_{n-1} = bq_n + r_n$.

$\diamondsuit$ *Existence :* We first note that there exists $j \in \mathbb{N}$ such that $q_j = 0$. Indeed, if this was not true, we would have for all $n \ge 0$, $q_n = bq_{n+1} + r_{n+1}$ with $0 \le r_{n+1}$, hence $bq_{n+1} \le q_n$. Since $q_{n+1} > 0$ and $b \ge 2$, we must have $q_{n+1} < q_n$. Hence, $(q_n)_{n \in \mathbb{N}}$ is a strictly decreasing sequence, which is impossible since $q_n \in \mathbb{N}$. Thus, $q_j = 0$ for some $j \in \mathbb{N}$.

So let n_0 be the smallest integer such that $q_{n_0} = 0$. [2] Then we have $n = r_0 + bq_0$, $q_k = r_k + bq_{k+1}$ for $0 \le k \le n_0$ and $q_{n_0} = 0$. Thus, $n = r_0 + r_1 b + r_2 b^2 + \ldots + r_{n_0} b^{n_0}$. Note that $r_{n_0} \neq 0$, since $q_{n_0 - 1} = bq_{n_0} + r_{n_0} = r_{n_0}$ and $q_{n_0-1} \neq 0$ by the choice of n_0.

$\diamondsuit$ *Uniqueness :* Suppose $n = a_0 + \ldots + a_m b^m$. Writing $n = a_0 + b(a_1 + \ldots + a_m b^{m-1})$, we see by the uniqueness of Euclidean division of n by b that $a_0 = r_0$ and $a_1 + \ldots + a_m b^{m-1} = q_0$. Continuing this way, we get $a_k = r_k$ for all $k \le m$ and $r_k = 0$ for all $k \ge m+1$. The integer m is thus the largest integer such that $r_k \neq 0$. $\square$

We now extend our base b expansion to real numbers. Let

$$\mathcal{B} = \{(p_n)_{n \in \mathbb{N}^*} : p_n \in \{0, \ldots, b-1\}\ \forall n \in \mathbb{N}^*\},$$
$$\mathcal{F} = \{(p_n)_{n \in \mathbb{N}^*} \in \mathcal{B} : \exists n_0 \text{ with } p_n = b-1\ \forall n \ge n_0\}, \quad \mathcal{P} = \mathcal{B} \setminus \mathcal{F}\,.$$

Lemma 4.2. *Given $x \in [0, 1[$, there exists a unique $(p_n)_n \in \mathcal{P}$ such that*

$$x = \lim_{n \to +\infty} \left(\frac{p_1}{b} + \frac{p_2}{b^2} + \ldots + \frac{p_n}{b^n} \right).$$

We denote this expansion by $x = (0\text{.}p_1 p_2 \ldots p_n p_{n+1} \ldots)_b$. When $b = 10$, we simply write $x = 0\text{.}p_1 \ldots p_n p_{n+1} \ldots$

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[2] n_0 exists by the well-ordering principle.

Proof. Since $xb \in [0, b[$, we have $0 \leq [xb] \leq b-1$. Let $p_1 = [xb]$. Then $p_1 \leq xb < p_1 + 1$, so dividing by b we get

$$x_1 := x - \frac{p_1}{b} \in \left[0, \frac{1}{b}\right[.$$

Thus, $x_1 b^2 \in [0, b[$, so $0 \leq [x_1 b^2] \leq b-1$. Let $p_2 = [x_1 b^2]$. Then

$$x_2 := x_1 - \frac{p_2}{b^2} = x - \frac{p_1}{b} - \frac{p_2}{b^2} \in \left[0, \frac{1}{b^2}\right[.$$

After n analogous steps, we obtain integers $p_1, p_2, \ldots p_n$ between 0 and $b-1$ such that

$$x_n := x - \frac{p_1}{b} - \frac{p_2}{b^2} - \cdots - \frac{p_n}{b^n} \in \left[0, \frac{1}{b^n}\right[.$$

Since $\frac{1}{b^n} \to 0$ and $0 \leq x_n \leq \frac{1}{b^n}$, we have $x_n \to 0$. Thus, the sequence

$$\frac{p_1}{b} + \frac{p_2}{b^2} + \cdots + \frac{p_n}{b^n} = x - x_n \to x .$$

The sequence $(p_n)_n$ cannot be in $\mathcal{F}$. Indeed, suppose there is an integer $n \geq 1$ such that $p_m = b-1$ for all $m \geq n$. In this case,

$$x_{n-1} = x - \frac{p_1}{b} - \frac{p_2}{b^2} - \cdots - \frac{p_{n-1}}{b^{n-1}} = \lim_{i\to\infty} \left(\frac{b-1}{b^n} + \frac{b-1}{b^{n+1}} + \cdots + \frac{b-1}{b^i}\right) = \frac{1}{b^{n-1}} ,$$

since $\sum \frac{1}{b^j} = \frac{1}{1-\frac{1}{b}} = \frac{b}{b-1}$. However, by construction, $x_{n-1} < \frac{1}{b^{n-1}}$, which is a contradiction. Thus, $(p_n) \in \mathcal{P}$.

It remains to prove the uniqueness of the sequence. Suppose there exists $(p_n)_n \in \mathcal{P}$ such that $x = \lim \left(\frac{p_1}{b} + \ldots + \frac{p_n}{b^n}\right)$. Then $x - \frac{p_1}{b} = \lim \left(\frac{p_2}{b^2} + \ldots + \frac{p_n}{b^n}\right)$. Since $(p_n)_n \in \mathcal{P}$, there exists $m \geq 1$ such that $0 \leq p_m \leq b-2$. If $n > m$, we have

$$\begin{aligned} 0 \leq \frac{p_2}{b^2} + \cdots + \frac{p_m}{b^m} + \cdots + \frac{p_n}{b^n} &\leq \frac{b-1}{b^2} + \cdots + \frac{b-2}{b^m} + \cdots + \frac{b-1}{b^n} \\ &= \frac{b-1}{b^2}\left(1 + \frac{1}{b} + \cdots + \frac{1}{b^{n-2}}\right) - \frac{1}{b^m} \\ &= \frac{b-1}{b^2}\left(\frac{1 - 1/b^{n-1}}{1 - 1/b}\right) - \frac{1}{b^m} . \end{aligned}$$

Taking $n \to \infty$, we get

$$0 \leq x - \frac{p_1}{b} \leq \frac{1}{b} - \frac{1}{b^m} .$$

The inequalities $0 \leq xb - p_1 \leq 1 - b^{1-m} < 1$ yield $p_1 = [xb]$. We thus showed that p_1 is the first term of the sequence we constructed. By induction, we verify that p_n corresponds to the nth term of the sequence we constructed. □

Given any $x \in \mathbb{R}$, we have $x = [x] + \{x\}$. We proved expansions $[x] = (q_m \ldots q_0)_b$ and $\{x\} = (0 \centerdot p_1 p_2 \ldots p_n p_{n+1} \ldots)_b$. In conclusion, we have $x = (q_m \ldots q_0 \centerdot p_1 \ldots p_n p_{n+1} \ldots)_b$.

In particular, in base 10, any $x \in \mathbb{R}$ takes the form $x = q_m \ldots q_0 \centerdot p_1 \ldots p_n p_{n+1} \ldots$ for some integers q_j and p_j. Defining $x_n := \frac{[10^n x]}{10^n}$, we obtain a sequence of rationals such that $x_n \to x$. Note that $x_1 = q_m \ldots q_0 \centerdot p_1$ and $x_2 = q_m \ldots q_0 \centerdot p_1 p_2$ and so on.

We conclude this chapter with a familiar property of rational numbers.

Definition 4.3. We say that $(p_n)_n$ is periodic if there exists $k \in \mathbb{N}$ such that $p_{n+k} = p_n$ for all $n \in \mathbb{N}$. We call k the *period* of this sequence. We say that $(p_n)_n$ is *eventually periodic* if there exists $k \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $p_{n+k} = p_n$ for all $n \geq n_0$.

Lemma 4.4. *A real number $x \in [0, 1[$ is rational if and only if its expansion in base b is eventually periodic.*

Proof. Suppose p/q is a rational, with p and q coprime satisfying $0 \leq p < q$. Dividing bp by q, we may find integers a_1 and r_1 satisfying $bp = a_1 q + r_1$ with $0 \leq r_1 < q$. By the last inequality, $a_1 = [bp/q]$. But if $p/q = \frac{\alpha_1}{b} + \frac{\alpha_2}{b^2} + \cdots$, then $[bp/q] = \alpha_1$, hence a_1 is the first term in the base b expansion of p/q. Now divide br_1 by q to get a_2 and r_2 satisfying $br_1 = a_2 q + r_2$ with $0 \leq r_2 < q$. Then $a_2 = [br_1/q] = [b^2 p/q - ba_1]$. But $[b^2 p/q - ba_1] = \alpha_2$, hence a_2 is the second term in the base b expansion of p/q. Proceeding this way, we obtain a sequence of quotients (a_n) which is the base b expansion of p/q, and a sequence r_n of remainders between 0 and $q-1$.

If $r_m = 0$ for some m, then $a_{m+1} = [br_m/q] = 0$ and $r_{m+1} = 0$. Thus $a_k = 0\ \forall k > m$, so the expansion of p/q terminates and is indeed eventually periodic, with period 1.

Suppose now that all the r_n are non-zero. Since $r_k \in \{1, \ldots, q-1\}$ for all $k \geq 1$, the integers $r_1, \ldots, r_q$ cannot be all distinct, say $r_h = r_i$ for some $1 \leq h < i \leq q$. Then $br_h = a_{h+1} q + r_{h+1}$ and $br_h = a_{i+1} q + r_{i+1}$, so by uniqueness, $a_{h+1} = a_{i+1}$ and $r_{h+1} = r_{i+1}$. Repeating, we find $a_{h+j} = a_{i+j}$ for all j. Thus, for $n \geq h$ we have $a_{n+i-h} = a_{i+(n-h)} = a_{h+(n-h)} = a_n$, so (a_n) is eventually periodic with period $i-h$. Note that the period $i-h$ is at most $q-1$.

Conversely, suppose that $x \in [0,1[$ has an eventually periodic expansion, say $x = (0{\bullet}q_1 q_2 \ldots q_n p_1 p_2 \ldots p_m p_1 p_2 \ldots p_m p_1 p_2 \ldots)_b$. Then

$$x = \sum_{j=1}^{n} \frac{q_j}{b^j} + \sum_{j=1}^{m} \frac{p_j}{b^{j+n}} + \sum_{j=1}^{m} \frac{p_j}{b^{j+n+m}} + \sum_{j=1}^{m} \frac{p_j}{b^{j+n+2m}} + \cdots.$$

Thus, $b^{n+m}x - b^n x = \sum_{j=1}^{n} q_j b^{n+m-j} + \sum_{j=1}^{m} p_j b^{m-j} - \sum_{j=1}^{n} q_j b^{n-j}$, which is an integer. Thus, x is rational. □

For example, writing in base 10, the number $x = 0{\bullet}12345678910111213141516\ldots$ is irrational. It is called the *Champernowne constant*.

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