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Principles of Mathematical Analysis

M 232

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Contents

Local Study of Functions

0.1 Reminders

Lemma 0.1. *Suppose that $f :]a, b[\to \mathbb{R}$ attains a maximum or minimum at $c \in]a, b[$. If f is differentiable at c, then $f'(c) = 0$.*

Proof. Suppose f has a maximum at c and let $g(x) = \frac{f(x)-f(c)}{x-c}$. Then
– if $x > c$, we have $g(x) \leq 0$, hence $f'(c) = \lim_{x\downarrow c} g(x) \leq 0$,
– if $x < c$, we have $g(x) \geq 0$, hence $f'(c) = \lim_{x\uparrow c} g(x) \geq 0$.
Hence, $f'(c) = 0$. If f has a minimum at c, then $-f$ has a maximum at c, hence $-f'(c) = 0$ and thus $f'(c) = 0$. □

Theorem 0.2 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$. If $f(a) = f(b) = 0$, then there exists $c \in]a, b[$ such that $f'(c) = 0$.*

Proof. Since $|f|$ is continuous on $[a, b]$, $|f|$ attains its maximum at a point $c \in [a, b]$. If $|f(c)| = 0$, then $|f(x)| = 0$ for all x (because $0 \leq |f(x)| \leq |f(c)|$), hence f is identically zero and the theorem is clear. If $|f(c)| > 0$, then $c \neq a, b$, moreover f has a maximum or minimum at c. Hence, $f'(c) = 0$ by the previous lemma. □

Theorem 0.3 (Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$. Then there exists $c \in]a, b[$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Proof. Let $g(x) = (b-a)f(x)+(f(a)-f(b))x+af(b)-bf(a)$. Then $g'(x) = (b-a)f'(x)+f(a) - f(b)$. Moreover, $g(b) = g(a) = 0$, so by Rolle's Theorem, there exists $c \in]a, b[$ such that $g'(c) = 0$, i.e. $f(b) - f(a) = f'(c)(b - a)$. □

Corollary 0.4. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$. Then*

(i) *f is constant iff $f' = 0$,*

(ii) *f is increasing (resp. decreasing) iff $f' \geq 0$ (resp. $f' \leq 0$),*

(iii) *if $f' > 0$ (resp. $f' < 0$), then f is strictly increasing (resp. strictly decreasing).*

Proof. If $f = C$ and $x \in]a, b[$, then $\frac{f(x+h)-f(x)}{h} = \frac{C-C}{h} = 0$, so $f'(x) = 0$. Conversely, if $f'(x) = 0$ for all $x \in]a, b[$, let $y > a$, then by Theorem 0.3, we may find $c \in]a, y[$ such that $f(y) - f(a) = (y - a)f'(c) = 0$. Hence, $f(y) = f(a)$ for all $y \in [a, b]$ and f is constant.

If f is increasing, then $\frac{f(x+h)-f(x)}{h} \geq 0$, hence $f'(x) \geq 0$. Conversely, if $f' \geq 0$, then given $a \leq x < y \leq b$, we have $f(y) - f(x) = (y - x)f'(c) \geq 0$ (if $f' > 0$, we also have $f(y) - f(x) > 0$). The case of decreasing functions is similar. □

0.2 Taylor-Young Formula

Definition 0.5 (Landau notation)**.** Let f, g be two functions defined near $x_0 \in \mathbb{R}$. Assume $g(x) \neq 0$ for $x \neq x_0$. We say that

1. $f = \mathrm{O}(g)$ near x_0 if there exists $M > 0$ such that $|f(x)| \leq M|g(x)|$ near x_0,
2. $f = \mathrm{o}(g)$ near x_0 if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$,
3. $f \sim g$ near x_0 if $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$.

When $f = \mathrm{O}(g)$, we say that f is *dominated* by g, when $f = \mathrm{o}(g)$, we say that f is *negligible* in front of g, when $f \sim g$, we say that f is *equivalent* to g.

Recall that f is differentiable at x_0 iff $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists; in this case we have by definition $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$. Hence, f is differentiable at x_0 iff

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \mathrm{o}(x - x_0) \quad \text{near } x_0 \,. \qquad (\star)$$

This means that $x \mapsto f(x_0) + f'(x_0)(x - x_0)$ is the best *linear approximation* of f near x_0. The aim of the Taylor-Young formula is to give the best *polynomial approximation* of f near x_0.

Lemma 0.6. *Suppose g is differentiable near 0 and satisfies $g'(h) = \mathrm{o}(h^k)$ near 0. Then $g(h) - g(0) = \mathrm{o}(h^{k+1})$ near 0.*

Proof. Let $\varepsilon > 0$. Since $g'(h)/h^k \to 0$ as $h \to 0$, we may find $h_0 > 0$ such that $|g'(h)| \leq \varepsilon|h|^k$ for $h \in \,]-h_0, h_0[\, \setminus \{0\}$. Let $h \in \,]-h_0, h_0[\, \setminus \{0\}$. Then by Theorem 0.3, we may find x between 0 and h such that $|g(h) - g(0)| = |g'(x)||h - 0| \leq \varepsilon|x|^k|h| \leq \varepsilon|h|^{k+1}$. Hence, $\frac{g(h)-g(0)}{h^{k+1}} \to 0$ as $h \to 0$. □

Theorem 0.7 (Taylor-Young formula)**.** *Let f be defined near x_0 and assume f is differentiable n times at x_0. Then near x_0 we have*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \mathrm{o}((x - x_0)^n) \,.$$

Proof. We prove the result by induction. When $n = 1$, the theorem is true by $(\star)$.

Suppose the result is true for $n - 1$ and apply it to f', which is differentiable $n - 1$ times at x_0. Then taking $h = x - x_0$, we have

$$f'(x_0 + h) = \sum_{j=0}^{n-1} \frac{(f')^{(j)}(x_0)}{j!} h^j + \mathrm{o}(h^{n-1}) = \sum_{j=0}^{n-1} \frac{f^{(j+1)}(x_0)}{j!} h^j + \mathrm{o}(h^{n-1}) \,.$$

Let $g(h) = f(x_0 + h) - f(x_0) - \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!} h^k$. Then

$$g'(h) = f'(x_0 + h) - \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{(k-1)!} h^{k-1} = f'(x_0 + h) - \sum_{j=0}^{n-1} \frac{f^{(j+1)}(x_0)}{j!} h^j = \mathrm{o}(h^{n-1}) \,.$$

Since $g(0) = 0$, we have $g(h) = \mathrm{o}(h^n)$ by Lemma 0.6, as asserted. □

Corollary 0.8. *The following expansions are valid near* 0 *:*

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \mathrm{o}(x^n) \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \mathrm{o}(x^{2n}) \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \mathrm{o}(x^{2n+1}) \\ (1+x)^a &= 1 + ax + \frac{a(a-1)}{2} x^2 + \ldots + \frac{a(a-1)\cdots(a-n+1)}{n!} x^n + \mathrm{o}(x^n) \\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n+1} \frac{x^n}{n} + \mathrm{o}(x^n) \end{aligned}$$

Remark 0.9. In Corollary 0.8,

(i) we may replace $\mathrm{o}(x^n)$, $\mathrm{o}(x^{2n})$ and $\mathrm{o}(x^{2n+1})$ by $\mathrm{O}(x^{n+1})$, $\mathrm{O}(x^{2n+2})$ and $\mathrm{O}(x^{2n+3})$, respectively. For example, $e^x = 1 + x + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \mathrm{o}(x^{n+1})$. Since $\frac{x^{n+1}}{(n+1)!} + \mathrm{o}(x^{n+1}) = \mathrm{O}(x^{n+1})$, we have the assertion.

(ii) We may remove $\mathrm{o}(x^n)$, $\mathrm{o}(x^{2n})$ and $\mathrm{o}(x^{2n+1})$ from the expansions, but replace $=$ by $\sim$. For example, $\ln(1+x) \sim x$ and $\ln(1+x) \sim x - \frac{x^2}{2}$ near 0. Indeed, $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{x+\mathrm{o}(x)}{x} = \lim_{x\to 0}(1 + \frac{\mathrm{o}(x)}{x}) = 1$, and $\lim_{x\to 0} \frac{\ln(1+x)}{x-\frac{x^2}{2}} = \lim_{x\to 0} \frac{x-\frac{x^2}{2}+\mathrm{o}(x^2)}{x-\frac{x^2}{2}} = \lim_{x\to 0}(1 + \frac{\mathrm{o}(x)}{1-\frac{x}{2}}) = 1$, since $\mathrm{o}(x) \to 0$ as $x \to 0$ (see Remark 5 below).

Remark 0.10. 1. If $f = \mathrm{o}(g)$, then $f = \mathrm{O}(g)$.

2. If $f \sim g$, then $f = \mathrm{O}(g)$ and $g = \mathrm{O}(f)$.
3. If $j \geq k$, then near 0, $\mathrm{o}(x^j) + \mathrm{o}(x^k) = \mathrm{o}(x^k)$. In particular, $\mathrm{o}(x) + \mathrm{o}(x^2) = \mathrm{o}(x)$.
4. $x \cdot \mathrm{o}(x^k) = \mathrm{o}(x^{k+1})$ and $\mathrm{o}(x^k) = x^k \mathrm{o}(1)$.
5. If $k \geq 0$, then $\lim_{x\to 0} \mathrm{o}(x^k) = 0$. In particular, $\lim_{x\to 0} \mathrm{o}(1) = 0$.
6. $\mathrm{o}(g) - \mathrm{o}(g) = \mathrm{o}(g)$ and $\mathrm{O}(g) - \mathrm{O}(g) = \mathrm{O}(g)$.
7. If $c \in \mathbb{R}$, then $c \cdot \mathrm{o}(g) = \mathrm{o}(g)$ and $c \cdot \mathrm{O}(g) = \mathrm{O}(g)$. [1]

Proof. 1. If $f = \mathrm{o}(g)$, then choosing $\varepsilon = 1$, we may find $\delta > 0$ such that $|\frac{f(x)}{g(x)}| \leq 1$, i.e. $|f(x)| \leq |g(x)|$ if $|x - x_0| \leq \delta$.

2. If $f \sim g$, then choosing $\varepsilon = \frac{1}{2}$, we may find $\delta > 0$ such that for $|x - x_0| \leq \delta$, $|\frac{f(x)}{g(x)} - 1| \leq \frac{1}{2}$, i.e. $\frac{1}{2} \leq \frac{f(x)}{g(x)} \leq \frac{3}{2}$, so $\frac{1}{2} \leq |\frac{f(x)}{g(x)}| \leq \frac{3}{2}$ and thus $|f(x)| \leq \frac{3}{2}|g(x)|$ and $|g(x)| \leq 2|f(x)|$.
3. If $f/x^j \to 0$ and $g/x^k \to 0$, then $(f+g)/x^k = (f/x^j)x^{j-k} + g/x^k \to 0$ as $x \to 0$.
4. If $f/x^k \to 0$, then $xf/x^{k+1} \to 0$. By induction we get $\mathrm{o}(x^k) = x^k \mathrm{o}(1)$.
5. If $g = \mathrm{o}(x^k)$, then $\lim_{x\to 0} g(x) = \lim_{x\to 0}(g(x)/x^k)x^k = 0$.
6. If $f_1/g \to 0$ and $f_2/g \to 0$, then $(f_1 - f_2)/g \to 0$. If $|f_1| \leq C_1|g|$ and $|f_2| \leq C_2|g|$ near x_0, then $|f_1 - f_2| \leq |f_1| + |f_2| \leq (C_1 + C_2)|g|$ near x_0.
7. If $f/g \to 0$, then $cf/g \to 0$. If $|f| \leq C|g|$, then $|cf| \leq C'|g|$. □

1. $\mathrm{O}(g)$ and $\mathrm{o}(g)$ are sets of functions, so the equality $f = \mathrm{O}(g)$ is not very precise (it should be $f \in \mathrm{O}(g)$ and $f \in \mathrm{o}(g)$). This is why we get some strange rules in Remark 0.10. Note also that the equality sign is not symmetric: $0 = \mathrm{o}(g)$, but $\mathrm{o}(g) \neq 0$. Still, these notations are very helpful in practice.

Example 0.11. (1) $\lim_{x\to 0}\frac{\cos(x)-1}{x\sin(x)} = -\frac{1}{2}$. Indeed, $\cos(x)-1 = -x^2/2 + o(x^2)$ and $x\sin(x) = x^2 + o(x^2)$, hence $\frac{\cos(x)-1}{x\sin(x)} = \frac{-1/2+o(1)}{1+o(1)} \to -1/2$.

(2) $\lim_{x\to 0}\frac{e^x-2\sqrt{1+x}+1}{x^2} = \frac{3}{4}$. Indeed, $\sqrt{1+x} = 1+x/2-x^2/8+o(x^2)$, hence $e^x-2\sqrt{1+x}+1 = (1+x+x^2/2+o(x^2)) - (2+x-x^2/4+o(x^2)) + 1 = \frac{3}{4}x^2 + o(x^2)$.

(3) Expand $f(x) = \frac{x^2\sin(x)}{1+x}$ to the order 5.
Let $g(x) = x^2\sin(x)$ and $h(x) = 1/(1+x)$. Then $g(x) = x^3 - x^5/6 + o(x^5)$ and $h(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + o(x^5)$. Hence, using remarks 3 and 4,

$$\begin{aligned} f(x) = g(x)h(x) &= (x^3 - x^5/6 + o(x^5))(1 - x + x^2 - x^3 + x^4 - x^5 + o(x^5)) \\ &= x^3 - x^4 + 5x^5/6 + o(x^5)\,. \end{aligned}$$

0.3 Estimation of the remainder

Theorem 0.12 (Taylor-Lagrange formula)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable $(n+1)$ times on $]a,b[$. Then there exists $c \in]a,b[$ such that*

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}\,.$$

Proof. Let $g(x) = f(x) - f(b) + \sum_{i=1}^n \frac{(b-x)^i}{i!} f^{(i)}(x)$. Then

$$g'(x) = f'(x) - \sum_{i=1}^n \frac{(b-x)^{i-1}}{(i-1)!} f^{(i)}(x) + \sum_{i=1}^n \frac{(b-x)^i}{i!} f^{(i+1)}(x) = f^{(n+1)}(x)\frac{(b-x)^n}{n!}\,.$$

Let $\lambda = -g(a)\frac{(n+1)!}{(b-a)^{n+1}}$ and $h(x) = g(x) + \lambda\frac{(b-x)^{n+1}}{(n+1)!}$. Then $h(a) = h(b) = 0$, so by Rolle's Theorem, we may find $c \in]a,b[$ such that $0 = h'(c) = \frac{(b-c)^n}{n!}(f^{(n+1)}(c) - \lambda)$. Hence, $\lambda = f^{(n+1)}(c)$. Inserting this in the equation $h(a) = 0$, we obtain Taylor's formula. □

Theorem 0.13 (Taylor formula with integral remainder)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuously differentiable $(n+1)$ times on $]a,b[$. Then*

$$f(b) = f(a) + (b-a)f'(a) + \ldots + \frac{(b-a)^n}{n!}f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t\,.$$

Proof. We prove the result by induction. For $n = 0$, the theorem is true by the fundamental theorem of integration ($f(b) = f(a) + \int_a^b f'(t)\,\mathrm{d}t$). Now integrate by parts with $u = f^{(n+1)}(t)$ and $v = \frac{-(b-t)^{n+1}}{(n+1)!}$ to get $\int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t = \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(a) + \int_a^b \frac{(b-t)^{n+1}}{(n+1)!} f^{(n+2)}(t)\,\mathrm{d}t$. Hence, if the result is true for n, it is also true for $n+1$. □

0.4 Exercises

1. Expand $f(x) = \frac{1}{1+x+\frac{x^2}{2}}$ to the order 2 near 0.

2. Expand $f(x) = e^{x+10}$ to the order 5 near 0.

3. Let $g(x) = \frac{1}{\cos x}$ on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Expand g to the order 4 near 0.

4. Expand $\tan x$ to the order 5 near 0.

5. Expand $f(x) = x(1+\cos x) - 2\tan x$ to the order 3 near 0. Deduce a function which is equivalent to f.

6. Calculate the following limits. Here $a, b > 0$.

- $\displaystyle\lim_{x\to 0} \frac{x(1+\cos x) - 2\tan x}{2x - \sin x - \tan x}$
- $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\ln(1+x)}\right)$
- $\displaystyle\lim_{x\to 0} \frac{a^x - b^x}{x}$
- $\displaystyle\lim_{x\to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$

7. Give an example of functions f, g, h such that $f \sim g$ near 0, but the relation $h+f \sim h+g$ is not true. Show that if h/g has a limit, then $h + f \sim h + g$.

Chapter 1

Numerical Sequences

1.1 Limits of numerical sequences

Definition 1.1. A *numerical sequence* is a map $u : \mathbb{N} \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote $u_n := u(n)$. The map itself is denoted by $(u_n)_{n\in\mathbb{N}}$.

If $\mathbb{K} = \mathbb{R}$, $(u_n)_{n\in\mathbb{N}}$ is *real sequence.* If $\mathbb{K} = \mathbb{C}$, $(u_n)_{n\in\mathbb{N}}$ is *complex sequence.*

More generally, a sequence may start from an index $k_0 \in \mathbb{N}$. In this case, we denote it by $(u_n)_{n\geq k_0}$. For simplicity, we shall often denote sequences by (u_n).

Definition 1.2. We say that a sequence (u_n) is *bounded* if

$$\exists M > 0 : \ |u_n| \leq M \qquad \forall n \in \mathbb{N} .$$

Definition 1.3. We say that $\ell \in \mathbb{C}$ is a *limit* for a numerical sequence (u_n) if

$$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} : n \geq n_0 \implies |u_n - \ell| \leq \varepsilon .$$

If (u_n) has a limit, we say that (u_n) is *convergent.* Otherwise, we say that (u_n) is *divergent.*

Lemma 1.4 (Uniqueness of the limit)**.** *Any sequence has at most one limit.*

Proof. Suppose on the contrary that a numerical sequence (u_n) has two limits $\ell \neq \ell'$ and choose $\varepsilon := \frac{|\ell-\ell'|}{3} > 0$. Then there exists an index n_1 such that $|u_n - \ell| \leq \varepsilon \ \forall n \geq n_1$, and an index n_2 such that $|u_n - \ell'| \leq \varepsilon \ \forall n \geq n_2$. Let $n_3 = \max(n_1, n_2)$. Then

$$|\ell - \ell'| = |(\ell - u_{n_3}) + (u_{n_3} - \ell')| \leq |\ell - u_{n_3}| + |u_{n_3} - \ell'| \leq 2\varepsilon = \frac{2}{3}|\ell - \ell'| .$$

This contradiction shows that $\ell = \ell'$. □

The unique limit of a sequence, when it exists, is denoted by $\lim\limits_{n\to\infty} u_n$, or more simply by $\lim u_n$. If (u_n) has a limit ℓ, we say it *converges to* ℓ and denote $u_n \to \ell$.

Remark 1.5. Given two sequences u and v, if $\exists n_0 : u_n = v_n \ \forall n \geq n_0$, and if u has a limit $\ell \in \mathbb{C}$, then v converges to the same ℓ. In other words, if a sequence converges, and if we change a finite number of its terms, then this does not change the limit.

Lemma 1.6. *Any convergent sequence is bounded.*

Proof. Suppose (u_n) converges to $\ell \in \mathbb{C}$. There there exists $n_0 \in \mathbb{N}$ such that $|u_n - \ell| \leq 1 \ \forall n \geq n_0$. Hence, $|u_n| \leq |u_n - \ell| + |\ell| \leq 1 + |\ell| \ \forall n \geq n_0$. Take $M := \max\{|u_0|, |u_1|, \ldots, |u_{n_0-1}|, 1 + |\ell|\}$. Then $|u_n| \leq M$ for all n, hence (u_n) is bounded. □

Lemma 1.7. *If* $\lim u_n = \ell \in \mathbb{C}$ *and* $\lim v_n = \ell' \in \mathbb{C}$, *then for any* $c \in \mathbb{C}$,

(i) $\lim_{n\to+\infty} cu_n = c\ell$, (ii) $\lim_{n\to+\infty}(u_n + v_n) = \ell + \ell'$,

(iii) $\lim_{n\to+\infty} u_n v_n = \ell\ell'$.

Proof. (i) Given $\varepsilon > 0$, there exists n_0 such that

$$n \geq n_0 \implies |cu_n - c\ell| = |c| \cdot |u_n - \ell| \leq |c| \cdot \varepsilon' \leq \varepsilon\,,$$

where $\varepsilon' = \varepsilon/|c|$ if $c \neq 0$ and $\varepsilon' = 1$ if $c = 0$.

(ii) Given $\varepsilon > 0$, we have

$$|(u_n + v_n) - (\ell + \ell')| = |(u_n - \ell) + (v_n - \ell')| \leq |u_n - \ell| + |v_n - \ell'|\,.$$

We know there exist $n_1, n_2 \in \mathbb{N}$ such that $|u_n - \ell| \leq \frac{\varepsilon}{2}$ $\forall n \geq n_1$ and $|v_n - \ell'| \leq \frac{\varepsilon}{2}$ $\forall n \geq n_2$. Take $n_3 = \max(n_1, n_2)$. Then $|(u_n + v_n) - (\ell + \ell')| \leq \varepsilon$ $\forall n \geq n_3$.

(iii) Given $\varepsilon > 0$, we have

$$\begin{aligned}|u_n v_n - \ell\ell'| &= |u_n v_n + u_n\ell' - u_n\ell' - \ell\ell'| \\ &= |u_n(v_n - \ell') + (u_n - \ell)\ell'| \leq |u_n||v_n - \ell'| + |u_n - \ell||\ell'|\,.\end{aligned}$$

Since (u_n) converges, it is bounded, so there exists $M > 0$ such that $|u_n| \leq M$ for all n. On the other hand, we may find $n_1, n_2 \in \mathbb{N}$ such that $|v_n - \ell'| \leq \frac{\varepsilon}{2M}$ for $n \geq n_1$ and $|u_n - \ell| \leq \varepsilon'$ for $n \geq n_2$, where $\varepsilon' = \frac{\varepsilon}{2|\ell'|}$ if $\ell' \neq 0$ and $\varepsilon = 1$ if $\ell' = 0$. Take $n_3 = \max(n_1, n_2)$. Then $|u_n v_n - \ell\ell'| \leq \varepsilon$ $\forall n \geq n_3$. □

Lemma 1.8. *Suppose* $u_n \to \ell \neq 0$. *Then there exists* k_1 *such that* $u_n \neq 0$ *for all* $n \geq k_1$. *Moreover, the sequence* $(1/u_n)_{n\geq k_1}$ *converges to* $1/\ell$.

Proof. Given $\varepsilon > 0$, we have $|\frac{1}{u_n} - \frac{1}{\ell}| = |\frac{u_n - \ell}{\ell u_n}|$. We start by giving a lower bound on the denominator. We have $|u_n| \geq |\ell| - |u_n - \ell|$. Choosing $\varepsilon' = \frac{|\ell|}{2}$, we know that $|u_n - \ell| \leq \varepsilon'$ starting from an index k_1. Hence, $|u_n| \geq |\ell| - \varepsilon' = \varepsilon'$ and we get

$$\Big|\frac{1}{u_n} - \frac{1}{\ell}\Big| = \Big|\frac{u_n - \ell}{\ell u_n}\Big| \leq \frac{1}{|\ell|\varepsilon'}|u_n - \ell| \qquad \text{for all } n \geq k_1\,.$$

But we may find k_2 such that $|u_n - \ell| \leq \varepsilon''$ for all $n \geq k_2$, where $\varepsilon'' = |\ell|\varepsilon'\varepsilon$. Hence, $|\frac{1}{u_n} - \frac{1}{\ell}| \leq \varepsilon$ for all $n \geq \max(n_1, n_2)$. □

Lemma 1.9. *If* (u_n) *is bounded and* (v_n) *converges to* 0, *then* $(u_n v_n)$ *converges to* 0.

Proof. Let $\varepsilon > 0$. We may find $M > 0$ such that $|u_n| \leq M$ for all n. On the other hand, choosing $\varepsilon' = \frac{\varepsilon}{M}$, we may find n_0 such that $|v_n| = |v_n - 0| \leq \varepsilon'$ for all $n \geq n_0$. Hence, $|u_n v_n| \leq M\varepsilon' = \varepsilon$ for all $n \geq n_0$. □

Definition 1.10. Let $z_0 \in \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $B_r(z_0) := \{z \in \mathbb{K} : |z - z_0| < r\}$ and let $\overline{B}_r(z_0) = \{z \in \mathbb{K} : |z - z_0| \leq r\}$. If $\mathbb{K} = \mathbb{R}$, then $B_r(z_0) = \,]z_0 - r, z_0 + r[$ and $\overline{B}_r(z_0) = [z_0 - r, z_0 + r]$. If $\mathbb{K} = \mathbb{C}$, then $B_r(z_0)$ is an *open disk* of radius $r > 0$, centered at z_0, while $\overline{B}_r(z_0)$ is a *closed disk*.

Definition 1.11. Let $f : B_r(z_0) \to \mathbb{C}$ and $p \in \overline{B}_r(z_0)$. We say that $\lim_{z\to p} f(z) = \ell$ if

$$\forall\varepsilon > 0\ \exists\delta > 0 : \forall z \in B_r(z_0),\ |z - p| \leq \delta \implies |f(z) - \ell| \leq \varepsilon\,.$$

Lemma 1.12. *Let $f : B_r(z_0) \to \mathbb{C}$ and $p \in \overline{B}_r(z_0)$. Then*

$$\lim_{z\to p} f(z) = \ell \iff \left(\forall \{u_n\} \subseteq B_r(z_0) : \ u_n \to p \implies f(u_n) \to \ell\right).$$

Proof. Let $\varepsilon > 0$ and suppose $\lim_{z\to p} f(z) = \ell$. Then we may find $\delta > 0$ such that $|f(z) - \ell| \leq \varepsilon$ for any $z \in B_r(z_0)$ satisfying $|z - p| \leq \delta$. But if $u_n \to p$, then $\exists n_0$ such that $|u_n - p| \leq \delta$ for all $n \geq n_0$. Hence, $|f(u_n) - \ell| \leq \varepsilon$ for all $n \geq n_0$ and thus $f(u_n) \to \ell$.

Conversely, suppose that $\lim_{z\to p} f(z) \neq \ell$. Then

$$\exists \varepsilon > 0\ \forall \delta > 0 : \exists z \in B_r(z_0),\ |z - p| \leq \delta \quad \text{but} \quad |f(z) - \ell| > \varepsilon.$$

Hence, for $\delta = 2^{-n}$ we may find $u_n \in B_r(z_0)$ such that $|u_n - p| \leq 2^{-n}$ but $|f(u_n) - \ell| > \varepsilon$. Hence, $u_n \to p$ but $f(u_n) \nrightarrow \ell$. The assertion follows by contraposition. □

Example 1.13. $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist. Indeed, if $x_n = \frac{2}{(2n+1)\pi}$, then $x_n \to 0$ but $\sin(\frac{1}{x_n}) = \sin(\frac{(2n+1)\pi}{2}) = (-1)^n$ has no limit. Hence, $\sin(\frac{1}{x})$ has no limit by Lemma 1.12.

Definition 1.14. We say that $f : B_r(z_0) \to \mathbb{C}$ is *continuous at $p \in B_r(z_0)$* if

$$\forall \varepsilon > 0\ \exists \delta > 0 : \forall z \in B_r(z_0),\ |z - p| < \delta \implies |f(z) - f(p)| \leq \varepsilon.$$

Lemma 1.15. *Let $f : B_r(z_0) \to \mathbb{C}$ and $p \in B_r(z_0)$. Then f is continuous at p iff $\forall \{u_n\} \subseteq B_r(z_0)$, if $u_n \to p$, then $f(u_n) \to f(p)$.*

Proof. Comparing Definition 1.11 and Definition 1.14, we see that f is continuous at p iff $\lim_{z\to p} f(z) = f(p)$. Hence, the assertion follows from Lemma 1.12. □

Example 1.16. If $u_n \to \ell$, then $|u_n| \to |\ell|$, since $x \mapsto |x|$ is a continuous function.

1.2 Basic properties of $\mathbb{R}$

Definition 1.17. We say that $r \in \mathbb{R}$ is an *upper bound* for a set $A \subset \mathbb{R}$ if

$$r \geq a \qquad \forall a \in A.$$

We say that $r \in \mathbb{R}$ is a *least upper bound* for $A \subset \mathbb{R}$ if
- ⋄ r is an upper bound for A,
- ⋄ any $s < r$ is not an upper bound for A.

The least upper bound is also called the *supremum* of A and is denoted by $r = \sup(A)$.

Theorem 1.18. *$\mathbb{R}$ has the least-upper-bound property. That is, if a nonempty $A \subset \mathbb{R}$ has an upper bound, then it has a least upper bound.*

Proof. Admitted. □

Theorem 1.19. *Given $x > 0$ and $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $nx \geq y$.*

We refer to this as the *archimedean property* of the reals.

Proof. Suppose on the contrary that $nx < y$ for all $n \in \mathbb{N}$. Then $A := \{nx : n \in \mathbb{N}\}$ is a nonempty set of real numbers which has an upper bound y. Hence, it has a least upper bound $r = \sup(A)$. Since $r - x < r$, $r - x$ is not an upper bound for A, so there exists $m \in \mathbb{N}$ such that $r - x < mx$. Hence, $r < (m + 1)x$, which is impossible, since r is an upper bound for A. This contradiction proves the theorem. □

Corollary 1.20. *Given $x > 1$ and $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x^n \geq y$.*

Proof. Let $x = 1 + h$, $h > 0$. Then by the binomial theorem, $(1+h)^n \geq 1 + nh$. By the archimedean property, we may find n such that $nh \geq y - 1$. Hence, $x^n \geq y$. □

Lemma 1.21. *Given $x \in \mathbb{R}$ and $a > 0$, there exists a unique $n \in \mathbb{Z}$ such that*

$$na \leq x < (n+1)a\,.$$

Proof. ◇ *Uniqueness :* If n and n' satisfy the assertion, then

$$\left.\begin{array}{lll} n \leq \frac{x}{a} < n' + 1 & \Longrightarrow n - n' < 1 & \Longrightarrow n - n' \leq 0 \\ n' \leq \frac{x}{a} < n + 1 & \Longrightarrow n' - n < 1 & \Longrightarrow n' - n \leq 0 \end{array}\right\} \Longrightarrow n = n'\,.$$

◇ *Existence :* By the archimedean property, we may find $n_1, n_2 \in \mathbb{N}$ such that $an_1 \geq x$ and $an_2 \geq -x$. Thus, $A := \{k \in \mathbb{Z} : ka \leq x\}$ is a nonempty subset of $\mathbb{Z}$ ($-n_2 \in A$) which has an upper bound n_1. Hence, A has a greatest element $n \in A$.[1] Since $n + 1 \notin A$, we get $na \leq x < (n+1)a$. □

The case $a = 1$ in the previous lemma is of special interest :

Definition 1.22. If $x \in \mathbb{R}$, the unique $n \in \mathbb{Z}$ satisfying $n \leq x < n+1$ is called the *integer part* of x, and denoted by $[x]$. The real $x - [x]$ is called the *fractional part* of x, and is denoted by $\{x\}$.

1.3 Limits of real sequences

We are now interested in certain notions that use the natural order of the reals. Let us first note the following result.

Lemma 1.23. (i) *If a real sequence (u_n) has a limit ℓ, then $\ell \in \mathbb{R}$.*

(ii) *A complex sequence (u_n) converges iff the sequences $(\operatorname{Re} u_n)$ and $(\operatorname{Im} u_n)$ converge.*

Proof. (i) Put $\ell = a + ib$, where $a, b \in \mathbb{R}$. Then for any $\varepsilon > 0$ we have $|u_n - \ell| \leq \varepsilon$ starting from an index n_0. Thus, $|b| \leq \sqrt{(u_{n_0} - a)^2 + b^2} = |u_{n_0} - \ell| \leq \varepsilon$ for any $\varepsilon > 0$, which means that $b = 0$.

(ii) If $(\operatorname{Re} u_n)$ and $(\operatorname{Im} u_n)$ converge, then $(u_n) = (\operatorname{Re} u_n + \operatorname{Im} u_n)$ converge by Lemma 1.7. Conversely, suppose $u_n \to \ell \in \mathbb{C}$ and put $\ell = a + ib$, $a, b \in \mathbb{R}$. Then

$$|\operatorname{Re} u_n - a| \leq \sqrt{(\operatorname{Re} u_n - a)^2 + (\operatorname{Im} u_n - b)^2} = |u_n - \ell|\,,$$

so the convergence of (u_n) to ℓ implies the convergence of $(\operatorname{Re} u_n)$ to a. Similarly, the sequence $(\operatorname{Im} u_n)$ converges to b. □

Lemma 1.24 (Passing to the limit)**.** *Let u and v be two real convergent sequences. If there exists $k_1 \in \mathbb{N}$ such that $u_n \geq v_n$ for all $n \geq k_1$, then $\lim u_n \geq \lim v_n$.*

In particular, taking the constant sequence $v \equiv 0$, if $u_n \geq 0$ for all n, then $\lim u_n \geq 0$.

1. This follows from the well-ordering principle, which says that any nonempty subset of $\mathbb{N}$ has a least element.

Proof. Suppose $u_n \to \ell_1$ and $v_n \to \ell_2$. By hypothesis, we have $u_n - v_n = |u_n - v_n|$ for all large n. Hence,

$$\ell_1 - \ell_2 = \lim(u_n - v_n) = \lim |u_n - v_n| = |\ell_1 - \ell_2| \geq 0\,. \qquad \square$$

Lemma 1.25. *Let (u_n) be a complex sequence and $\ell \in \mathbb{C}$. If there exists a real sequence (v_n) converging to 0 such that $|u_n - \ell| \leq v_n$ for all n, then $u_n \to \ell$.*

Proof. Let $\varepsilon > 0$. Since $v_n \to 0$, there exists n_0 such that $|v_n| \leq \varepsilon$ for all $n \geq n_0$, and thus $|u_n - \ell| \leq v_n \leq \varepsilon$ for all $n \geq n_0$. $\square$

Example 1.26. Let f be a bounded function defined on $X \subseteq \mathbb{R}$. If $M = \sup_X f$, then we may find a sequence (x_n) of elements of X such that $f(x_n) \to M$.

Indeed, $M - \frac{1}{n}$ is not an upper bound, so there exists $x_n \in X$ such that $M - \frac{1}{n} < f(x_n)$. But M is an upper bound, so $f(x_n) \leq M$. Hence, $-\frac{1}{n} < f(x_n) - M \leq 0$ and thus $|f(x_n) - M| \leq \frac{1}{n}$. Hence, $f(x_n) \to M$ by Lemma 1.25.

Corollary 1.27 (Sandwich Theorem)**.** *Let (u_n), (v_n) and (w_n) be real sequences. If there exists k such that $u_n \leq v_n \leq w_n$ for all $n \geq k$, and if $\lim u_n = \lim w_n = \ell \in \mathbb{R}$, then (v_n) converges, and $\lim v_n = \ell$.*

Proof. We have hypothesis $0 \leq v_n - u_n \leq w_n - u_n$ for any $n \geq k$. Since $\lim(w_n - u_n) = 0$, the sequence $(v_n - w_n)$ tends to 0 by Lemma 1.25. Since $v_n = u_n + (v_n - u_n)$, we have $\lim v_n = \ell + 0 = \ell$. $\square$

Definition 1.28. We say that a real sequence (u_n)

- ◇ tends to $+\infty$ if
$$\forall M \in \mathbb{R}\ \exists n_0 \in \mathbb{N} : n \geq n_0 \implies u_n \geq M\,,$$
- ◇ tends to $-\infty$ if $(-u_n)$ tends to $+\infty$, i.e. if
$$\forall M \in \mathbb{R}\ \exists n_0 \in \mathbb{N} : n \geq n_0 \implies u_n \leq M\,.$$

Note that if u and v are real sequences, and if there exists $k \in \mathbb{N}$ such that $u_n \leq v_n$ for all $n \geq k$, then $\big(u \to +\infty \implies v \to +\infty\big)$, and $\big(v \to -\infty \implies u \to -\infty\big)$. This is clear from the definitions, and may be seen as an appendix to the sandwich theorem.

Lemma 1.29. *Let (u_n) be a real sequence. Then $u_n \to +\infty$ iff $\exists k_1 \in \mathbb{N} : u_n > 0\ \forall n \geq k_1$, and the sequence $(\frac{1}{u_n})_{n \geq k_1}$ tends to 0.*

Proof. Suppose $u_n \to +\infty$ and let $\varepsilon > 0$. Then we may find k_1 such that $u_n \geq 1/\varepsilon$ for all $n \geq k_1$. Thus, for $n \geq k_1$, we have $u_n > 0$ and $0 < \frac{1}{u_n} \leq \varepsilon$. Thus, $\frac{1}{u_n} \to 0$.

Conversely, let $M > 0$. If $\frac{1}{u_n} \to 0$, then we may find $n_0 \geq k_1$ such that $0 < \frac{1}{u_n} \leq \frac{1}{M}$, since $u_n > 0$. Hence, $u_n \geq M$ and $u_n \to +\infty$. $\square$

Definition 1.30. We say that a real sequence (u_n) is

- ◇ *bounded from above* if there exists $M \in \mathbb{R}$ such that $u_n \leq M$ for all n,
- ◇ *bounded from below* if there exists $m \in \mathbb{R}$ such that $u_n \geq m$ for all n,
- ◇ *increasing* if $u_{n+1} - u_n \geq 0$ for all n,
- ◇ *decreasing* if $u_{n+1} - u_n \leq 0$ for all n,
- ◇ *monotone* if it is either increasing or decreasing.

We say it is *strictly increasing, strictly decreasing* or *strictly monotone* if the corresponding inequality is strict.

Theorem 1.31. *Let u be an increasing real sequence.*

1. *If it is bounded from above, then it converges to $\ell = \sup\{u_n : n \in \mathbb{N}\}$.*
2. *If it is not bounded from above, then it tends to $+\infty$.*

If v is decreasing and bounded from below, then it converges to $\ell' = \inf\{v_n : n \in \mathbb{N}\}$.

Proof. 1. If u is bounded from above, then $A := \{u_n : n \in \mathbb{N}\}$ has a least upper bound ℓ by Theorem 1.18. Let $\varepsilon > 0$. Then $\ell - \varepsilon$ is not an upper bound, so $\exists n_0 \in \mathbb{N}$ such that $\ell - \varepsilon < u_{n_0}$. Since u is increasing and bounded from above by ℓ, we get

$$n \geq n_0 \implies \ell - \varepsilon < u_{n_0} \leq u_n \leq \ell\,,$$

hence $u_n \to \ell$.

2. If u is not bounded from above, then given $M > 0$, we may find n_0 such that $u_{n_0} > M$. Since u is increasing, we thus have $u_n \geq u_{n_0} > M$ for all $n \geq n_0$. Thus, $u_n \to +\infty$.

Finally, if $u = -v$, then u is increasing and bounded from above, hence converges to $\sup u = \sup(-v) = -\inf v$, so that $v = -u$ converges to $-\sup u = \inf v$. □

Definition 1.32. We say that two real sequences u and v are *adjacent* if

- ◇ $\forall n \in \mathbb{N}$, $u_n \leq v_n$,
- ◇ u is increasing and v is decreasing,
- ◇ $(v_n - u_n)$ tends to 0.

Lemma 1.33. *If u and v are two adjacent sequences, then they converge to a common limit ℓ satisfying $u_n \leq \ell \leq v_n$ for all n.*

Proof. We have $u_n \leq v_n \leq v_0$ for all n, so u is increasing and bounded from above. Hence, u converges and $u_n \leq \lim u$ by Theorem 1.31. Similarly, v is decreasing and bounded from below, hence converges and we have $v_n \geq \lim v$. Finally, $v = u + (v - u)$ and $\lim(v - u) = 0$, so $\lim u = \lim v$. □

Definition 1.34. We say that (I_n) is a sequence of *nested intervals* if each I_n is an interval of $\mathbb{R}$ and $I_{n+1} \subseteq I_n$.

Corollary 1.35 (The Nested Interval Theorem)**.** *If $([a_n, b_n])$ is a sequence of nested closed intervals whose length tends to 0, then the set $\cap_{n\in\mathbb{N}} [a_n, b_n]$ consists of a single point.*

Proof. By hypothesis, the sequences (a_n) and (b_n) are adjacent, so they possess a common limit ℓ satisfying $a_n \leq \ell \leq b_n$ for all n. The point ℓ thus belongs to each $[a_n, b_n]$, hence the intersection of these intervals contains ℓ and is nonempty.

On the other hand, if $x \in \cap_{n\in\mathbb{N}} [a_n, b_n]$, then $a_n \leq x \leq b_n$ for all n, so passing to the limit we get $\ell = \lim a_n \leq x \leq \lim b_n = \ell$. Hence, $x = \ell$ and $\cap_{n\in\mathbb{N}} [a_n, b_n] = \ell$. □

1.4 Some standard limits

Definition 1.36. Let (u_n) and (v_n) be two sequences of numbers. We say that

1. $u_n = \mathrm{O}(v_n)$ if there exists $n_0 \in \mathbb{N}$ and $C > 0$ such that $|u_n| \leq C|v_n|$ for all $n \geq n_0$.
2. $u_n = \mathrm{o}(v_n)$ if there exists a sequence $w_n \to 0$ such that $u_n = w_n v_n$.
3. $u_n \sim v_n$ if there exists a sequence $w_n \to 1$ such that $u_n = w_n v_n$.

Note that if v is never 0, then $u_n = \mathrm{o}(v_n)$ iff $\frac{u_n}{v_n} \to 0$ and $u_n \sim v_n$ iff $\frac{u_n}{v_n} \to 1$.

Remark 1.37. If $u_n = \mathrm{o}(v_n)$, then $u_n = \mathrm{O}(v_n)$. If $u_n \sim v_n$, then $u_n = \mathrm{O}(v_n)$ and $v_n = \mathrm{O}(u_n)$. This is proved as in Chapter 0.

Theorem 1.38. (a) *If* $\alpha > 0$, *then* $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$.

(b) *If* $a \in \mathbb{C}$ *and* $|a| < 1$, *then* $\lim_{n\to\infty} a^n = 0$.

(c) *If* $p > 0$, *then* $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

(d) $\lim \sqrt[n]{n} = 1$.

(e) *If* $a > 1$ *and* $\alpha \in \mathbb{R}$, *then* $\lim_{n\to\infty} \frac{n^\alpha}{a^n} = 0$.

(f) *If* $a > 1$, *then* $\lim_{n\to\infty} a^n = +\infty$.

(g) *For any* $\alpha, \beta > 0$, *we have* $\lim_{n\to\infty} \frac{(\ln n)^\alpha}{n^\beta} = 0$.

Proof. (a) Let $u_n = \frac{1}{n^\alpha}$. Given $\varepsilon > 0$, we use the archimedean property to find $n_0 \in \mathbb{N}$ such that $n_0 \varepsilon^{1/\alpha} \geq 1$. Then $0 \leq u_n \leq \frac{1}{n_0^\alpha} \leq \varepsilon$ for any $n \geq n_0$.

(b) If $a = 0$, this is clear, otherwise $\frac{1}{|a|} > 1$, so given $\varepsilon > 0$ we may use Corollary 1.20 to find n_0 such that $(\frac{1}{|a|})^{n_0} \geq \frac{1}{\varepsilon}$. Hence, $|a^n| \leq |a|^{n_0} \leq \varepsilon$ for any $n \geq n_0$.

(c) If $p = 1$, this is clear. If $p > 1$, put $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$ and by the binomial theorem, $1 + nx_n \leq (1 + x_n)^n = p$. Thus, $0 < x_n \leq \frac{p-1}{n}$ and $x_n \to 0$.
If $0 < p < 1$, we obtain the assertion by considering $q = \frac{1}{p}$.

(d) Put $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$ and $n = (1 + x_n)^n \geq \frac{n(n-1)}{2} x_n^2$. Hence, $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$ for $n \geq 2$. Hence, $x_n \to 0$.

(e) Put $a = 1 + h$, $h > 0$. Let k be an integer such that $k > \alpha$ and $k > 0$. Then for $n > 2k$, we have $n = \frac{n}{2} + \frac{n}{2} > \frac{n}{2} + k$, so by the binomial formula,

$$a^n = (1+h)^n > \binom{n}{k} h^k = \frac{n(n-1)\cdots(n-k+1)}{k!} h^k > \frac{n^k h^k}{2^k k!}.$$

Thus, $0 < \frac{n^\alpha}{a^n} < \frac{2^k k!}{h^k} n^{\alpha - k}$ for any $n > 2k$. Since $\alpha - k < 0$, we have $n^{\alpha-k} \to 0$.

(f) This follows from (e) by taking $\alpha = 0$ and applying Lemma 1.29.

(g) We first show that $(\ln n)^\alpha \leq C n^{\beta/2}$ for some $C > 0$. Given $\gamma > 0$, let $f(x) = \gamma^{-1} x^\gamma - \ln x$. Then $f'(x) = x^{\gamma - 1} - \frac{1}{x} = \frac{x^\gamma - 1}{x} \geq 0$ for any $x \geq 1$. Furthermore, $f(1) = \gamma^{-1} > 0$. Hence, $f(x) > 0$ for any $x \geq 1$. Taking $\gamma = \frac{\beta}{2\alpha}$, we thus get the required inequality for any $n \in \mathbb{N}^*$. The value of the limit now follows from (a). □

Lemma 1.39 (Decimal approximation of a real number)**.** *For any* $x \in \mathbb{R}$, *the sequence* $u_n = \frac{[10^n x]}{10^n}$ *converges to* x.

In particular, any real number is the limit of a sequence of rationals. We say that $\mathbb{Q}$ is *dense* in $\mathbb{R}$.

Proof. By definition, $[10^n x] \leq 10^n x < [10^n x] + 1$, hence $u_n \leq x < u_n + \frac{1}{10^n}$. Thus, $0 \leq x - u_n < \frac{1}{10^n}$ and thus $u_n \to x$. □

Example 1.40. If $x = \sqrt{2}$, then $u_0 = 1$, $u_1 = 1.4$, $u_2 = 1.41$ and $u_3 = 1.414$.

1.5 Subsequences and the Bolzano-Weierstrass Theorem

Definition 1.41. A sequence $(v_n)_{n\in\mathbb{N}}$ is said to be a *subsequence* of $(u_n)_{n\in\mathbb{N}}$ if there exists a strictly increasing map $\varphi : \mathbb{N} \to \mathbb{N}$ such that $v_n = u_{\varphi(n)}$ for all $n \in \mathbb{N}$.

Example 1.42. $(u_{n+1})_{n\in\mathbb{N}}$, $(u_{2n})_{n\in\mathbb{N}}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ are subsequences of $(u_n)_{n\in\mathbb{N}}$.

Note that if $\varphi : \mathbb{N} \to \mathbb{N}$ is strictly increasing, then we see by induction that

$$\varphi(n) \geq n \qquad \forall n \in \mathbb{N}\,.$$

Lemma 1.43. *If $u_n \to \ell$, then any subsequence (v_n) of (u_n) also converges to ℓ.*

Proof. Given $\varepsilon > 0$, we may find n_0 such that $|u_n - \ell| \leq \varepsilon$ for any $n \geq n_0$. But $v_n = u_{\varphi(n)}$ for some strictly increasing $\varphi : \mathbb{N} \to \mathbb{N}$. Hence, $\varphi(n) \geq n$ and

$$n \geq n_0 \implies \varphi(n) \geq n_0 \implies |u_{\varphi(n)} - \ell| \leq \varepsilon\,. \qquad \square$$

Example 1.44. 1. The sequence $(-1)^n$ is divergent, since the subsequence (u_{2n}) tends to 1, while the subsequence (u_{2n+1}) tends to -1.

2. The sequence $u_n = \cos(n\pi/4)$ is divergent, since the subsequence $u_{4n} = (-1)^n$ diverges.

Lemma 1.45. *If the subsequences (u_{2n}) and (u_{2n+1}) of u converge to a common limit ℓ, then u also converges to ℓ.*

Proof. Given $\varepsilon > 0$, we may find $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \implies |u_{2n} - \ell| \leq \varepsilon \quad \text{and} \quad n \geq n_2 \implies |u_{2n+1} - \ell| \leq \varepsilon\,.$$

Take $n_0 = \max(2n_1, 2n_2 + 1)$. Then for $n \geq n_0$, we have $|u_n - \ell| \leq \varepsilon$, hence $u_n \to \ell$. $\square$

Lemma 1.46 (cf. [12])**.** *Any real sequence has a monotone subsequence.*

Proof. Let (u_n) be a real sequence. Call u_{n_0} *dominant* if $u_{n_0} > u_n$ for all $n > n_0$.

If (u_n) has an infinite number of dominant terms with indices $n_0 < n_1 < \ldots$, then the subsequence $(u_{n_j})_{j\in\mathbb{N}}$ is decreasing, since $u_{n_k} > u_{n_{k+1}}$. So the result is true in this case.

If (u_n) has a finite number of dominant terms, let u_N be the last one and put $m_1 = N+1$. Then u_{m_1} is not dominant, so there exists $m_2 > m_1$ such that $u_{m_1} \leq u_{m_2}$. Similarly, u_{m_2} is not dominant since $m_2 > N$, so there exists $m_3 > m_2$ such that $u_{m_2} \leq u_{m_3}$. By induction, there exists a subsequence $u_{m_1} \leq u_{m_2} \leq u_{m_3} \leq \ldots$. This completes the proof. $\square$

Corollary 1.47 (Bolzano-Weierstrass Theorem)**.** *Any bounded sequence has a convergent subsequence.*

Proof. 1. Let u be a real bounded sequence. By Lemma 1.46, it has a monotone subsequence v, which is of course bounded too. Hence, v converges by Theorem 1.31.

2. Let u be a complex bounded sequence and put $u_n = x_n + iy_n$, with $x_n, y_n \in \mathbb{R}$. Since $|x_n| \leq \sqrt{x_n^2 + y_n^2} = |u_n|$, the sequence (x_n) is real and bounded, so it has a convergent subsequence $(x_{\varphi(n)})$ with limit a. Similarly, $|y_{\varphi(n)}| \leq |u_{\varphi(n)}|$, so $(y_{\varphi(n)})$ has a convergent subsequence $(y_{\varphi(\psi(n))})$ with limit b. Finally, $(x_{\varphi(\psi(n))})$ is a subsequence of $(x_{\varphi(n)})$, so $x_{\varphi(\psi(n))} \to a$ by Lemma 1.43. Thus, $u_{\varphi(\psi(n))} = x_{\varphi(\psi(n))} + iy_{\varphi(\psi(n))} \to a + ib$. $\square$

1.6 Cauchy criterion

Definition 1.48. We say that (u_n) is a *Cauchy sequence* if

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} : p, q \geq n_0 \implies |u_p - u_q| \leq \varepsilon \, .$$

Remark 1.49. Equivalently, (u_n) is a Cauchy sequence if

$$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} : r \geq n_0, s \geq 0 \implies |u_r - u_{r+s}| \leq \varepsilon \, .$$

Indeed, to go from the first definition to the second, take $p = r$ and $q = r + s$. For the converse, take $r = \min(p, q)$ and $s = \max(p, q) - \min(p, q)$.

Theorem 1.50. *A sequence converges iff it is a Cauchy sequence.*

Proof. Suppose $u_n \to \ell$ and let $\varepsilon > 0$. Choose n_0 such that $|u_n - \ell| \leq \varepsilon/2$ for all $n \geq n_0$. Then for $p, q \geq n_0$ we have

$$|u_p - u_q| = |(u_p - \ell) - (u_q - \ell)| \leq |u_p - \ell| + |u_q - \ell| \leq \varepsilon \, .$$

Conversely, suppose (u_n) is Cauchy. We first show that (u_n) is bounded. Taking $\varepsilon = 1$, we may find n_0 such that $|u_p - u_q| \leq 1$ for all $p, q \geq n_0$. In particular, $|u_n| \leq 1 + |u_{n_0}|$ for any $n \geq n_0$. Put $M = \max\{|u_0|, \ldots, |u_{n_0-1}|, 1 + |u_{n_0}|\}$. Then $|u_n| \leq M$ for all n, hence (u_n) is bounded.

It follows from the Bolzano-Weierstrass theorem that (u_n) has a convergent subsequence $(u_{\varphi(n)})$ with limit ℓ. So $\exists n_1$ such that $|u_{\varphi(n)} - \ell| \leq \varepsilon/2$ for all $n \geq n_1$. But (u_n) is Cauchy, so $\exists n_2$ such that $|u_p - u_q| \leq \varepsilon/2$ for all $p, q \geq n_2$. Since $\varphi(n) \geq n$, we thus have in particular $|u_n - u_{\varphi(n)}| \leq \varepsilon/2$ for all $n \geq n_2$. Hence, if $n_3 = \max(n_1, n_2)$, we have

$$n \geq n_3 \implies |u_n - \ell| \leq |u_n - u_{\varphi(n)}| + |u_{\varphi(n)} - \ell| \leq \varepsilon \, ,$$

hence (u_n) converges to ℓ. □

1.7 Recursive sequences

In the following, $u_n : \mathbb{N} \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

1.7.1 Arithmetico-geometric sequences

We say that (u_n) is an *arithmetico-geometric sequence* if there exist $a, b \in \mathbb{K}$ such that the following recurrence relation is satisfied :

$$\forall n \in \mathbb{N} : u_{n+1} = a u_n + b \, .$$

If $a = 1$, we have an arithmetic sequence, hence

$$\forall n \in \mathbb{N} : u_n = u_0 + nb \, .$$

Suppose $a \neq 1$, we show that

$$\forall n \in \mathbb{N} : u_n = a^n (u_0 - r) + r \, , \qquad \text{where } r = \frac{b}{1 - a} \, .$$

Indeed, put $v_n = u_n - r$. Then $v_{n+1} = a u_n + b - r = a v_n + a r + b - r = a v_n$. Hence, (v_n) is a geometric sequence with common ratio a. Hence, $u_n = v_n + r = a^n v_0 + r = a^n (u_0 - r) + r$.

If $u_0 = r$, the sequence is thus constant and converges to r. If $|a| < 1$, we also have $u_n \to r$. If $u_0 \neq r$ and $|a| > 1$, then $|u_n| \geq |a|^n |u_0 - r| - |r| \to +\infty$, so (u_n) is divergent (if (u_n) was convergent, $(|u_n|)$ would also be convergent). We will see in Exercise 7 that if $|a| = 1$, then (a^n) converges iff $a = 1$.

Conclusion : if $|a| < 1$, then $u_n \to r$. If $|a| \geq 1$, then (u_n) converges iff $(u_0 = r)$ or $(a = 1$ and $b = 0)$, in which case (u_n) is constant.

1.7.2 Linear recursive sequences of order 2

We say that (u_n) is a *linear recursive sequence of order 2* if

$$\forall n \in \mathbb{N} : u_{n+2} = a u_{n+1} + b u_n \qquad (\star)$$

for some $a, b \in \mathbb{K}$.

Lemma 1.51. *Let $P(X) = X^2 - aX - b$.*

(a) *If P has two distinct roots r_1 and r_2, then the solutions of $(\star)$ are the sequences $(\alpha r_1^n + \beta r_2^n)_{n\in\mathbb{N}}$, with $\alpha, \beta \in \mathbb{K}$.*

(b) *If P has a double root r, then the solutions of $(\star)$ are the sequences $((\alpha + \beta n) r^n)_{n\in\mathbb{N}}$, with $\alpha, \beta \in \mathbb{K}$.*

(c) *If $\mathbb{K} = \mathbb{R}$ and P has no real roots, there exist $\rho > 0$ and $\theta \notin \{k\pi : k \in \mathbb{Z}\}$ such that the solutions of $(\star)$ take the form $(\rho^n(\lambda\cos(n\theta) + \mu\sin(n\theta)))_{n\in\mathbb{N}}$, with $\lambda, \mu \in \mathbb{R}$.*

Proof. Let $E = \{(u_n)_{n\in\mathbb{N}} : (u_n)_{n\in\mathbb{N}} \text{ satisfies } (\star)\}$. Using elementary linear algebra, we see that E is a vector space of dimension 2 (the map $f : E \to \mathbb{K}^2$ given by $f : (u_n)_{n\in\mathbb{N}} \mapsto (u_0, u_1)$ is an isomorphism).

Hence, there are exactly two linearly independent sequences which satisfy $(\star)$.

In case (a), the sequences (r_1^n) and (r_2^n) both satisfy $(\star)$, and they are not proportional. Hence, they form a basis for E, and this proves (a).

In case (b), the sequence $v_n = r^n$ satisfies $(\star)$. Let us show that $w_n = nr^n$ also satisfies $(\star)$. Since P has a double root r, the discriminant $\Delta = a^2 + 4b$ is zero and we have $r = \frac{a}{2}$. Thus,

$$\begin{aligned} w_{n+2} &= (n+2)r^n \cdot r^2 = (n+2)r^n(ar + b) = a(n+1)r^{n+1} + b(nr^n) + ar^{n+1} + 2br^n \\ &= aw_{n+1} + bw_n + (\tfrac{a^2}{2} + 2b)r^n = aw_{n+1} + bw_n \end{aligned}$$

because $r = \frac{a}{2}$ and $\Delta = 0$. Hence, (v_n) and (w_n) form a basis for E, and this proves (b).

Finally, in case (c), P has roots in $\mathbb{C}$ of the form $r_\pm = \rho e^{\pm i\theta}$, with $\rho > 0$ and $\theta \notin \{k\pi : k \in \mathbb{Z}\}$. Since the sequences (r_+^n) and (r_-^n) both satisfy $(\star)$, their linear combination $(\rho^n \cos(n\theta))$ and $(\rho^n \sin(n\theta))$ also satisfy $(\star)$. Moreover, they are real and form a basis for E. This yields (c). □

1.7.3 General recursive sequences of order 1

We conclude this chapter with a study of real sequences $(u_n)_{n\in\mathbb{N}}$ satisfying a recurrence relation of the form

$$\forall n \in \mathbb{N} : u_{n+1} = f(u_n)\,,$$

where $f : I \to \mathbb{R}$ is a function defined on an interval $I \subseteq \mathbb{R}$, with $u_0 \in I$. We also suppose that I is *stable by* f, i.e. $f(I) \subseteq I$, to ensure that the sequence is well defined (indeed, if say $u_1 = f(u_0) \notin I$, then $u_2 = f(u_1)$ is undefined).

Definition 1.52. We say that an interval $I \subseteq \mathbb{R}$ is *closed* if it has the form

$$I = [a, b] \ , \quad I = [a, +\infty[\ , \quad I =]-\infty, b] \quad \textit{or} \quad I = \mathbb{R} \, .$$

Lemma 1.53. *If I is closed, $u_n \in I$ $\forall n$ and $u_n \to \ell \in \mathbb{R}$, then $\ell \in I$.*

Proof. This is trivial if $I = \mathbb{R}$. If $I = [a, +\infty[$, then $u_n \geq a$ $\forall n$, hence $\lim u_n \geq a$ by Lemma 1.24 and thus $\ell \in I$. The other cases are proved similarly. $\square$

Lemma 1.54. *Let $f : I \to I$ be a continuous function on a closed interval I. If the sequence $u_{n+1} = f(u_n)$, $u_0 \in I$ has a limit ℓ, then $\ell \in I$ and $\ell = f(\ell)$.*

Thus, the limit of such a sequence must be a *fixed point* of f.

Proof. Since $u_n \in I$ $\forall n$ and I is closed, then $\ell \in I$ by Lemma 1.53.

On the other hand, $u_n \to \ell$ implies $f(u_n) \to f(\ell)$ by Lemma 1.15, hence $u_{n+1} \to f(\ell)$. But $(u_{n+1})_{n \in \mathbb{N}}$ is a subsequence of $(u_n)_{n \in \mathbb{N}}$, so $u_{n+1} \to \ell$. The uniqueness of the limit now shows that $\ell = f(\ell)$. $\square$

Example 1.55. If $u_{n+1} = u_n^2 + 2$, then u_n diverges, because the equation $x = f(x)$ has no solution. Indeed, $\forall x \in \mathbb{R}$, $x < x^2 + 2$. Taking $x = u_n$, we see that (u_n) is strictly increasing. Since it diverges, it tends to $+\infty$.

Lemma 1.56. *If there exists $\ell \in I$ and $k \in [0, 1[$ such that the map $f : I \to I$ satisfies*

$$\forall x \in I : |f(x) - \ell| \leq k|x - \ell| \, ,$$

then given $u_0 \in I$, the sequence $u_{n+1} = f(u_n)$ converges to ℓ.

Proof. We see by induction that $|u_n - \ell| \leq k^n |u_0 - \ell|$, so $u_n \to \ell$, since $|k| < 1$. $\square$

Example 1.57. Let $u_0 \geq 0$. We define the sequence $u_{n+1} = \frac{5u_n+3}{u_n+5}$, whose terms are nonnegative because the interval $\mathbb{R}_+$ is stable by $f : x \mapsto \frac{5x+3}{x+5}$.

The equation $x = \frac{5x+3}{x+5}$ has two roots $\pm\sqrt{3}$. Since all the u_n are nonnegative, the limit of u must be nonnegative. Hence, if u converges, it must tend to $\sqrt{3}$ by Lemma 1.54.

Given $x \geq 0$, the inequality

$$|f(x) - \sqrt{3}| = \frac{|5x + 3 - x\sqrt{3} - 5\sqrt{3}|}{x + 5} = \frac{|(x - \sqrt{3})(5 - \sqrt{3})|}{x + 5} \leq \Big(\frac{5 - \sqrt{3}}{5}\Big)|x - \sqrt{3}|$$

proves that $u_n \to \sqrt{3}$ by Lemma 1.56.

Definition 1.58. A map $f : I \to \mathbb{R}$ is a *contraction* if it is k-Lipschitz continuous on I, with $k \in [0, 1[$. That is, if $\exists k \in [0, 1[$ such that $\forall x, y \in I : |f(x) - f(y)| \leq k|x - y|$.

Corollary 1.59. *If $\ell \in I$ is a fixed point of f and if $f : I \to I$ is a contraction, then the sequence converges to ℓ.*

Remark 1.60. If f is differentiable on I, and if $|f'(x)| \leq k$ on I, where $k \in [0, 1[$, then f is a contraction on I by the mean value theorem.

Example 1.61. Let $u_0 = 0$ and $u_{n+1} = \frac{1}{4}\sin u_n + \frac{1}{2}$. Since the interval $[0,1]$ is stable by $f(x) = \frac{1}{4}\sin x + \frac{1}{2}$, this defines a sequence of elements in $[0,1]$.

The function $g(x) = x - f(x)$ satisfies $g(0) < 0$ and $g(1) > 0$. Since it is continuous and strictly increasing, there must exist a unique point ℓ such that $g(\ell) = 0$, i.e. $\ell = f(\ell)$.

The function f is $\frac{1}{4}$-Lipschitz continuous, since it is differentiable and $|f'(x)| \leq \frac{1}{4}$. Hence, the sequence converges to ℓ.

We can actually estimate the speed of convergence :

$$|u_n - \ell| \leq \left(\frac{1}{4}\right)^n |u_0 - \ell| \leq \left(\frac{1}{4}\right)^n .$$

Sometimes the preceding criteria are not sufficient to conclude. We can then try to study the monotonicity of the sequence. For instance, if we show that (u_n) is increasing and bounded from above, then we can conclude that it converges (Theorem 1.31). We have the following criteria.

Lemma 1.62. *If $f(x) - x \geq 0$ $\forall x \in I$, then the sequence $u_{n+1} = f(u_n)$ is increasing. If $f(x) - x \leq 0$ $\forall x \in I$, then the sequence is decreasing.*

Proof. We have $u_{n+1} - u_n = f(u_n) - u_n$. □

Lemma 1.63. *If f is increasing, then the sequence $u_{n+1} = f(u_n)$ is monotone.*

Proof. If $u_0 \leq u_1$, then $u_1 = f(u_0) \leq f(u_1) = u_2$ and by induction, $u_n \leq u_{n+1}$ $\forall n$. Similarly, if $u_0 \geq u_1$, then (u_n) is decreasing. □

Remark 1.64. If f is increasing, we can use the fixed points of f to bound the sequence. For instance, if $f(a) = a$ and $u_n \leq a$, then $u_{n+1} = f(u_n) \leq f(a) = a$.

Example 1.65. Let $u_0 \geq 0$ and $u_{n+1} = u_n^2 - u_n + 1$. Since $f(x) = x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4}$, we see that $\mathbb{R}_+$ is stable by f. The equation $f(x) = x$ has a unique solution $x = 1$.

The estimate $|f(x) - 1| = |x||x-1|$ gives no information. So we will argue by monotonicity.

We have $\forall x \in \mathbb{R}_+$, $f(x) \geq x$, so the sequence (u_n) is increasing. Hence, if $u_0 > 1$, the sequence cannot converge to 1, the only fixed point of f, so it diverges to $+\infty$. If $u_0 \leq 1$, the sequence is increasing and bounded above by 1 (use Remark 1.64). Hence, the sequence converges, and its limit must be 1.

We conclude this chapter with an important theorem. In contrast to Corollary 1.59, this theorem *proves* the existence of a fixed point under certain assumptions.

Theorem 1.66 (The Fixed Point Theorem[2], cf. [13])**.** *If $f : I \to I$ is a contraction on a closed interval I, then f has a unique fixed point $\ell \in I$. Moreover, for any $c \in I$, the sequence $u_0 = c$, $u_{n+1} = f(u_n)$ converges to ℓ. The speed of convergence may be estimated by*

$$|u_n - \ell| \leq \frac{k^n}{1-k}|u_1 - u_0| ,$$

where $k \in [0,1[$ is the Lipschitz constant of f.

2. of Banach or Picard.

Proof. For any $x, y \in I$ we have

$$\begin{aligned} |x-y| &\leq |x-f(x)| + |f(x)-f(y)| + |f(y)-y| \\ &\leq |x-f(x)| + k|x-y| + |f(y)-y|, \end{aligned}$$

and thus

$$|x-y| \leq \frac{|f(x)-x| + |f(y)-y|}{1-k}. \qquad (*)$$

This already shows the uniqueness of the eventual fixed point : if x and y are two fixed points, then $|x-y|=0$.

To prove its existence, let $f^n = f \circ \cdots \circ f$ be the map f composed n times with itself, with the convention $f^0(x) = x$. Let $c \in I$. We may write the sequence $u_0 = c$, $u_{n+1} = f(u_n)$ in the form $(f^n(c))_{n\in\mathbb{N}}$. Let us show that this is a Cauchy sequence. It will then follow that it converges to a fixed point $\ell \in I$ by Lemma 1.54, which is the required result.

We first note by induction that $|f^n(x) - f^n(y)| \leq k^n|x-y|$ for any $x, y \in I$. Taking $x = f^n(c)$ and $y = f^m(c)$ in $(*)$, we thus get

$$\begin{aligned} |f^n(c) - f^m(c)| &\leq \frac{|f(f^n(c)) - f^n(c)| + |f(f^m(c)) - f^m(c)|}{1-k} \\ &= \frac{|f^n(f(c)) - f^n(c)| + |f^m(f(c)) - f^m(c)|}{1-k} \\ &\leq \frac{k^n|f(c)-c| + k^m|f(c)-c|}{1-k} = \frac{k^n + k^m}{1-k}|u_1 - u_0|. \end{aligned}$$

Since $k \in [0,1[$, this tends to 0 as $n, m \to +\infty$, hence $(f^n(c))_{n\in\mathbb{N}}$ is a Cauchy sequence. Keeping n fixed and letting $m \to +\infty$, we obtain the estimate on the speed of convergence. □

1.8 Exercises

1. Calculate the limits of the following sequences, if they exist :

- $\dfrac{3^n - (-2)^n}{3^n + (-2)^n}$
- $\sqrt{n^2+n+1} - \sqrt{n^2-n+1}$
- $\dfrac{n - \sqrt{n^2+1}}{n + \sqrt{n^2-1}}$
- $\left(1 + \dfrac{1}{n}\right)^n$
- $\sin^{\frac{1}{n}}\left(\dfrac{1}{n}\right)$
- $\dfrac{\sin n}{n + (-1)^n}$
- $\dfrac{n!}{n^n}$
- $\dfrac{e^n}{n^n}$
- $(2 + (-1)^n)^{\frac{1}{n}}$
- $(1 + 2^n + 3^n + 7^n)^{1/n}$
- $\sqrt[n]{n!}$
- $\sin(n)$.

2. Let $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$, $\mathbf{v} = (v_n)_{n\in\mathbb{N}}$ be two real sequences. Are the following assertions correct ? Justify your answer.

(a) If $\mathbf{u}$ is increasing and bounded from above, then it converges.

(b) If $\mathbf{u}$ is bounded from above and convergent, then it is increasing.

(c) If $\mathbf{u}$ converges to $l \in \mathbb{R}$, then it is bounded.

(d) If $\mathbf{u}$ is decreasing and nonnegative, then it converges.

(e) If $\mathbf{u}$ is decreasing and strictly positive, then it converges to a strictly positive limit.

(f) If $\mathbf{u}$ diverges to $-\infty$ and $\mathbf{v}$ diverges to $+\infty$, then $\mathbf{u}+\mathbf{v}$ converges to 0.

(g) If $\mathbf{u}$ tends to 0 and the u_n are nonzero, then $1/\mathbf{u}$ tends to $+\infty$ or $-\infty$.

(h) If $\mathbf{u}$ and $\mathbf{v}$ diverge to $-\infty$, then $\mathbf{u}+\mathbf{v}$ diverges to $-\infty$.

(i) If $\mathbf{u}$ converges and $\mathbf{v}$ diverges to $+\infty$, then $\mathbf{uv}$ diverges.

(j) If $\mathbf{u}$ tends to 0, then $\mathbf{uv}$ tends to 0.

(k) If $\mathbf{u}$ is increasing and $\mathbf{v}$ converges to 0, then $\mathbf{u}-\mathbf{v}$ is increasing.

(l) If $\mathbf{u}$ tends to $+\infty$ and $\mathbf{v}$ is bounded from below, then $\mathbf{u}+\mathbf{v}$ tends to $+\infty$.

(m) If $\mathbf{u}$ converges and $\mathbf{v}$ does not converge, then $\mathbf{uv}$ does not converge.

(n) If $\mathbf{u}$ tends to $+\infty$ and $\mathbf{v}$ is bounded from below beyond a certain index by a strictly positive real number, then $\mathbf{uv}$ tends to $+\infty$.

3. Let $(u_n)_{n\geq 1}$, $(v_n)_{n\geq 1}$ be two real sequences defined by

$$u_n := 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \quad \text{and} \quad v_n := u_n + \frac{1}{n \cdot n!}.$$

Show that $(u_n)_n$ and $(v_n)_n$ are adjacent. Deduce that they converge to a common limit. We will admit that this limit is the real number e.

We would now like to prove that $e \notin \mathbb{Q}$. For this, show that we can write $u_n = \frac{a_n}{n!}$ with $a_n \in \mathbb{N}$ and deduce the inequality

$$a_n < en! < a_n + \frac{1}{n} \qquad \text{for all } n \geq 1.$$

Suppose that $e = \frac{p}{q} \in \mathbb{Q}$ and deduce a contradiction by choosing $n = q$ in the previous inequality.

4. Let $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$, $\mathbf{v} = (v_n)_{n\in\mathbb{N}}$ be two real sequences which do not vanish beyond a certain index. Are the following asserions correct ? Justify your answer.

(a) If $u_n \sim v_n$, then for all $k \in \mathbb{Z}$, we have $u_n^k \sim v_n^k$;

(b) If $u_n \sim v_n$, then $e^{u_n} \sim e^{v_n}$;

(c) If $u_n \sim v_n$, then $\ln u_n \sim \ln v_n$;

(d) If $u_n = 1 + \mathrm{o}(1)$, then $e^{u_n} \sim e$;

(e) If $u_n \sim v_n$, then $|u_n| \sim |v_n|$;

(f) If $u_n \sim v_n$, then $u_n^n \sim v_n^n$;

(g) If $u_n = \mathrm{o}(v_n)$, then $u_n = \mathrm{o}(v_n^2)$.

5. Let a and b be strictly positive, $a \leq b$, and let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences defined by

$$\begin{aligned} u_0 &= a & &\text{and} & v_0 &= b \\ u_{n+1} &= \frac{2u_n v_n}{u_n + v_n} & &\text{and} & v_{n+1} &= \frac{u_n + v_n}{2}. \end{aligned}$$

(1) Show that for all $n \in \mathbb{N}$, $u_n \leq v_n$.

(2) Show that $(u_n)_n$ and $(v_n)_n$ converge and that their limits are equal.

(3) Using the product $u_n v_n$, determine the value of this limit.

(4) Application: give rational approximations to $\sqrt{2}$ and $\sqrt{3}$.

6. (Cesàro means). Let $(u_n)_{n\geq 1}$ be a numerical sequence and $(v_n)_{n\geq 1}$ the sequence of *Cesàro means* defined by

$$v_n := \frac{1}{n}\sum_{k=1}^{n} u_k\,.$$

(1) Show that if $u_n \to 0$, then $v_n \to 0$.

(2) Deduce that if $u_n \to l \in \mathbb{C}$ then $v_n \to l$.

(3) Does the convergence of v_n imply the convergence of u_n?

(4) Application: Let $(x_n)_{n\geq 1}$ be a sequence with strictly positive terms. Suppose the sequence $y_n = \frac{x_{n+1}}{x_n}$ converges to $l > 0$. Show that the sequence $(z_n)_{n\geq 1}$ defined by $z_n = \sqrt[n]{x_n}$ converges to l.

(5) Application: Determine the limit of the sequence $\binom{2n}{n}^{1/n}$.

(6) Give an example of a sequence $(x_n)_{n\geq 1}$ such that $\sqrt[n]{x_n} \to l$ but the limit of $\frac{x_{n+1}}{x_n}$ does not exist.

7. Let z be a complex number with $|z| = 1$ such that the sequence $(z^n)_{n\geq 0}$ converge. Determine the limit of this sequence.

Hint. You may use Exercise 6.

8. Let (u_n) be a real sequence such that, for all $p \in \mathbb{N}$,

$$\lim_{n\to\infty}(u_{n+p} - u_n) = 0.$$

Does the sequence (u_n) converge ?

9. Study the convergence of the following sequences, and give their limits if possible.

- $u_0 = 1$ and $u_{n+1} = \sqrt{2u_n}$
- $u_0 = 0$ and $u_{n+1} = \sqrt{2 - u_n}$
- u_0, u_1 and $u_{n+2} = \sqrt{2}u_{n+1} - u_n$
- $u_0 = \dfrac{1}{3}$ and $u_{n+1} = \dfrac{1-u_n}{3}\exp(u_n)$
- $u_0 \geq 0$ and $u_{n+1} = \dfrac{u_n + 2}{u_n + 1}$
- $u_0 = 0$ and $u_{n+1} = \dfrac{3u_n - 2}{2u_n - 1}$
- $u_0 = 0, u_1 = 1$ and $u_{n+2} = 4u_{n+1} - 4u_n + 2$

10. Let (u_n) be a real sequence and put

$$S_n = \sum_{k=1}^{n} u_k \qquad \text{and} \qquad T_n = \sum_{k=1}^{n} \frac{u_k}{k}.$$

(1) Show that if S_n converges, then $u_n \to 0$.

(2) Show that if T_n converges, then u_n tends to 0 in the sense of Cesàro, i.e. $\frac{1}{n}\sum_{k=1}^{n} u_k \to 0$. *Hint.* Express u_k as a function of T_j then use Exercise 6.

11. Let $n \in \mathbb{N}$ and let $f_n : [0,1] \to \mathbb{R}$ be the function defined by $f_n(x) := x^n + x^2 + x - 1$.

(1) Fix $n \in \mathbb{N}$. Show that $f_n(x)$ is a continuous, strictly increasing function.

(2) Deduce that f_n has exactly one zero. We denote $x_n \in [0,1]$ the zero of f_n.

(3) Fix $x \in [0,1]$. Show that the sequence $(f_n(x))_n$ is decreasing. Deduce that the sequence (x_n) is increasing.

(4) Show that (x_n) converges. We denote its limit by l.

(5) Show that for all $n \in \mathbb{N}$, $f_n\left(\frac{3}{4}\right) > 0$. Deduce that $x_n < \frac{3}{4}$ and that $x_n^n \to 0$ as $n \to \infty$.

(6) Conclude that $l^2 + l - 1 = 0$. What is the value of l ?

12. (Fekete's Lemma)**.** Let $(u_n)_{n\geq 0}$ be a real sequence satisfying $u_{n+m} \leq u_n + u_m$ for any $n, m \in \mathbb{N}$. Show that $(\frac{u_n}{n})_{n\geq 1}$ tends to $\ell = \inf\limits_{n\geq 1} \frac{u_n}{n} \in \mathbb{R} \cup \{-\infty\}$.

13. Let (u_n) be a nonnegative, decreasing sequence that tends to 0. We put

$$v_n := \Big(\sum_{k=1}^{n} u_k\Big) - nu_n.$$

(a) Let $p \in \mathbb{N}$. Show that for all $n \geq p$, we have $v_n \geq \sum_{k=1}^{p}(u_k - u_n)$.

(b) Deduce that if (v_n) is a bounded sequence, then the sequence $S_m = \sum_{k=1}^{m} u_k$ converges.

(c) Study the converse.

14. Let $(u_n)_n$ be a real sequence. We say that $\ell \in \mathbb{R}$ is a *limit point* of the sequence if there exists a subsequence of (u_n) which converges to ℓ.

a) What are the limit points of the sequence $u_n = (-1)^n$? the sequence $u_n = \sin(\frac{n\pi}{3})$?

b) Show that if a sequence converges, then it has a unique limit point. Is the converse true ?

c) Show that every bounded real sequence which has a unique limit point is convergent.

d) Application: Let $(u_n)_n$ be a bounded real sequence such that $u_n + \frac{u_{2n}}{2}$ converges. Show that $(u_n)_n$ converges.

e) Application: Let (a_n) and (b_n) be two real sequences such that $a_n + b_n \to 0$ and $e^{a_n} + e^{b_n} \to 2$. Show that the two sequences converge.

Chapter 2

Numerical Series

2.1 Generalities

Definition 2.1. Given a sequence (u_n), we call $\sum_{n=0}^{\infty} u_n = u_0 + u_1 + \ldots$ a *series with general term* u_n. We call $S_k := \sum_{n=0}^{k} u_n$ the *partial sums* of the series. We say that the series *converges* if the sequence S_k converges. In this case, $S = \lim_{k\to\infty} S_k = \sum_{n=0}^{\infty} u_n$ is called the *sum* of the series. $R_n = S - S_n = \sum_{k=n+1}^{\infty} u_k$ is called the *remainder* of the series.

Example 2.2. The *geometric series* $\sum \lambda^n$ converges iff $|\lambda| < 1$. Indeed, if $\lambda = 1$, the series diverges. If $\lambda \neq 1$, we have $S_k = 1 + \lambda + \ldots + \lambda^k$ and $\lambda S_k = \lambda + \ldots + \lambda^{k+1}$. Hence, $S_k - \lambda S_k = 1 - \lambda^{k+1}$, i.e. $S_k = \frac{1-\lambda^{k+1}}{1-\lambda}$. We see that (S_k) converges iff $|\lambda| < 1$. In this case, $\sum_{n=0}^{\infty} \lambda^n = \lim_{k\to\infty} S_k = \frac{1}{1-\lambda}$.

Remarks 2.3.
- ◇ If $\sum u_n$ and $\sum v_n$ converge, then $\lambda \sum u_n + \mu \sum v_n := \sum(\lambda u_n + \mu v_n)$ converges. The converse is clearly not true.
- ◇ The convergence of a series $\sum u_n$ is unchanged if we modify a finite number of u_n.

Lemma 2.4. *If $\sum u_n$ converges, then $u_n \to 0$.*

Proof. If $n \geq 1$, we have $u_n = S_n - S_{n-1}$. Hence, $S_n \to S \implies u_n \to S - S = 0$. □

The converse is not true. For example, $u_n = \frac{1}{n} \to 0$. However, using the identity $\ln(1+x) \leq x$ with $x = \frac{1}{n}$, we have

$$S_n = \sum_{k=1}^{n} \frac{1}{k} \geq \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right) = \ln(n+1)\,,$$

which tends to ∞ as $n \to \infty$.

Lemma 2.5. *If $\sum u_n$ converges, then $R_n \to 0$.*

Proof. Assume $S_n \to S$. Then $R_n = S - S_n \to S - S = 0$. □

Example 2.6. $\sum_{n\geq 1} \frac{1}{n(n+1)}$ converges. Indeed, $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, hence $\sum_{n=1}^{k} \frac{1}{n(n+1)} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \ldots + (\frac{1}{k} - \frac{1}{k+1}) = 1 - \frac{1}{k+1} \to 1$ as $k \to \infty$.

Such series of the form $\sum(v_n - v_{n+1})$ are called *telescoping series*. In this case, $\sum_{n=0}^{k}(v_n - v_{n+1}) = v_0 - v_{k+1}$.

2.2 Series with nonnegative terms

Lemma 2.7. *Let $\sum u_n$ be a series with $u_n \in \mathbb{R}_+$. Then $\sum u_n$ converges iff (S_n) is bounded.*

Proof. If $u_n \geq 0$ for all n, then S_n is an increasing sequence of real numbers, which converges iff it is bounded. □

Theorem 2.8 (Comparison Test)**.** *Let $\sum u_n$ and $\sum v_n$ be two series with nonnegative terms.*

(1) *Suppose $u_n = \mathrm{O}(v_n)$. Then the convergence of $\sum v_n$ implies the convergence of $\sum u_n$, and the divergence of $\sum u_n$ implies the divergence of $\sum v_n$.*

(2) *Suppose $u_n = \mathrm{o}(v_n)$. Then the convergence of $\sum v_n$ implies the convergence of $\sum u_n$, and the divergence of $\sum u_n$ implies the divergence of $\sum v_n$.*

(3) *Suppose $u_n \sim v_n$. Then $\sum u_n$ converges iff $\sum v_n$ converges.*

Proof. If $u_n = \mathrm{O}(v_n)$, then there exists $n_0 \in \mathbb{N}$ and $C > 0$ such that $0 \leq u_n \leq C\, v_n$ for all $n \geq n_0$. Let $S_n(u) = \sum_{k=0}^{n} u_n$ and $S_n(v) = \sum_{k=0}^{n} v_n$. Suppose that $\sum v_n$ converges and let $S_v = \sum_{n=0}^{\infty} v_n$. Then for $n > n_0$, we have

$$S_n(u) = S_{n_0}(u) + \sum_{k=n_0+1}^{n} u_n \leq S_{n_0}(u) + C \sum_{k=n_0+1}^{n} v_n \leq S_{n_0}(u) + C \cdot S_v\,.$$

The RHS does not depend on n. Hence, $(S_n(u))$ is bounded and $\sum u_n$ converges by Lemma 2.7.

If $u_n = \mathrm{O}(v_n)$ and $\sum u_n$ does not converge, then $\sum v_n$ does not converge by contraposition.

We thus proved (1). Result (2) follows because $u_n = \mathrm{o}(v_n)$ implies $u_n = \mathrm{O}(v_n)$. Result (3) follows because $u_n \sim v_n$ implies $u_n = \mathrm{O}(v_n)$ and $v_n = \mathrm{O}(u_n)$. □

Example 2.9. 1. If $\sum u_n$ and $\sum v_n$ are two convergent series with nonnegative terms, then $\sum(\max(u_n, v_n))$ converges. Indeed, $\max(u_n, v_n) \leq u_n + v_n$.

2. Under the same hypotheses, if $0 \leq \alpha, \beta \leq 1$ are such that $\alpha + \beta = 1$, then $\sum u_n^\alpha v_n^\beta$ converges. Indeed, $u_n^\alpha v_n^\beta \leq \max(u_n, v_n)^\alpha \max(u_n, v_n)^\beta = \max(u_n, v_n)$.

To get more concrete applications for the comparison test, we first need to have good reference series with which to compare. A fundamental one is the following.

Theorem 2.10 (Riemann's Rule)**.** *Let $\alpha \in \mathbb{R}$. Then $\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges iff $\alpha > 1$.*

Proof. We already saw that $\sum \frac{1}{n}$ diverges in the counter-example of Lemma 2.4.

Suppose $\alpha \neq 1$. Then $\frac{1}{n^\alpha} \sim \frac{1}{\alpha-1}(\frac{1}{n^{\alpha-1}} - \frac{1}{(n+1)^{\alpha-1}})$. Indeed,

$$\begin{aligned}
\frac{n^\alpha}{\alpha-1}\Big(\frac{1}{n^{\alpha-1}} - \frac{1}{(n+1)^{\alpha-1}}\Big) &= \frac{n}{\alpha-1}\Big(1 - (\frac{n}{n+1})^{\alpha-1}\Big) = \frac{n}{\alpha-1}\Big(1 - (1 - \frac{1}{n+1})^{\alpha-1}\Big) \\
&= \frac{n}{\alpha-1}\Big(1 - (1 - \frac{\alpha-1}{n+1} + \mathrm{o}(\frac{1}{n+1}))\Big) \\
&= \frac{n}{n+1} + \frac{n}{n+1}\mathrm{o}(1) \to 1\,.
\end{aligned}$$

But $\sum(\frac{1}{n^{\alpha-1}} - \frac{1}{(n+1)^{\alpha-1}})$ is a telescoping series, with $\sum_{n=1}^{k}(\frac{1}{n^{\alpha-1}} - \frac{1}{(n+1)^{\alpha-1}}) = 1 - \frac{1}{(k+1)^{\alpha-1}}$, which converges iff $\alpha > 1$. Hence, $\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges iff $\alpha > 1$. □

Here are two other proofs of the divergence of $\sum \frac{1}{n}$:

1) $\frac{1}{n} \sim \ln(n+1) - \ln(n)$. Indeed, $\frac{\ln(n+1)-\ln(n)}{1/n} = \frac{\ln(1+1/n)}{1/n} = \frac{1/n+\mathrm{o}(1/n)}{1/n} = 1 + \mathrm{o}(1) \to 1$. But $\sum \big(\ln(n+1) - \ln(n)\big)$ is a telescoping series with $\sum_{n=1}^{k} \big(\ln(n+1) - \ln(n)\big) = \ln(k+1) - \ln(1) \to \infty$ as $k \to \infty$, hence $\sum \frac{1}{n}$ diverges.
2) Let $S_k = \sum_{n=1}^{k} \frac{1}{n}$. Then $S_{2k} - S_k = \frac{1}{k+1} + \ldots + \frac{1}{2k}$. But $\frac{1}{j} \geq \frac{1}{2k}$ for each $k+1 \leq j \leq 2k$. Hence, $S_{2k} - S_k \geq \frac{1}{2k} + \ldots + \frac{1}{2k} = k\frac{1}{2k} = \frac{1}{2}$. But if $S_k \to S$, then $S_{2k} \to S$ because (S_{2k}) is a subsequence of S_k, so that $S_{2k} - S_k \to 0$, a contradiction. Hence, $\sum \frac{1}{n}$ diverges.

We now make an important remark :

Lemma 2.11. *Let $\sum u_n$ be a series with nonnegative terms.*

(i) *If $\sqrt[n]{u_n} \geq 1$ for infinitely many n, then $\sum u_n$ diverges.*
(ii) *If there exists n_0 and $0 \leq \lambda < 1$ such that $\sqrt[n]{u_n} \leq \lambda$ for all $n \geq n_0$, then $\sum u_n$ converges.*

Proof. In case (i), $u_n \nrightarrow 0$, so $\sum u_n$ diverges by Lemma 2.4.

In case (ii), $u_n \leq \lambda^n$ for all $n \geq n_0$, i.e. $u_n = \mathrm{O}(\lambda^n)$. But $\sum \lambda^n$ is a geometric series, which converges because $|\lambda| < 1$, so $\sum u_n$ converges. □

Corollary 2.12 (Root test)**.** *Let $\sum u_n$ be a series with nonnegative terms. Suppose that $\sqrt[n]{u_n} \to \mu \neq 1$. Then $\sum u_n$ converges if $\mu < 1$, and $\sum u_n$ diverges if $\mu > 1$.*

Proof. If $\mu < 1$, then for $\varepsilon = \frac{1-\mu}{2}$, we may find n_0 such that $\sqrt[n]{u_n} - \mu \leq \varepsilon\ \forall n \geq n_0$, hence $\sqrt[n]{u_n} \leq \mu + \varepsilon = \frac{1+\mu}{2} < 1\ \forall n \geq n_0$, so $\sum u_n$ converges by Lemma 2.11.

If $\mu > 1$, then $\exists n_1 \in \mathbb{N}$ with $\sqrt[n]{u_n} \geq 1\ \forall n \geq n_1$, so $u_n \nrightarrow 0$ and $\sum u_n$ diverges. □

Note that the root test fails when $\mu = 1$. For example, $\sqrt[n]{\frac{1}{n}} = \frac{1}{\sqrt[n]{n}} \to 1$ and $\sqrt[n]{\frac{1}{n^2}} = (\frac{1}{\sqrt[n]{n}})^2 \to 1$, but $\sum \frac{1}{n}$ diverges while $\sum \frac{1}{n^2}$ converges.

Example 2.13. Study the convergence of $\sum_{n\geq 1} u_n$, where $u_n = \left(1 + \frac{a^2}{n}\right)^{-n^2}$.

Solution : $\ln u_n = -n^2 \ln\left(1 + \frac{a^2}{n}\right) \sim -n^2(\frac{a^2}{n}) = -na^2$. Hence, $\frac{1}{n}\ln u_n \to -a^2$ and thus $\sqrt[n]{u_n} = e^{\ln \sqrt[n]{u_n}} \to e^{-a^2} < 1$. Hence, $\sum u_n$ converges.

Here is another criterion :

Lemma 2.14. *Suppose there exists n_0 such that for all $n \geq n_0$, we have $u_n > 0$, $v_n > 0$ and $\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}$. Then the convergence of $\sum v_n$ implies the convergence of $\sum u_n$, and the divergence of $\sum u_n$ implies the divergence of $\sum v_n$.*

Proof. For $n \geq n_0$ we have $\frac{u_{n+1}}{v_{n+1}} \leq \frac{u_n}{v_n}$, hence $(\frac{u_n}{v_n})$ is decreasing. In particular, $\frac{u_n}{v_n} \leq \frac{u_{n_0}}{v_{n_0}}$ and $u_n \leq \frac{u_{n_0}}{v_{n_0}} v_n\ \forall n \geq n_0$. Thus, $u_n = \mathrm{O}(v_n)$ and the result follows from the comparison test. □

Corollary 2.15 (Ratio test)**.** *Let $\sum u_n$ be a series with strictly positive terms. Suppose that $\frac{u_{n+1}}{u_n} \to \lambda \neq 1$. Then $\sum u_n$ converges if $\lambda < 1$, and $\sum u_n$ diverges if $\lambda > 1$.*

Proof. If $\lambda < 1$, then for $\varepsilon = \frac{1-\lambda}{2}$, we may find n_0 such that $\frac{u_{n+1}}{u_n} - \lambda \leq \varepsilon\ \forall n \geq n_0$, hence $\frac{u_{n+1}}{u_n} \leq \lambda + \varepsilon = \frac{1+\lambda}{2} < 1$. Let $\mu = \frac{1+\lambda}{2}$ and $v_n = \mu^n$. Then $\sum v_n$ converges and $\frac{v_{n+1}}{v_n} = \mu$. Hence, $\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}\ \forall n \geq n_0$ and $\sum u_n$ converges by Lemma 2.14.

If $\lambda > 1$, then for $\varepsilon = \frac{\lambda-1}{2}$, we may find n_1 such that $\frac{u_{n+1}}{u_n} - \lambda \geq -\varepsilon\ \forall n \geq n_0$, hence $\frac{u_{n+1}}{u_n} \geq \lambda - \varepsilon = \frac{1+\lambda}{2} > 1$. In particular, $u_n \geq u_{n_0}$ for all $n \geq n_0$. But $\sum u_{n_0} = \infty$, hence $\sum u_n$ diverges by the comparison test. □

Again, the ratio test fails if $\lambda = 1$.

Example 2.16. Study the convergence of $\sum u_n$, where $u_n = \frac{1}{(2n)!}\Big(\prod_{k=1}^{n}(a+k)^2\Big)$, $a \in \mathbb{R}$.

Solution : If $a = -m$ for some $m \in \mathbb{N}^*$, then $u_n = 0$ for all $n \geq m$, hence $\sum u_n$ converges and equals $\sum_{n=0}^{m} u_n$.

Otherwise, $\frac{u_{n+1}}{u_n} = \frac{(a+n+1)^2}{(2n+1)(2n+2)} \to \frac{1}{4}$, hence $\sum u_n$ is also convergent.

Theorem 2.17 (Integral test)**.** *Let $f : [1,\infty[\to \mathbb{R}_+$ be continuous and decreasing. Put $u_n = f(n)$. Then $\sum_{n\geq 1} u_n$ converges iff the sequence $\big(\int_1^n f(t)\,\mathrm{d}t\big)$ converges.*

Proof. Since f decreases, we have $\int_k^{k+1} f(t)\,\mathrm{d}t \leq f(k)$ for all $k \geq 1$, and $\int_{k-1}^{k} f(t)\,\mathrm{d}t \geq f(k)$ for all $k \geq 2$. Hence,

$$\int_1^{n+1} f(t)\,\mathrm{d}t = \sum_{k=1}^{n}\int_k^{k+1} f(t)\,\mathrm{d}t \leq \sum_{k=1}^{n} f(k) \leq f(1) + \sum_{k=2}^{n}\int_{k-1}^{k} f(t)\,\mathrm{d}t = f(1) + \int_1^n f(t)\,\mathrm{d}t\,.$$

Hence, if $\int_1^n f(t)\,\mathrm{d}t$ converges, $S_n = \sum_{k=1}^{n} f(k)$ is bounded, hence $\sum f(n)$ converges.

If $\int_1^n f(t)\,\mathrm{d}t$ diverges, then $\int_1^{n+1} f(t)\,\mathrm{d}t \to \infty$, hence $S_n \to \infty$ and $\sum f(n)$ diverges. $\square$

Remarks 2.18. 1. In the previous proof, we established that

$$\int_1^{n+1} f(t)\,\mathrm{d}t \leq \sum_{k=1}^{n} f(k) \leq f(1) + \int_1^n f(t)\,\mathrm{d}t.$$

Here is an application : study the convergence of $a_n = \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$.

Solution : let $f(x) = \frac{1}{\sqrt{x}}$. Then

$$\int_1^{n+1}\frac{1}{\sqrt{t}}\,\mathrm{d}t = 2\sqrt{t}\big|_1^{n+1} \leq \sum_{k=1}^{n}\frac{1}{\sqrt{k}} \leq 1 + \int_1^n \frac{1}{\sqrt{k}} = 1 + (2\sqrt{t}\big|_1^n)\,.$$

Thus, $\frac{2(\sqrt{n+1}-1)}{\sqrt{n}} \leq u_n \leq \frac{1+2(\sqrt{n}-1)}{\sqrt{n}}$ and $u_n \to 2$ by the Sandwich theorem.

2. In case of convergence, if we sum instead over $k \geq n+1$ in the previous proof, we obtain

$$\int_{n+1}^{\infty} f(t)\,\mathrm{d}t \leq \sum_{k=n+1}^{\infty} f(k) \leq \int_n^{\infty} f(t)\,\mathrm{d}t\,.$$

Here is an application : estimate the remainder of the series $\sum_{n\geq 1}\frac{1}{n^\alpha}$ when $\alpha > 1$.

Solution : let $f(x) = \frac{1}{x^\alpha}$ and $R_n = \sum_{k=n+1}^{\infty}\frac{1}{n^\alpha}$. Then

$$\frac{-1}{\alpha-1}\frac{1}{x^{\alpha-1}}\Big|_{n+1}^{\infty} \leq R_n \leq \frac{-1}{\alpha-1}\frac{1}{x^{\alpha-1}}\Big|_{n}^{\infty}\,.$$

Thus, $\frac{1}{\alpha-1}\frac{1}{(n+1)^{\alpha-1}} \leq R_n \leq \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}}$, and $R_n \sim \frac{1}{\alpha-1}\frac{1}{n^{\alpha-1}}$.

2.3 General series

Theorem 2.19 (Cauchy criterion for series)**.** *A series of numbers $\sum u_n$ converges iff it satisfies the Cauchy criterion :*

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N} : n \geq N, p \geq 0 \implies |S_{n+p} - S_n| = |u_{n+1} + \ldots + u_{n+p}| \leq \varepsilon\,.$$

Proof. This follows immediately from the Cauchy criterion for sequences, since $\sum u_n$ converges iff (S_n) converges. $\square$

Definition 2.20. A series of numbers $\sum u_n$ is *absolutely convergent* if $\sum |u_n|$ converges.

Theorem 2.21. *If $\sum u_n$ is absolutely convergent, then $\sum u_n$ is convergent.*

Proof. Suppose $\sum |u_n|$ converges and let $\varepsilon > 0$. By the Cauchy criterion, we may find $N \in \mathbb{N}$ such that $\sum_{k=n+1}^{n+p} |u_k| \le \varepsilon$ for all $n \ge N$, $p \ge 0$. Hence, $|\sum_{k=n+1}^{n+p} u_k| \le \sum_{k=n+1}^{n+p} |u_k| \le \varepsilon$. Hence, $\sum u_n$ converges by the Cauchy criterion. $\square$

Example 2.22. $\sum \frac{\sin(\sqrt{3}n)}{3^n}$ is absolutely convergent.

If $u_n = (-1)^n(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}})$, then $\sum_{n\ge 1} u_n$ is not absolutely convergent, since $|u_n| \sim \frac{2}{\sqrt{n}}$. However, $\sum u_n$ is convergent. Indeed, $u_n = v_n - v_{n+1}$, where $v_n = \frac{(-1)^n}{\sqrt{n}}$. Hence, $\sum_{k=1}^n u_k = \sum_{k=1}^n (v_k - v_{k+1}) = v_1 - v_{n+1} = -1 + \frac{(-1)^n}{\sqrt{n+1}} \to -1$.

Remark 2.23. The comparison test of Section 2.2 fails when the reference sequence changes sign. For example, if $u_n = (-1)^n(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}})$, then $\frac{1}{n} = \mathrm{o}(u_n)$. However, $\sum u_n$ converges, but $\sum \frac{1}{n}$ diverges.

Definition 2.24. A *real* series $\sum u_n$ is *alternating* if $(-1)^n u_n$ has a fixed sign.

For example, $\sum \frac{(-1)^n}{n}$ is alternating, since $(-1)^n \frac{(-1)^n}{n} = \frac{1}{n}$ is always positive.

Theorem 2.25 (Aternating series criterion)**.** *Suppose $\sum u_n = \sum (-1)^n v_n$ is an alternating series such that (v_n) is nonnegative, decreasing and convergent to 0. Then $\sum u_n$ converges, and its sum S satisfies $S_{2n+1} \le S \le S_{2n}$ for all n. Moreover,*

$$|R_n| = \left| \sum_{k=n+1}^{\infty} u_k \right| \le v_{n+1}\,.$$

Proof. We prove that S_{2n+1} and S_{2n} are adjacent. We have

$$\begin{aligned} S_{2n+1} &= S_{2n-1} + u_{2n} + u_{2n+1} = S_{2n-1} + v_{2n} - v_{2n+1} \ge S_{2n-1}, \\ S_{2n+2} &= S_{2n} + u_{2n+1} + u_{2n+2} = S_{2n} - v_{2n+1} + v_{2n+2} \le S_{2n}\,. \end{aligned}$$

Thus, (S_{2n+1}) is increasing, (S_{2n}) is decreasing and $S_{2n} - S_{2n+1} = -u_{2n+1} = v_{2n+1} \to 0$. Thus, (S_{2n+1}) and (S_{2n}) converge to a common limit S satisfying $S_{2n+1} \le S \le S_{2n}$ for all n. Hence, $0 \le S_{2n} - S \le S_{2n} - S_{2n+1} = v_{2n+1}$, and $0 \le S - S_{2n-1} \le S_{2n} - S_{2n-1} = v_{2n}$, which yields $|R_{2n}| \le v_{2n+1}$ and $|R_{2n-1}| \le v_{2n}$. $\square$

Remark 2.26. If $u_n = (-1)^{n-1} v_n$ with (v_n) nonnegative, decreasing and converging to 0, then $\sum u_n$ is also convergent, but now $S_{2n} \le S \le S_{2n+1}$. The estimate on R_n is still valid.

Example 2.27. If $\alpha > 0$, then $\sum \frac{(-1)^n}{n^\alpha}$ converges and $|\sum_{k=n+1}^{\infty} \frac{(-1)^k}{k^\alpha}| \le \frac{1}{(n+1)^\alpha}$.

In particular, $\sum \frac{(-1)^n}{\sqrt{n}}$ converges.

If $u_n = \frac{(-1)^n}{\sqrt{n}+(-1)^n}$, then $u_n \sim \frac{(-1)^n}{\sqrt{n}}$. However, $\sum u_n$ diverges. Indeed,

$$u_n = \frac{(-1)^n}{\sqrt{n}}\Big(\frac{1}{1+\frac{(-1)^n}{\sqrt{n}}}\Big) = \frac{(-1)^n}{\sqrt{n}}\Big(1 - \frac{(-1)^n}{\sqrt{n}} + \mathrm{O}(\frac{1}{n})\Big) = \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + \mathrm{O}(\frac{1}{n\sqrt{n}})\,.$$

The first term gives a convergent series, the third one also yields a convergent series by Riemann's rule. Hence $\sum u_n$ diverges due to the divergence of $\sum \frac{1}{n}$.

This example shows again that the comparison test fails when the reference sequence (here $\frac{(-1)^n}{\sqrt{n}}$) changes its sign.

Remark 2.28. The condition that (v_n) is decreasing in Theorem 2.25 is necessary. For example, if $u_{2n} = \frac{1}{3^n}$ and $u_{2n+1} = -\frac{1}{n+5}$, then $u_n = (-1)^n v_n$ with (v_n) positive and convergent to 0. However, $\sum u_n$ diverges, since $S_{2n+1} = \sum_{k=0}^{n} \frac{1}{3^k} - \sum_{k=1}^{n} \frac{1}{k+5} \to -\infty$.

2.4 Summation by packets

Theorem 2.29. *Let $\sum u_n$ be a series of numbers and $\varphi : \mathbb{N} \to \mathbb{N}$ be strictly increasing. Put $v_0 = \sum_{k=0}^{\varphi(0)} u_k$ and $v_n = \sum_{k=\varphi(n-1)+1}^{\varphi(n)} u_k$ for $n \geq 1$. Then*

(1) *If $\sum u_n$ converges, then $\sum v_n$ converges and has the same sum.*

(2) *If $\sum v_n$ converges, then $\sum u_n$ converges in the following cases :*

  (a) *$u_n \to 0$ and $(\varphi(n+1) - \varphi(n))$ is bounded,*

  (b) *(u_n) is real, and the u_k of each packet $\{\varphi(n-1)+1 \leq k \leq \varphi(n)\}$ have the same sign.*

Proof. Let $U_n = \sum_{j=0}^{n} u_j$ and $V_n = \sum_{k=0}^{n} v_k$. Then

$$V_n = \sum_{k=0}^{\varphi(0)} u_k + \sum_{k=\varphi(0)+1}^{\varphi(1)} u_k + \ldots + \sum_{k=\varphi(n-1)+1}^{\varphi(n)} u_k = \sum_{k=0}^{\varphi(n)} u_k = U_{\varphi(n)}\,.$$

Thus, (V_n) is a subsequence of (U_n). So if $U_n \to \sum u_n$, then V_n converges to the same limit. This proves (1)

Now suppose that $\sum v_n$ converges. Given $n \in \mathbb{N}$, let p be the unique natural number satisfying $\varphi(p-1) < n \leq \varphi(p)$. Then

$$V_p - U_n = U_{\varphi(p)} - U_n = \sum_{k=n+1}^{\varphi(p)} u_k\,.$$

To prove (2.a), assume $S = \sum_{n \geq 0} v_n$ and let K such that $\varphi(n+1) - \varphi(n) \leq K$ for all n. Now let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $|u_n| \leq \frac{\varepsilon}{2K}$ for $n \geq N$. Then for $n \geq N$ we have $|V_p - U_n| \leq \sum_{k=n+1}^{\varphi(p)} |u_k| \leq (\varphi(p) - n)\frac{\varepsilon}{2K} \leq \frac{\varepsilon}{2}$. But there exists N' such that $|S - V_q| \leq \frac{\varepsilon}{2}$ for $q \geq N'$. Given $n \geq \max(N, \varphi(N'))$, we have $\varphi(p) \geq \varphi(N')$, so $p \geq N'$ and thus

$$|S - U_n| \leq |S - V_p| + |V_p - U_n| \leq \varepsilon\,,$$

so $\sum u_n$ converges.

To prove (2.b), note that if all u_k have the same sign in each packet, then

$$|V_p - U_n| = \sum_{k=n+1}^{\varphi(p)} |u_k| \leq \sum_{k=\varphi(p-1)+1}^{\varphi(p)} |u_k| = \left| \sum_{k=\varphi(p-1)+1}^{\varphi(p)} u_k \right| = |v_p|\,.$$

Since $\sum v_n$ converges, we may find N such that $|v_q| \leq \frac{\varepsilon}{2}$ for $q \geq N$. Hence, for $n \geq \varphi(N)$, we have $\varphi(p) \geq \varphi(N)$, so $p \geq N$ and thus $|V_p - U_n| \leq |v_p| \leq \frac{\varepsilon}{2}$. It now follows as before that $\sum u_n$ converges. $\square$

Remark 2.30. If $u_n = (-1)^n$ and we take packets of size 2, i.e. $\varphi(n) = 2n+1$, then $v_n = 0$ for all n, hence $\sum v_n$ converges but $\sum u_n$ diverges. So an additional condition like (2.a) or (2.b) above is necessary to guarantee the convergence of $\sum u_n$.

Example 2.31. Let $u_n = \frac{(-1)^n}{n+(-1)^n}$. Then $|u_n|$ is not decreasing, so we cannot apply the criterion of alternating series. However, if we take $\varphi(n) = 2n+1$, then

$$v_n = u_{2n} + u_{2n+1} = \frac{1}{2n+1} - \frac{1}{2n} = -\frac{1}{2n(2n+1)} \sim -\frac{1}{4n^2}\,,$$

and since $\sum \frac{1}{n^2}$ converges, then $\sum v_n$ converges. Since (2.a) is satisfied, $\sum u_n$ is also convergent.

Example 2.32. Let $u_n = \frac{\cos(2n\pi/3)}{n}$. We take packets of size 3, i.e. $\varphi(n) = 3n+2$. Then noting that $\cos(2\pi/3) = \cos(4\pi/3) = -1/2$, we get

$$v_n = u_{3n} + u_{3n+1} + u_{3n+2} = \frac{1}{3n} - \frac{1}{2(3n+1)} - \frac{1}{2(3n+2)} \sim \frac{1}{6n^2}$$

as can be easily checked. Again, (2.a) is satisfied, so $\sum u_n$ converges.

2.5 Rearrangements

Theorem 2.33. *Suppose $\sum u_n$ is absolutely convergent and let $\sigma : \mathbb{N} \to \mathbb{N}$ be a permutation, i.e. a bijective map. Then $\sum u_{\sigma(n)}$ is also convergent, and $\sum u_{\sigma(n)} = \sum u_n$.*

Proof. Let $\varepsilon > 0$. By the Cauchy criterion, we may find $n_0 \in \mathbb{N}$ such that

$$n \geq n_0, p \geq 1 \implies |u_{n+1}| + \ldots + |u_{n+p}| \leq \varepsilon\,.$$

Choose n_1 such that the terms $u_0, \ldots, u_{n_0}$ appear in the partial sum $S'_{n_1} = \sum_{n=0}^{n_1} u_{\sigma(n)}$. Then S'_{n_1} contains at least n_0+1 terms, so $n_1 \geq n_0$. Hence, for $m \geq n_1$, the terms $u_0, \ldots, u_{n_0}$ disappear from the difference $S_m - S'_m$. So there exists $k \in \mathbb{N}$ such that

$$|S_m - S'_m| \leq |u_{n_0+1}| + \ldots + |u_{n_0+k}| \leq \varepsilon\,.$$

Hence, $S_m - S'_m \to 0$, so S'_m converges to the same limit of S_m. ☐

Remark 2.34. Absolute convergence is necessary for the previous theorem to hold. For example, consider the series $\sum u_n$ where u_n is given by

$$1, -1, \frac{1}{2}, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{4}, \ldots$$

Then taking packets of size 2, we see that $\sum u_n$ converges since $u_n \to 0$. However, consider the following rearrangement :

$$1, \frac{1}{2}, -1, \frac{1}{3}, \frac{1}{4}, \frac{-1}{2}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{-1}{3}, \ldots$$
$$\frac{1}{2^{p-1}+1}, \frac{1}{2^{p-1}+2}, \frac{1}{2^{p-1}+3}, \ldots, \frac{1}{2^p}, \frac{-1}{p}, \ldots$$

Then this series diverges since it does not satisfy the Cauchy criterion. Indeed,

$$\frac{1}{2^{p-1}+1} + \frac{1}{2^{p-1}+2} + \frac{1}{2^{p-1}+3} + \ldots + \frac{1}{2^p} \geq (2^p - 2^{p-1})\frac{1}{2^p} = \frac{1}{2}\,.$$

2.6 Double series

Definition 2.35. A *double sequence* is a map $u : \mathbb{N}^2 \to \mathbb{C}$. We denote $u_{p,q} := u(p,q)$. The map itself is denoted by $(u_{p,q})_{(p,q)\in\mathbb{N}^2}$.

A *double series* is a sequence $S_{n,m} = \sum_{p=0}^{n} \sum_{q=0}^{m} u_{p,q}$.

Clearly, $S_{n,m} = \sum_{q=0}^{m} \sum_{p=0}^{n} u_{p,q}$. But do we have $\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} u_{p,q} = \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} u_{p,q}$?

Example 2.36. Consider the double sequence

$$u_{p,q} = \begin{cases} 1 & \text{if } q = p \\ -1 & \text{if } q = p-1 \\ 0 & \text{otherwise.} \end{cases} \qquad \begin{array}{cccccc} \vdots & \vdots & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \end{array}$$

Then $\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} u_{p,q} = 1 + 0 + 0 + \ldots = 1$ but $\sum_{q=0}^{\infty} \sum_{p=0}^{\infty} u_{p,q} = 0 + 0 + 0 + \ldots = 0$. So we cannot interchange the order of summation here.

Definition 2.37. We say that $\sum_{k=0}^{\infty} v_k$ is a *linear arrangement* of $\sum_{(p,q)\in\mathbb{N}^2} u_{p,q}$ if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}^2$ such that $v_k = u_{\sigma(k)}$.

Example 2.38. The series $\sum_{k=0}^{\infty} \sum_{p+q=k} u_{p,q} = (u_{0,0}) + (u_{1,0} + u_{0,1}) + (u_{2,0} + u_{1,1} + u_{0,2}) + \ldots$ is a linear arrangement of $\sum u_{p,q}$. Here, $\sigma(0) = (0,0)$, $\sigma(1) = (1,0)$, $\sigma(2) = (0,1)$, $\sigma(3) = (2,0)$ and so on.

Theorem 2.39. *Suppose there exists $B \geq 0$ such that $\sum_{p=0}^{m} \sum_{q=0}^{m} |u_{p,q}| \leq B$ for all $m \geq 0$. Then the series $\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} u_{p,q}$ and $\sum_{q=0}^{\infty} \sum_{p=0}^{\infty} u_{p,q}$ both converge to the same sum. Moreover, every linear arrangement of $\sum_{(p,q)\in\mathbb{N}^2} u_{p,q}$ converges to the same sum.*

Proof. Let $\sum_{k=0}^{\infty} v_k$ be a linear arrangement of $\sum u_{p,q}$. The sequence $\left(\sum_{k=0}^{n} |v_k|\right)$ is increasing and bounded by B, hence $\sum |v_k|$ and $\sum v_k$ both converge. Similarly, we see that $R_p = \sum_{q=0}^{\infty} u_{p,q}$ and $L_q = \sum_{p=0}^{\infty} u_{p,q}$ converge.

Let $\varepsilon > 0$. By the Cauchy criterion, we may find n_0 such that

$$n \geq n_0, k \geq 1 \implies |v_{n+1}| + \ldots + |v_{n+k}| \leq \varepsilon\,.$$

Choose n_1 such that the terms $v_0, \ldots, v_{n_0}$ appear in the partial sum $\sum_{p=0}^{n_1} \sum_{q=0}^{n_1} u_{p,q}$. Then for $l \geq n_0$ and $n, m \geq n_1$, the terms $v_0, \ldots, v_{n_0}$ disappear from the difference $D_{n,m,l} = \sum_{p=0}^{m} \sum_{q=0}^{n} u_{p,q} - \sum_{k=0}^{l} v_k$ and there exists $k \in \mathbb{N}$ such that

$$|D_{n,m,l}| \leq |v_{n_0+1}| + \ldots + |v_{n+k}| \leq \varepsilon\,.$$

Put $S = \sum_{k=0}^{\infty} v_k$. Then as $l, n \to \infty$ we get $|\sum_{p=0}^{m} R_p - S| \leq \varepsilon$. If we interchange $\sum_{p=0}^{m} \sum_{q=0}^{n} = \sum_{q=0}^{n} \sum_{p=0}^{m}$, we obtain similarly $|\sum_{q=0}^{n} L_q - S| \leq \varepsilon$. Hence, $\sum_{p=0}^{\infty} R_p$ and $\sum_{q=0}^{\infty} L_q$ both converge to the same sum S. □

If $u_{p,q} \geq 0$ for all p, q, we have a much stronger result. Namely, we can always interchange the order of summation :

Theorem 2.40. *If $u_{p,q} \geq 0$ for all $p, q \geq 0$, then*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} u_{p,q} = \sum_{q=0}^{\infty}\sum_{p=0}^{\infty} u_{p,q}\,.$$

This means that even if one of the double series diverges, then both will be $+\infty$.

Proof. Admitted. See for example [15], page 23. □

2.7 Cauchy product

Definition 2.41. The Cauchy product of two series $\sum_{p=0}^{\infty} u_p$ and $\sum_{q=0}^{\infty} v_q$ is defined to be the series $\sum_{n=0}^{\infty}\Big(\sum_{q=0}^{n} u_{n-q}v_q\Big) = u_0v_0 + (u_0v_1 + u_1v_0) + (u_0v_2 + u_1v_1 + u_2v_0) + \ldots$.

Example 2.42. The Cauchy product of $\sum_{n\geq 0}\frac{(-1)^n}{\sqrt{n+1}}$ with itself is divergent. Indeed,

$$\left|\sum_{q=0}^{n} u_{n-q}v_q\right| = \sum_{q=0}^{n}\frac{1}{\sqrt{n-q+1}\sqrt{j+1}} \geq (n+1)\frac{1}{n+1} = 1\,.$$

Thus, the Cauchy product of two convergent series may diverge.

Theorem 2.43. *If $\sum_{p=0}^{\infty} u_p$ and $\sum_{q=0}^{\infty} v_q$ are both absolutely convergent, then their Cauchy product converges, and we have*

$$\Big(\sum_{p=0}^{\infty} u_p\Big)\Big(\sum_{q=0}^{\infty} v_q\Big) = \sum_{n=0}^{\infty}\Big(\sum_{q=0}^{n} u_{n-q}v_q\Big)\,.$$

Proof. By hypothesis, there exist B_1, B_2 such that $\sum_{p=0}^{\infty}|u_p| \leq B_1$ and $\sum_{q=0}^{\infty}|v_q| \leq B_2$. Hence, $\sum_{p=0}^{m}\sum_{q=0}^{m}|u_p||v_q| \leq B_1B_2$ for any m, and we may apply Theorem 2.39. Here we have $R_p = u_p\sum_{q=0}^{\infty} v_q$ and $\sum_{p=0}^{\infty} R_p = \Big(\sum_{p=0}^{\infty} u_p\Big)\Big(\sum_{q=0}^{\infty} v_q\Big)$. By the theorem, the Cauchy product, which is a linear arrangement of $\sum_{(p,q)\in\mathbb{N}^2} u_pv_q$, converges to the same sum. □

Example 2.44. Let $|a| < 1$. The Cauchy product of $\frac{1}{1-a} = \sum_{p=0}^{\infty} a^p$ with $\frac{1}{1+a} = \sum_{q=0}^{\infty}(-1)^q a^q$ is

$$\sum_{n=0}^{\infty}\sum_{q=0}^{n} a^{n-q}(-1)^q a^q = \sum_{n=0}^{\infty} a^n\sum_{q=0}^{n}(-1)^q = \sum_{k=0}^{\infty} a^{2k} = \frac{1}{1-a^2}\,.$$

This is not surprising since we know that $\frac{1}{1-a}\frac{1}{1+a} = \frac{1}{1-a^2}$.

Example 2.45. Let us calculate the Taylor series of $\frac{1}{(1-a)^2}$ using the Cauchy product of $\frac{1}{1-a}$ with itself. We have

$$\frac{1}{(1-a)^2} = \sum_{n=0}^{\infty}\sum_{q=0}^{n} a^{n-q}a^q = \sum_{n=0}^{\infty}(n+1)a^n\,.$$

Example 2.46. The series $\sum\frac{z^n}{n!}$ is absolutely convergent for any $z \in \mathbb{C}$ (use the ratio test). The sum is by definition e^z. Let us show that $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for $z_j \in \mathbb{C}$. We have

$$e^{z_1}e^{z_2} = \sum_{n=0}^{\infty}\sum_{q=0}^{n}\frac{z_1^{n-q}}{(n-q)!}\frac{z_2^q}{q!} = \sum_{n=0}^{\infty}\sum_{q=0}^{n}\frac{1}{n!}\binom{n}{q} z_1^{n-q}z_2^q = \sum_{n=0}^{\infty}\frac{1}{n!}(z_1+z_2)^n = e^{z_1+z_2}\,.$$

where we used the binomial theorem.

Remark 2.47. Theorem 2.43 remains true if only one series is absolutely convergent. For a proof, see e.g. [9].

2.8 Exercises

1. Determine the nature of the following series, and calculate the sum in case of convergence.

- $\displaystyle\sum_{n\geq 1}\frac{1}{n(n+1)}$
- $\displaystyle\sum_{n\geq 1}\frac{1}{n+1}\left(\ln\left(1+\frac{1}{n}\right)-\frac{\ln(n)}{n}\right)$
- $\displaystyle\sum_{n\geq 2}\frac{1}{n^3-n}$
- $\displaystyle\sum_{n\geq 0}\frac{1}{\sqrt{n}+\sqrt{n+1}}$
- $\displaystyle\sum_{n\geq 1}\sin\left(\frac{\pi}{4n^2-1}\right)\sin\left(\frac{2n\pi}{4n^2-1}\right)$.

Hint. Show that they are telescoping series.

2. Let $\sum_{n\in\mathbb{N}} u_n$ and $\sum_{n\in\mathbb{N}} v_n$ be series with $u_n \geq 0$ and $v_n > 0$ beyond some index. Are the following assertions correct ? Justify your answer.

(a) If the partial sums of the series $\sum_{n\in\mathbb{N}} u_n$ are bounded, then the series converges.

(b) If $u_n \to 0$, then the series $\sum_{n\in\mathbb{N}} u_n$ converges.

(c) If $\sum_{n\in\mathbb{N}} u_n$ converges, then $u_n \to 0$.

(d) If $\sum_{n\in\mathbb{N}} u_n$ converges, then for all $n_0 \in \mathbb{N}$ the series $\sum_{n\geq n_0} u_n$ converges.

(e) Let $n_0 \in \mathbb{N}$. If $\sum_{n\geq n_0} u_n$ converges, then the series $\sum_{n\in\mathbb{N}} u_n$ converges.

(f) If $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ converge, then $\sum_{n\geq 0} u_n v_n$ converges.

(g) If $\sum v_n$ diverges and $u_n \sim v_n$, then $\sum_{n\in\mathbb{N}} u_n$ diverges.

(h) If $u_n = \mathrm{o}(v_n)$ and $\sum v_n$ converges, then $\sum_{n\in\mathbb{N}} u_n$ converges.

(i) If $u_n = \mathrm{o}(v_n)$ and $\sum v_n$ diverges, then $\sum_{n\in\mathbb{N}} u_n$ diverges.

(j) If $u_n = \mathrm{O}(v_n)$, then $\sum_{n\in\mathbb{N}} u_n$ and $\sum_{n\in\mathbb{N}} v_n$ have the same nature.

(k) If $u_n = \mathrm{o}\left(\frac{1}{n}\right)$, then $\sum_{n\in\mathbb{N}} u_n$ diverges.

(l) If $u_n = \mathrm{o}\left(\frac{1}{n}\right)$, then $\sum_{n\in\mathbb{N}} u_n$ converges.

(m) If $u_n \sim \frac{1}{n^2}$, then $\sum_{n\in\mathbb{N}} u_n$ converges.

3. Study the convergence of the following series (here $a > 0$ and $\alpha \in \mathbb{R}$)

- $\displaystyle\sum_{n\geq 3}\frac{2^n+5}{3^n-11}$
- $\displaystyle\sum_{n\geq 1} n^{1+\frac{1}{n}}$
- $\displaystyle\sum_{n\geq 1}\left(\frac{1}{2}+\frac{1}{n}\right)^n$
- $\displaystyle\sum_{n\geq 3}\frac{1}{n\cdot\ln n\cdot\ln\ln n}$
- $\displaystyle\sum_{n\geq 1}\frac{(n!)^3}{(3n)!}$
- $\displaystyle\sum_{n\geq 1}\left(1-\frac{1}{n}\right)^n$
- $\displaystyle\sum_{n\geq 1}\left(1-\frac{1}{n}\right)^{n^2}$
- $\displaystyle\sum_{n\geq 1}\frac{\sqrt{n+1}-\sqrt{n}}{n}$
- $\displaystyle\sum_{n\geq 1}\frac{n^n}{n!}$
- $\displaystyle\sum_{n\geq 1}\frac{1!+\ldots+(n-1)!}{n!}$
- $\displaystyle\sum_{n\geq 1}\frac{2^n+5^n}{a^n}$
- $\displaystyle\sum_{n\geq 1}\frac{n^{\ln n}}{(\ln n)^n}$
- $\displaystyle\sum_{n\geq 0} e^{-\sqrt{n}}$
- $\displaystyle\sum_{n\geq 1}\ln\left(\cos\frac{1}{n}\right)$
- $\displaystyle\sum_{n\geq 1}\frac{n^2}{2^n}$
- $\displaystyle\sum_{n\geq 1} n^2\sin 2^{-n}$
- $\displaystyle\sum_{n\geq 1} 1-\cos\left(\frac{1}{n}\right)$
- $\displaystyle\sum_{n\geq 2}\frac{1!+\ldots+(n-2)!}{n!}$
- $\displaystyle\sum_{n\geq 1}(n^6+3)^\alpha-(n^2+2)^{3\alpha}$
- $\displaystyle\sum_{n\geq 1} n^{\frac{1}{n^2+1}}-1$
- $\displaystyle\sum_{n\geq 1}\frac{n^\alpha}{(1+a)\cdots(1+a^n)}$

4. (Cauchy condensation test)**.** Suppose $u_1 \geq u_2 \geq u_3 \geq \ldots \geq 0$. Show that $\sum_{n=1}^{\infty} u_n$ converges iff $\sum_{k=0}^{\infty} 2^k u_{2^k}$ converges.

Deduce new proofs about the nature of the series $\sum \frac{1}{n^p}$ and $\sum \frac{1}{n(\ln n)^p}$.

5. Let $\sum_{n\in\mathbb{N}} u_n$, $\sum_{n\in\mathbb{N}} v_n$ be series with $u_n, v_n \in \mathbb{R}$. Suppose $\exists k_0$ such that $v_n \neq 0$ $\forall n \geq k_0$. Are the following assertions correct ? Justify your answer.

(a) If $u_n \sim \frac{(-1)^n}{n^2}$, then $\sum_{n\in\mathbb{N}} u_n$ converges.

(b) If $u_n \sim \frac{(-1)^n}{n}$, then $\sum_{n\in\mathbb{N}} u_n$ converges.

(c) If $\sum_{n\geq 0} u_n$ converges, then $\sum_{n\geq 0} u_n$ is absolutely convergent.

(d) If $\sum_{n\geq 0} u_n$ is absolutely convergent, then $\sum_{n\geq 0} u_n$ converges.

(e) If $\sum_{n\geq 0} u_n$ is absolutely convergent, then $\sum_{n\geq 0} u_n^2$ converges.

(f) If $\sum_{n\geq 0} u_n$ is absolutely convergent, then $\sum_{n\geq 0} \sqrt{|u_n|}$ converges.

(g) If $\sum_{n\geq 0} u_n$ converges, then $\sum_{n\geq 0} u_n^2$ converges.

6. Study the convergence and absolute convergence of the following series:

- $\displaystyle\sum_{n\geq 1} \sqrt{n+(-1)^n} - \sqrt{n}$
- $\displaystyle\sum_{n\geq 2} \frac{(-1)^n}{2n+(-1)^n}$
- $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{(2n-1)^3}$
- $\displaystyle\sum_{n\geq 1} (-1)^n \cosh\left(\frac{1}{n}\right) \sin\left(\frac{1}{n}\right)$
- $\displaystyle\sum_{n\geq 1} 2^{-n} \sin(n\theta)$
- $\displaystyle\sum_{n\geq 1} \ln\left(1+\frac{(-1)^n}{n+1}\right)$
- $\displaystyle\sum_{n\geq 1} \sin\left(\pi\sqrt{n^2+1}\right)$

7. (1) Let $f(x) = \frac{e^{i\sqrt{x}}}{\sqrt{x}}$. Estimate $f(n+1) - f(n)$ using Taylor's formula with integral remainder. Deduce the nature of the series $\sum_{n\geq 1} f'(n)$.

(2) Determine a primitive F of f. Estimate $F(n+1) - F(n)$ by the same method. Deduce the nature of the series $\sum_{n\geq 1} f(n)$.

8. (Raabe-Duhamel rule)**.** Let $\sum u_n$ be a series with strictly positive terms.

(1) Show that if $\frac{u_{n+1}}{u_n} = 1 - \frac{\alpha}{n} + \mathrm{O}(\frac{1}{n^\beta})$ with $\beta > 1$, then the sequence $(n^\alpha u_n)$ has a finite non-zero limit, and
 ◇ if $\alpha > 1$, then $\sum u_n$ converges;
 ◇ if $\alpha \leq 1$, then $\sum u_n$ diverges.

(2) Application: Study the convergence of the series $\sum u_n$, where $u_n = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\frac{1}{\sqrt{n}}$.

(3) Application: Prove the Stirling approximation: there exists $K > 0$ such that $n! \sim K n^n e^{-n}\sqrt{n}$. One can show that $K = \sqrt{2\pi}$.

(4) Show that if $\frac{u_{n+1}}{u_n} = 1 - \frac{\alpha}{n} + \mathrm{o}(\frac{1}{n})$ with
 ◇ $\alpha > 1$, then $\sum u_n$ converges;
 ◇ $\alpha < 1$, then $\sum u_n$ diverges.

9. Let $(a_n)_{n\geq 0}$ be a sequence with strictly positive terms. Suppose $\sum a_n$ diverges and put $S_n := \sum_{k=0}^{n} a_k$. Study the nature of the series $\sum \frac{a_{n+1}}{S_n}$.

10. (Riemann rearrangement theorem)**.** Let $\sum_{k\geq 1} a_k$ be a real series which converges, but not absolutely. Suppose $-\infty \leq \alpha \leq \infty$. Show that there exists a permutation σ of $\mathbb{N}$ such that $\sum_{k=1}^{\infty} a_{\sigma(k)} = \alpha$.

11. Let (u_n) be a complex sequence and $\sum v_n$ a series with nonnegative terms.

(1) Suppose that $\sum v_n$ diverges. Show that

- ◇ if $u_n = \mathrm{O}(v_n)$, then $\sum_{p=0}^{n} u_p = \mathrm{O}\Big(\sum_{p=0}^{n} v_p\Big)$;
- ◇ if $u_n = \mathrm{o}(v_n)$, then $\sum_{p=0}^{n} u_p = \mathrm{o}\Big(\sum_{p=0}^{n} v_p\Big)$;
- ◇ if $u_n \sim v_n$, then $\sum_{p=0}^{n} u_p \sim \sum_{p=0}^{n} v_p$.

Applications: Show that if $u_n \to \ell$, then $u_n \to \ell$ in the sense of Cesàro.

Show that $\sum_{p=1}^{n} \frac{1}{k} \sim \ln n$.

(2) Suppose that $\sum v_n$ converges. Show that

- ◇ if $u_n = \mathrm{O}(v_n)$, then $\sum_{p=n}^{+\infty} u_p = \mathrm{O}\Big(\sum_{p=n}^{+\infty} v_p\Big)$;
- ◇ if $u_n = \mathrm{o}(v_n)$, then $\sum_{p=n}^{+\infty} u_p = \mathrm{o}\Big(\sum_{p=n}^{+\infty} v_p\Big)$;
- ◇ if $u_n \sim v_n$, then $\sum_{p=n}^{+\infty} u_p \sim \sum_{p=n}^{+\infty} v_p$.

12. (Abel's test)**.** Let $\sum a_n v_n$ be a series such that $A_n := \sum_{k=0}^{n} a_k$ is bounded and v_n decreases and tends to 0. Show that $\sum a_n v_n$ converges.

Application: Study the convergence of the series $\sum \frac{\cos n\theta}{n^\alpha}$ and $\sum \frac{\sin n\theta}{n^\alpha}$, where $\alpha > 0$ and θ is a real which is not an integer multiple of π.

Variant: Show that if A_n converges and if v_n is monotone and convergent, then $\sum a_n v_n$ converges.

13. Let (u_n) be a decreasing sequence of strictly positive reals such that the series $\sum u_n$ converges. Show that $u_n = \mathrm{o}(\frac{1}{n})$.

Give an example of a sequence (u_n) of strictly positive reals such that the series $\sum u_n$ converges but the sequence nu_n does not tend to 0.

14. Let (a_n) be a nonnegative sequence such that the series $\sum a_n$ converges. Show that if $\alpha > \frac{1}{2}$, then the series $\sum \frac{\sqrt{a_n}}{n^\alpha}$ converges. What can we say if $\alpha = \frac{1}{2}$?

Chapter 3

Sequences and Series of Functions

In this chapter, all functions f, f_n are defined on a ball $I = B_r(z_0) \subseteq \mathbb{K}$. Recall that $B_r(z_0) =]z_0 - r, z_0 + r[$ if $\mathbb{K} = \mathbb{R}$ and $B_r(z_0)$ is an open disk of radius of r if $\mathbb{K} = \mathbb{C}$.

We follow [14] closely in Sections 3.1 to 3.3, then we follow [2] in Sections 3.4 and 3.5.

3.1 Pointwise convergence

Definition 3.1. We say that a sequence of functions (f_n) *converges pointwise* on I if for any $x \in I$, the sequence $(f_n(x))$ converges. In this case we define the *limit function* f by

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad x \in I\,.$$

We say that $\sum f_n$ converges pointwise on I if the sequence of partial sums $S_n(x) = \sum_{k=0}^{n} f_k(x)$ converges pointwise on I. The limit function of (S_n) is then called the *sum* of the series and is denoted by $S(x) = \sum_{n=0}^{\infty} f_n(x)$ for $x \in I$.

If the functions f_n are continuous, differentiable, or integrable, is the same true for the limit function ?

Recall that f is continuous at x if $\lim\limits_{t\to x} f(t) = f(x)$. But $\lim\limits_{t\to x} f(t) = \lim\limits_{t\to x}\lim\limits_{n\to\infty} f_n(t)$, and if f_n are continuous at x, then $f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}\lim\limits_{t\to x} f_n(t)$. Hence, if (f_n) are continuous, then limit function f will be continuous if

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)\,,$$

i.e. if we can interchange limits without affecting the result.

Let us first illustrate that the order of the limits is important in general. Afterward, we will give some conditions under which the limits can be safely interchanged.

Example 3.2. For $m, n \in \mathbb{N}^*$, let $s_{m,n} = \frac{m}{m+n}$. Then for any fixed n, we have $\lim\limits_{m\to\infty} s_{m,n} = 1$, hence $\lim\limits_{n\to\infty}\lim\limits_{m\to\infty} s_{m,n} = 1$. However, for any fixed m, $\lim\limits_{n\to\infty} s_{m,n} = 0$, so $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty} s_{m,n} = 0$. So the result changes when we interchange limits.

Example 3.3. Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Since $f_n(0) = 0$ for all n, we have $\sum_{n=0}^{\infty} f_n(0) = 0$. For $x \neq 0$, we have $\sum_{n=0}^{\infty} f_n(x) = x^2 \sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n} = x^2 \frac{1}{1-\frac{1}{1+x^2}} = 1 + x^2$. Thus, $\sum f_n$ converges pointwise on $\mathbb{R}$ to the function

$$S(x) = \begin{cases} 0 & \text{if } x = 0\,, \\ 1 + x^2 & \text{if } x \neq 0\,. \end{cases}$$

Thus, a convergent series of continuous functions may have a discontinuous sum.

Example 3.4. For $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$, let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ and $f(x) = \lim_{n\to\infty} f_n(x) = 0$. Then $f'(x) = 0$ for all x. However, $f_n'(x) = \sqrt{n}\cos nx$, so $f_n'(0) = \sqrt{n} \to \infty$. Hence, (f_n') does not converge pointwise to f'.

Example 3.5. Let $f_n(x) = n^2 x(1-x^2)^n$ for $x \in [0,1]$ and $n \in \mathbb{N}^*$. Then $f_n(0) = 0$, and for $x > 0$, $\lim_{n\to\infty} f_n(x) = 0$. Thus, (f_n) converges pointwise on $[0,1]$ to $f \equiv 0$. However, taking $y = x^2$, we have $\int_0^1 x(1-x^2)^n \,\mathrm{d}x = \frac{1}{2}\int_0^1 (1-y)^n \,\mathrm{d}y = \frac{1}{2}\frac{-(1-y)^{n+1}}{n+1}\Big|_0^1 = \frac{1}{2n+2}$. So $\int_0^1 f_n(x)\,\mathrm{d}x = \frac{n^2}{2n+2} \to \infty$, although $\int_0^1 f(x)\,\mathrm{d}x = 0$.

Note that if $g_n(x) = nx(1-x^2)^n$, then g_n still converges pointwise to $g \equiv 0$, but we now have $\int_0^1 g_n(x)\,\mathrm{d}x = \frac{n}{2n+2} \to \frac{1}{2}$. Thus, even if $\big(\int_a^b g_n(x)\,\mathrm{d}x\big)_{n\in\mathbb{N}}$ converges, its limit may not be equal to $\int_a^b g(x)\,\mathrm{d}x$.

3.2 Uniform convergence

Definition 3.6. We say that a sequence of functions (f_n) *converges uniformly* to a function f on I if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} : n \geq N \implies |f_n(x) - f(x)| \leq \varepsilon \qquad \forall x \in I\,.$$

We denote this by $f_n \rightrightarrows f$ on I.

We say that a series of functions $\sum f_n$ converges uniformly on I if the sequence of partial sums $S_n(x) = \sum_{k=0}^n f_k(x)$ converges uniformly on I.

Remark 3.7. Recall that (f_n) converges pointwise to f on I if

$$\forall x \in I\ \forall \varepsilon > 0\ \exists N \in \mathbb{N} : n \geq N \implies |f_n(x) - f(x)| \leq \varepsilon\,.$$

The difference is that in uniform convergence, there is *one* N which works *for any* $x \in I$. In pointwise convergence, *for each* x, we may find a corresponding N (which may depend on x). Hence, uniform convergence implies pointwise convergence.

Lemma 3.8. *Let f, g be bounded functions on I and denote $\|f\|_\infty := \sup_{x\in I}|f(x)|$. Then*

1. $\|f\|_\infty \geq 0, \quad \|f\|_\infty = 0 \iff f \equiv 0$ *on I,*
2. $\|\alpha f\|_\infty = |\alpha| \cdot \|f\|_\infty \quad \forall \alpha \in \mathbb{C}$,
3. $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.
4. $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$.

We call $\|f\|_\infty$ the supremum norm *of f.*

Proof. Clearly, $\|f\|_\infty \geq 0$. Moreover, $\|f\|_\infty = 0 \iff \sup_{x\in I}|f(x)| = 0 \iff |f(x)| = 0\ \forall x \in I \iff f(x) = 0\ \forall x \in I \iff f \equiv 0$ on I.

Next, $\|\alpha f\|_\infty = \sup_{x\in I}|\alpha f(x)| = \sup_{x\in I}|\alpha||f(x)| = |\alpha|\sup_{x\in I}|f(x)| = |\alpha|\|f\|_\infty$.

Next, if $x \in I$ we have $|f(x)+g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$. Taking the supremum in the LHS we get $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

Finally, if $x \in I$ we have $|f(x)g(x)| = |f(x)||g(x)| \leq \|f\|_\infty\|g\|_\infty$, so taking the sup in the LHS we get (4). □

Remark 3.9. Note that

$$f_n \rightrightarrows f \text{ on } I \iff \lim_{n\to\infty}\|f_n - f\|_\infty = 0\,.$$

Lemma 3.10. *Suppose* $f_n \rightrightarrows f$, $g_n \rightrightarrows g$ *on* I *and let* $\alpha, \beta \in \mathbb{C}$. *Then* $\alpha f_n + \beta g_n \rightrightarrows \alpha f + \beta g$ *on* I. *Moreover, if each* f_n *and* g_n *is bounded on* I, *then* f *and* g *are bounded on* I, *and* $f_n g_n \rightrightarrows fg$ *on* I.

Proof. By Lemma 3.8,

$$\|\alpha f_n + \beta g_n - (\alpha f + \beta g)\|_\infty \leq |\alpha| \cdot \|f_n - f\|_\infty + |\beta| \cdot \|g_n - g\|_\infty \to 0\,,$$

which proves that $\alpha f_n + \beta g_n \rightrightarrows \alpha f + \beta g$ on I.

Next, note that if each f_n is bounded on I, then f must be bounded on I since for $\varepsilon = 1$ we may find N such that $\|f\|_\infty \leq \|f_N - f\|_\infty + \|f_N\|_\infty \leq 1 + C$, where $C = \|f_N\|_\infty < \infty$ because f_N is bounded. Similarly, g is bounded on I. Hence,

$$\begin{aligned}\|f_n g_n - fg\|_\infty &= \|(f_n - f)(g_n - g) + g(f_n - f) - f(g - g_n)\|_\infty \\ &\leq \|f_n - f\|_\infty \|g_n - g\|_\infty + \|g\|_\infty \|f_n - f\|_\infty + \|f\|_\infty \|g - g_n\|_\infty \to 0\,. \quad \square\end{aligned}$$

Remark 3.11. The last result is not true if f_n or g_n is unbounded. For example, if $f_n(x) = \frac{1}{n}$ and $g_n(x) = x$ on $\mathbb{R}$, then $f_n \rightrightarrows 0$, $g_n \rightrightarrows g$, where $g(x) = x$, but $f_n g_n = \frac{x}{n}$ does not converge uniformly to $0 \cdot g = 0$. Indeed, $\|f_n g_n - 0\|_\infty \geq f_n(n^2) g_n(n^2) = n \to \infty$.

Lemma 3.12 (Cauchy criterion)**.** *A sequence of functions* (f_n) *converges uniformly on* I *iff*

$$\forall \varepsilon > 0 \;\exists N \in \mathbb{N} : n, m \geq N \implies |f_n(x) - f_m(x)| \leq \varepsilon \qquad \forall x \in I\,. \qquad (\star)$$

Proof. Suppose $f_n \rightrightarrows f$ on I. Then there exists $N \in \mathbb{N}$ such that, if $n \geq N$, then $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2}$ for any $x \in I$. Hence,

$$n, m \geq N \implies |f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \varepsilon$$

for any $x \in I$. Hence, (f_n) satisfies the Cauchy condition.

Conversely, suppose $(\star)$ holds. Then for any fixed $x \in I$, $(f_n(x))$ is a Cauchy sequence of numbers, hence converges to some $\ell_x \in \mathbb{C}$. Define $f(x) := \ell_x$. Then, (f_n) converges pointwise to f on I.

To see that $f_n \rightrightarrows f$ on I, let $\varepsilon > 0$ and take N such that $(\star)$ holds. Fix n and let $m \to \infty$ in $(\star)$. Since $f_m(x) \to f(x)$, we get

$$n \geq N \implies |f_n(x) - f(x)| \leq \varepsilon \qquad \forall x \in I\,. \qquad \square$$

Corollary 3.13. *If* $\sum f_n$ *converges uniformly on* I, *then* $f_n \rightrightarrows 0$ *on* I.

Proof. Since $\sum f_n$ is Cauchy, given $\varepsilon > 0$, we may find $N \in \mathbb{N}$ such that $|S_n(x) - S_{n-1}(x)| \leq \varepsilon$ for any $n \geq N$ and $x \in I$. Thus, $|f_n(x)| \leq \varepsilon$ for any $n \geq N$ and $x \in I$, hence $f_n \rightrightarrows 0$. $\square$

Lemma 3.14. *Let* $\sum f_n$ *be a pointwise convergent series and put* $R_n(x) = \sum_{k=n+1}^{\infty} f_n(x)$. *Then* $\sum f_n$ *is uniformly convergent iff* $R_n \rightrightarrows 0$ *on* I.

Proof. If $S_n \rightrightarrows S$ on I, then $R_n = S - S_n \rightrightarrows 0$ on I.

Conversely, if $\sum f_n$ converges pointwise to S on I and $R_n \rightrightarrows 0$ on I, then $S_n = S - R_n \rightrightarrows S$ on I. $\square$

Corollary 3.15. *Let* $\sum (-1)^n g_n$ *be a series of functions on* I *such that*

- ◇ $\forall x \in I$, *the numerical sequence* $(g_n(x))_{n \in \mathbb{N}}$ *is nonnegative and decreasing,*
- ◇ $g_n \rightrightarrows 0$ *on* I.

Then $\sum (-1)^n g_n$ *converges uniformly on* I.

Proof. By the alternating series criterion, for any $x \in I$, the series $\sum(-1)^n g_n(x)$ is convergent and $|R_n(x)| \leq g_{n+1}(x)$. Hence, $\|R_n\|_\infty \leq \|g_{n+1}\|_\infty \to 0$, so the series converges uniformly by Lemma 3.14. $\square$

Definition 3.16. We say that $\sum f_n$ *converges absolutely* on I if the numerical series $\sum |f_n(x)|$ converges for each $x \in I$.

We say that $\sum f_n$ *converges normally* on I if the numerical series $\sum \|f_n\|_\infty$ converges.

Lemma 3.17 (Weierstrass M-test)**.** *If $\sum f_n$ is normally convergent, then it is absolutely and uniformly convergent.*

Proof. Given $x \in I$, $|f_n(x)| = \mathrm{O}(\|f_n\|_\infty)$, hence $\sum |f_n(x)|$ converges. Thus, $\sum f_n$ is absolutely convergent.

Moreover, given $\varepsilon > 0$, we may find $N \in \mathbb{N}$ such that $\sum_{i=n}^m \|f_i\|_\infty \leq \varepsilon$ for $n, m \geq N$. Hence,

$$n, m \geq N \implies \Big|\sum_{i=n}^m f_i(x)\Big| \leq \sum_{i=n}^m |f_i(x)| \leq \sum_{i=n}^m \|f_i\|_\infty \leq \varepsilon \qquad \forall x \in I\,.$$

Hence, $\sum f_n$ satisfies the Cauchy criterion and thus converges uniformly. $\square$

Remark 3.18. The converse of the previous lemma is not true. For example, let

$$f_n(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x < \frac{1}{n+1}, \\ \frac{1}{n} & \text{if} \ \frac{1}{n+1} \leq x < \frac{1}{n}, \\ 0 & \text{if} \quad \frac{1}{n} \leq x \leq 1\,. \end{cases}$$

Then $\sum f_n$ converges absolutely on $[0,1]$ because for any $x \in [0,1]$, we have $f_n(x) = 0$ for all large n. $\sum f_n$ also converges uniformly on $[0,1]$ because $\|\sum_{k=n+1}^{n+p} f_k\|_\infty \leq \frac{1}{n+1}$. However, $\sum f_n$ does not converge normally, since $\|f_n\|_\infty = \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges.

3.3 Conservation of properties by uniform convergence

Definition 3.19. A *limit point* of a ball I is a point in I or on its boundary.

A *neighborhood* of a point $x \in I$ is a ball $V = B_s(x)$ for some $s > 0$.

A neighborhood of $+\infty$ and $-\infty$ is an interval of the form $]M, +\infty[$ and $]-\infty, M[$, respectively, for some $M \in \mathbb{R}$.

Definition 3.20. Let a be a limit point of I. We say that $\lim_{x\to a} f(x) = \ell$ if for any $\varepsilon > 0$, we may find a neighborhood V of a such that $x \in V \cap I \implies |f(x) - \ell| \leq \varepsilon$.

Theorem 3.21. *Suppose $f_n \rightrightarrows f$ on I. Let x be a limit point of I and suppose that*

$$\lim_{t\to x} f_n(t) = A_n\,.$$

Then (A_n) converges and

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n\,.$$

Thus, we have $\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$.

Proof. Given $\varepsilon > 0$, we may find by Cauchy's condition $N \in \mathbb{N}$ such that

$$n, m \geq N \implies |f_n(t) - f_m(t)| \leq \varepsilon \qquad \forall t \in I\,.$$

Letting $t \to x$ we obtain $|A_n - A_m| \leq \varepsilon$ for all $n, m \geq N$, hence (A_n) is Cauchy and converges, say to A. Now for any $n \in \mathbb{N}$,

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|\,. \qquad (\dagger)$$

Choose $n_0 \in \mathbb{N}$ such that $|f(t) - f_{n_0}(t)| \leq \varepsilon/3$ for any $t \in I$ and $|A_{n_0} - A| \leq \varepsilon/3$. Next, choose a neighborhood V of x such that $|f_{n_0}(t) - A_{n_0}| \leq \varepsilon/3$ for all $t \in V \cap I$. Then taking $n = n_0$ in (†) we get $|f(t) - A| \leq \varepsilon$ for any $t \in V \cap I$. □

Corollary 3.22. *If (f_n) is a sequence of continuous functions on I, and if $f_n \rightrightarrows f$ on I, then f is continuous on I.*

Proof. Let $x \in I$. Since f_n is continuous at x, we have $A_n = \lim_{t\to x} f_n(t) = f_n(x)$. Using Theorem 3.21, we get $\lim_{t\to x} f(t) = \lim_{n\to\infty} f_n(x) = f(x)$. Hence, f is continuous at x. □

Theorem 3.23. *Let (f_n) be a sequence of Riemann integrable functions on $[a, b]$ such that $f_n \rightrightarrows f$ on $[a, b]$. Then f is Riemann integrable on $[a, b]$ and*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x\,.$$

Proof. We admit that f is Riemann integrable [1]. Next,

$$\begin{aligned}\Big|\int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x\Big| &= \Big|\int_a^b (f_n(x) - f(x))\,\mathrm{d}x\Big| \\ &\leq \int_a^b |f_n(x) - f(x)|\,\mathrm{d}x \leq (b-a)\|f_n - f\|_\infty \to 0\,.\end{aligned}$$ □

Corollary 3.24 (Term by term integration)**.** *Let (f_n) be a sequence of integrable functions on $[a, b]$ such that $\sum f_n \rightrightarrows S$ on $[a, b]$. Then S is integrable on $[a, b]$ and*

$$\int_a^b S(x)\,\mathrm{d}x = \sum \int_a^b f_n(x)\,\mathrm{d}x\,.$$

Proof. Let $S_n(x) = \sum_{k=0}^n f_k(x)$. Since $S_n \rightrightarrows S$, we have by Theorem 3.23,

$$\int_a^b S(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b S_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=0}^n \int_a^b f_k(x)\,\mathrm{d}x = \sum_{k=0}^\infty \int_a^b f_k(x)\,\mathrm{d}x\,.$$ □

Theorem 3.25. *Let (f_n) be a sequence of differentiable functions on $[a, b]$. If $(f_n(x_0))$ converges for some $x_0 \in [a, b]$, and if (f_n') converges uniformly on $[a, b]$, then (f_n) converges uniformly on $[a, b]$ to a function f such that*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad \forall x \in [a, b]\,.$$

1. Here is a proof: Let $X = (x_k)_{k=0}^n$ be a subdivision of $[a, b]$, let $s(f) = \sup_X \sum_{k=0}^{n-1}(x_{k+1} - x_k)\inf_{[x_k, x_{k+1}]} f$ and $S(f) = \inf_X \sum_{k=0}^{n-1}(x_{k+1} - x_k)\sup_{[x_k, x_{k+1}]} f$. Since f_n is integrable, $s(f_n) = S(f_n)$. Let $\varepsilon_n = \|f_n - f\|_\infty$. Then $f_n - \varepsilon_n \leq f \leq f_n + \varepsilon$ on $[a, b]$, so $\int_a^b (f_n - \varepsilon_n) = s(f_n - \varepsilon_n) \leq s(f) \leq S(f) \leq S(f_n + \varepsilon_n) = \int_a^b (f_n + \varepsilon_n)$. Thus, $0 \leq S(f) - s(f) \leq \int_a^b (f_n + \varepsilon_n) - s(f) \leq \int_a^b (f_n + \varepsilon_n - f_n + \varepsilon_n) = 2\varepsilon_n(b-a)$. Taking $n \to \infty$, we get $s(f) = S(f)$, so f is Riemann integrable.

Proof. Given $\varepsilon > 0$, choose N such that for $n, m \geq N$, we have

$$|f_n(x_0) - f_m(x_0)| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \|f_n' - f_m'\|_\infty \leq \frac{\varepsilon}{2(b-a)}\,.$$

Applying the mean value theorem to the function $f_n - f_m$, we get

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{\varepsilon}{2(b-a)}|x-t| \leq \frac{\varepsilon}{2} \qquad (\ddagger)$$

for any $n, m \geq N$ and $x, t \in [a, b]$. Hence,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| \leq \varepsilon$$

for any $x \in [a, b]$ and $n, m \geq N$. Hence, (f_n) converges uniformly on $[a, b]$. Put $f(t) = \lim_{n\to\infty} f_n(t)$, fix $x \in [a, b]$ and define

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x} \quad \text{and} \quad \phi(t) = \frac{f(t) - f(x)}{t - x} \qquad \text{for } t \in [a, b] \setminus \{x\}\,.$$

Then ϕ_n converges pointwise to ϕ on $[a, b] \setminus \{x\}$. Moreover, by $(\ddagger)$,

$$|\phi_n(t) - \phi_m(t)| \leq \frac{\varepsilon}{2(b-a)}$$

for any $n, m \geq N$, so that (ϕ_n) converges uniformly on $[a, b] \setminus \{x\}$. Thus, $\phi_n \rightrightarrows \phi$ on $[a, b] \setminus \{x\}$.

Finally, apply Theorem 3.21 to ϕ_n with $A_n = \lim_{t\to x} \phi_n(t) = f_n'(x)$ to get

$$f'(x) = \lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x)\,. \qquad \square$$

Definition 3.26. We say that $f \in C^p\,[a, b]$ if the derivatives $f', f'', \ldots f^{(p)}$ exist and are continuous. If $f \in C^p\,[a, b]$, we say that f *is of class C^p on* $[a, b]$.

We say that $f \in C^\infty\,[a, b]$ if $f^{(k)}$ exists for all $k \geq 1$.

Corollary 3.27. *Let (f_n) be a sequence of functions in $C^p\,[a, b]$, where $1 \leq p \leq \infty$. If (f_n) converges pointwise on $[a, b]$, and if for each k, $1 \leq k \leq p$, the sequence $(f_n^{(k)})$ is uniformly convergent, then (f_n) converges uniformly to a function $f \in C^p\,[a, b]$ such that*

$$f^{(k)}(x) = \lim_{n\to\infty} f_n^{(k)}(x) \qquad \forall x \in [a, b]\,.$$

Proof. Follows by induction from Theorem 3.25. $\square$

Corollary 3.28 (Term by term differentiation)**.** *Let (f_n) be a sequence of functions in $C^p\,[a, b]$, where $1 \leq p \leq \infty$. If $\sum f_n$ is pointwise convergent on $[a, b]$, and if for each k, $1 \leq k \leq p$, the series $\sum f_n^{(k)}$ converges uniformly on $[a, b]$, then $\sum f_n$ converges uniformly to a function $S \in C^p\,[a, b]$ such that*

$$S^{(k)}(x) = \sum f_n^{(k)}(x) \qquad \forall x \in [a, b]\,.$$

3.4 Uniform continuity

Definition 3.29. Let $I \subset \mathbb{R}$ be an interval. We say that f is *uniformly continuous* on I if

$$\forall \varepsilon > 0\ \exists \delta > 0 : \forall x, y \in I,\ |x-y| \le \delta \implies |f(x)-f(y)| \le \varepsilon\,.$$

Remark 3.30. Recall that f is continuous on I if

$$\forall x \in I\ \forall \varepsilon > 0\ \exists \delta > 0 : \forall y \in I,\ |x-y| \le \delta \implies |f(x)-f(y)| \le \varepsilon\,.$$

The difference is that the δ in uniform continuity depends only on ε, while in continuity, it may also depend on x. Hence, uniform continuity implies continuity.

Reminder 3.31. We say that f is *Lipschitz continuous* on I (Lip) if there exists $K > 0$ such that $\forall x, y \in I : |f(x)-f(y)| \le K|x-y|$.

If f is Lip, then given $\varepsilon > 0$, if we choose $\delta = \varepsilon/K$, we see that f is uniformly continuous.

Example 3.32. (a) $f(x) = x^2$ is uniformly continuous on $I = [0,1]$, since it is Lip on $[0,1]$. Indeed, given $x, y \in I$, $|x^2-y^2| = |x+y||x-y| \le 2|x-y|$.

(b) $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. Indeed, take $\varepsilon = 2$ and any $\delta > 0$. Now choose $x = \frac{1}{\delta}$ and $y = x+\delta$. Then $|x-y| \le \delta$, but $|x^2-y^2| = y^2-x^2 = 2+\delta^2 > \varepsilon$.

(c) $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$, since it is Lip on $\mathbb{R}$. Indeed, for any $x, y \in \mathbb{R}$, we may find $z \in \,]x,y[$ such that $|\sin x - \sin y| = |\cos z| \cdot |x-y| \le |x-y|$.

(d) $f(x) = \sqrt{x}$ is uniformly continuous on $\mathbb{R}_+$, but not Lip on $\mathbb{R}_+$.
Indeed, given $K > 0$, choose $x = \frac{1}{(2K)^2}$. Then $|f(x)-f(0)| = \sqrt{x} = \frac{1}{\sqrt{x}}|x-0| = 2K|x-0| > K|x-0|$. Hence, $f(x) = \sqrt{x}$ is not Lipschitz on $\mathbb{R}_+$.
On the other hand, note that if $0 \le x \le y$, then $x+y = (y-x)+2x \le |x-y|+2\sqrt{xy}$. Similarly, if $0 \le y \le x$, then $x+y = (x-y)+2y \le |x-y|+2\sqrt{xy}$. Thus, for any $x, y \in \mathbb{R}_+$ we have $x-2\sqrt{xy}+y \le |x-y|$, i.e. $(\sqrt{x}-\sqrt{y})^2 \le |x-y|$. Hence, $|\sqrt{x}-\sqrt{y}| \le \sqrt{|x-y|}$. Given $\varepsilon > 0$, choosing $\delta = \varepsilon^2$, we thus have

$$\forall x, y \in \mathbb{R}_+ : |x-y| \le \delta \implies |\sqrt{x}-\sqrt{y}| \le \sqrt{|x-y|} \le \varepsilon\,,$$

which shows that f is uniformly continuous on $\mathbb{R}_+$.

We thus have

$$f \text{ Lipshitz continuous} \implies f \text{ uniformly continuous} \implies f \text{ continuous},$$

and neither implication is an equivalence.

Theorem 3.33 (Heine-Cantor Theorem)**.** *Let $[a,b]$ be a closed bounded interval. If f is continuous on $[a,b]$, it is uniformly continuous on $[a,b]$.*

Proof. Suppose f is continuous on $I = [a,b]$. Suppose on the contrary that f is not uniformly continuous on I. Then

$$\exists \varepsilon_0 > 0 : \forall \delta > 0,\ \exists x, y \in I,\ |x-y| \le \delta \text{ but } |f(x)-f(y)| \ge \varepsilon_0\,.$$

Take this ε_0. For $\delta = 2^{-n}$, we may thus choose $x_n, y_n \in I$ such that $|x_n - y_n| \le 2^{-n}$ but $|f(x_n)-f(y_n)| \ge \varepsilon_0$. Hence,

$$\lim(x_n - y_n) = 0 \text{ but } \forall n \in \mathbb{N},\ |f(x_n)-f(y_n)| \ge \varepsilon_0\,. \qquad (*)$$

Since I is bounded, the sequence (x_n) is bounded and has a subsequence $(x_{\varphi(n)})$ which converges to some α. Since I is closed, we have $\alpha \in I$. Since $y_{\varphi(n)} = x_{\varphi(n)} + (y_{\varphi(n)} - x_{\varphi(n)})$, we have $y_{\varphi(n)} \to \alpha$.

Since f is continuous, we thus get $\lim \left(f(x_{\varphi(n)}) - f(y_{\varphi(n)})\right) = f(\alpha) - f(\alpha) = 0$. This contradicts $(*)$. Hence, f is uniformly continuous on I. □

3.5 Approximation theorems

Definition 3.34. A map $f : [a,b] \to \mathbb{C}$ is *piecewise continuous* if there exists a subdivision $a = a_0 < a_1 < \ldots < a_n = b$ such that the restriction of f to each $]a_{i-1}, a_i[$ can be extended to a continuous function on $[a_{i-1}, a_i]$. The set of such functions is denoted by $CP\,[a,b]$.

Remark 3.35. If there is a subdivision $a = a_0 < a_1 < \ldots < a_n = b$ such that
- ◇ g is continuous on each $]a_{i-1}, a_i[$,
- ◇ g has a right-sided limit at $a_0, \ldots, a_{n-1}$,
- ◇ g has a left-sided limit at $a_1, \ldots, a_n$,

then $g \in CP\,[a,b]$.

Definition 3.36. A map $\varphi : [a,b] \to \mathbb{C}$ is a *step function* if there exists a subdivision $a = a_0 < a_1 < \ldots < a_n = b$ of $[a,b]$ such that the restriction of φ to each interval $]a_{i-1}, a_i[$ is constant. The set of such functions is denoted by $E\,[a,b]$.

Lemma 3.37. *For any $f \in CP\,[a,b]$, we may find $g \in C\,[a,b]$ and $\varphi \in E\,[a,b]$ such that $f = g + \varphi$.*

Proof. We prove the lemma by induction over the number of points of discontinuity of f.

If f has no point of discontinuity, take $g = f$ and $\varphi = 0$.

Suppose the result is true for piecewise continuous functions having n points of discontinuity and let f be discontinuous at $n+1$ points. Let c be one of them. We define

$$h(x) = \begin{cases} f(x) & \text{if } x \in [a,c[\\ \lim_{x\to c^-} f(x) & \text{if } x = c \\ f(x) - \lim_{x\to c^+} f(x) + \lim_{x\to c^-} f(x) & \text{if } x \in]c,b] \,. \end{cases}$$

Then h is piecewise continuous on $[a,b]$ and has at most n points of discontinuity. Moreover, $\varphi := f - h$ is a step function on $[a,b]$. By hypothesis, h may be written as $h = g + \psi$ for some $g \in C\,[a,b]$ and $\psi \in E\,[a,b]$. Hence, $f = \underbrace{g}_{\in C[a,b]} + \underbrace{(\varphi + \psi)}_{\in E[a,b]}$. □

Theorem 3.38. *Any $f \in CP\,[a,b]$ is the uniform limit on $[a,b]$ of a sequence of step functions.*

Proof. By the preceding lemma, we may assume $f \in C\,[a,b]$. Indeed, if $f = g + \varphi$ and $\varphi_n \rightrightarrows g$, then $\varphi_n + \varphi \rightrightarrows f$.

So let $f \in C\,[a,b]$. By Heine's theorem, f is uniformly continuous on $[a,b]$, so for $\varepsilon = \frac{1}{n}$, $\exists \delta_n > 0$ such that $\forall x, y \in [a,b] : |x - y| \le \delta_n \implies |f(x) - f(y)| \le \frac{1}{n}$.

Let $a = a_0 < a_1 < \ldots < a_p = b$ be a subdivision of $[a,b]$ such that $a_{k+1} - a_k \le \delta_n$ and let ψ_n be the unique step function on $[a,b]$ such that $\psi_n(a_p) = f(a_p)$ and $\psi_n(t) = f(a_k)$ for $t \in [a_k, a_{k+1}[$, $0 \le k \le p-1$. Then

$$\forall t \in [a_k, a_{k+1}[\, : |f(t) - \psi_n(t)| = |f(t) - f(a_k)| \le 1/n\,, \qquad 0 \le k \le p-1\,.$$

Since the inequality is clearly true for $t = a_p$, the proof is complete. □

Definition 3.39. A function $f : [a, b] \to \mathbb{C}$ is *piecewise linear* if there exists a subdivision $a = a_0 < a_1 < \ldots < a_n = b$ of $[a, b]$ such that f is linear on each interval $]a_{i-1}, a_i[$.

Theorem 3.40. *Any $f \in C\,[a, b]$ is the uniform limit on $[a, b]$ of a sequence (f_n) of piecewise linear continuous functions satisfying $f_n(a) = f(a)$ and $f_n(b) = f(b)$ for all $n \in \mathbb{N}$.*

Proof. Let $f \in C\,[a, b]$, so f is uniformly continuous on $[a, b]$. Hence, for $\varepsilon = \frac{1}{n}$ we may find $\delta_n > 0$ such that for any $x, y \in [a, b]$, if $|x - y| \le \delta_n$, then $|f(x) - f(y)| \le \frac{1}{n}$.

Let $a = a_0 < a_1 < \ldots < a_p = b$ be a subdivision of $[a, b]$ such that $a_{k+1} - a_k < \delta_n$ and let g_n be the unique piecewise linear function on $[a, b]$ such that $g_n(a_k) = f(a_k)$ for all k.

We may express g_n as follows :

$$g_n(t) = \lambda_{t,k} f(a_k) + (1 - \lambda_{t,k}) f(a_{k+1}) \qquad \text{if } t \in [a_k, a_{k+1}]\,,\ 0 \le k \le p - 1\,,$$

where $\lambda_{t,k} = \frac{a_{k+1} - t}{a_{k+1} - a_k}$. Then $g_n(a) = f(a)$, $g_n(b) = f(b)$ and

$$\begin{aligned} |f(t) - g_n(t)| &= |\lambda_{t,k} f(t) + (1 - \lambda_{t_k}) f(t) - g_n(t)| \\ &\le \lambda_{t,k} |f(t) - f(a_k)| + (1 - \lambda_{t,k}) |f(t) - f(a_{k+1})| \le \frac{1}{n}\,. \end{aligned}$$ □

Corollary 3.41. *Any $f \in CP\,[a, b]$ is the uniform limit on $[a, b]$ of a sequence of piecewise linear functions.*

Theorem 3.42. *Let $f \in C\,[0, 1]$. The sequence of polynomials (B_n) defined by*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\Big(\frac{k}{n}\Big)$$

converges uniformly to f on $[0, 1]$.

We call B_n the nth Bernstein polynomial associated to f.

Proof. Let $\varepsilon > 0$. By Heine's theorem, f is uniformly continuous, so there exists $\delta > 0$ such that for any $x, y \in [0, 1]$ satisfying $|x - y| \le \delta$, we have $|f(x) - f(y)| \le \frac{\varepsilon}{2}$. On the other hand, if we put $M = \max_{x \in [0,1]} |f(x)|$, we always have $|f(x) - f(y)| \le 2M$. Thus, for any $x \in [0, 1]$ and $n \in \mathbb{N}$,

– if k satisfies $|\frac{k}{n} - x| \le \delta$, then $|f(x) - f(\frac{k}{n})| \le \frac{\varepsilon}{2}$,

– if k satisfies $|\frac{k}{n} - x| \ge \delta$, then $|f(x) - f(\frac{k}{n})| \le 2M \le 2M \frac{(k - nx)^2}{n^2 \delta^2}$, since in this case, $\frac{(k - nx)^2}{n^2 \delta^2} \ge 1$.

Hence, for any index k we have $|f(x) - f(\frac{k}{n})| \le \frac{\varepsilon}{2} + 2M \frac{(k - nx)^2}{n^2 \delta^2}$. Thus,

$$\begin{aligned} \forall x \in [0, 1] : |f(x) - B_n(x)| &= \Big| \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} \Big\{ f(x) - f\Big(\frac{k}{n}\Big) \Big\} \Big| \\ &\le \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} \Big(\frac{\varepsilon}{2} + 2M \frac{(k - nx)^2}{n^2 \delta^2} \Big)\,, \end{aligned}$$

where in the first equality, we used the fact that $\sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} = 1$ by the binomial theorem. We now note that

$$(k - nx)^2 = k(k - 1) + (1 - 2nx)k + n^2 x^2\,.$$

But $k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = n\binom{n-1}{k-1}$ if $k \geq 1$, and 0 if $k = 0$. Hence,

$$\begin{aligned}\sum_{k=0}^{n} k\binom{n}{k}x^k(1-x)^{n-k} &= \sum_{k=1}^{n} n\binom{n-1}{k-1}x^k(1-x)^{n-k} = n\sum_{j=0}^{n-1}\binom{n-1}{j}x^{j+1}(1-x)^{n-j-1} \\ &= nx\sum_{j=0}^{n-1}\binom{n-1}{j}x^j(1-x)^{(n-1)-j} = nx\,.\end{aligned}$$

Similarly, $k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}$ if $k \geq 2$ and 0 if $k = 0, 1$. Hence,

$$\begin{aligned}\sum_{k=0}^{n} k(k-1)\binom{n}{k}x^k(1-x)^{n-k} &= \sum_{k=2}^{n} n(n-1)\binom{n-2}{k-2}x^k(1-x)^{n-k} \\ &= n(n-1)\sum_{j=0}^{n-2}\binom{n-2}{j}x^{j+2}(1-x)^{n-j-2} = n(n-1)x^2\,.\end{aligned}$$

We thus showed that

$$\sum_{k=0}^{n}(k-nx)^2\binom{n}{k}x^k(1-x)^{n-k} = n(n-1)x^2+(1-2nx)nx+n^2x^2 = -nx^2+nx = nx(1-x)\,,$$

so

$$\forall x \in [0,1] : |f(x) - B_n(x)| \leq \frac{\varepsilon}{2} + \frac{2M}{n^2\delta^2}\cdot nx(1-x) = \frac{\varepsilon}{2} + \frac{2Mx(1-x)}{n\delta^2} \leq \frac{\varepsilon}{2} + \frac{2M}{n\delta^2}\,,$$

since $x \in [0,1]$. Choosing n_0 such that $\frac{2M}{n\delta^2} \leq \frac{\varepsilon}{2}$ for all $n \geq n_0$, we obtain the result. □

Corollary 3.43 (Weierstrass approximation theorem)**.** *Any $f \in C[a,b]$ is the uniform limit on $[a,b]$ of a sequence of polynomial functions.*

Proof. Let $\varphi : [0,1] \to \mathbb{R}$ be given by $\varphi(x) = a + (b-a)x$. Then $f \circ \varphi \in C[0,1]$, so it is the uniform limit of a sequence (B_n) of associated Bernstein polynomials. Hence, f is the uniform limit of the sequence $B_n \circ \varphi^{-1}$, which is indeed a sequence of polynomials since $B_n \circ \varphi^{-1}(x) = B_n(\frac{x-a}{b-a})$. □

3.6 Exercises

1. Study the pointwise and uniform convergence of the following sequences of functions:

- $f_n(x) = x^n$ on $[0,1]$
- $f_n(x) = \begin{cases} n^2x & \text{if } |x| \leq \frac{1}{n}, \\ \frac{1}{x} & \text{if } |x| > \frac{1}{n}. \end{cases}$
- $f_n(x) = n^\alpha xe^{-nx}$ on $\mathbb{R}_+$
- $f_n(x) = \dfrac{x}{1+nx}$ on $[0,1]$
- $f_n(x) = \dfrac{1}{1+nx}$ on $[0,1]$
- $f_n(x) = \dfrac{\ln(1+nx)}{1+nx}$ on $\mathbb{R}_+$
- $f_n(x) = (1-x)x^n$ on $[0,1]$
- $f_n(x) = nx^2e^{-nx}$ on $[0,\pi]$
- $f_n(x) = e^{-nx}$ on $[0,\pi]$
- $f_n(x) = (1-|x|)^n$ on $]-1,1[$
- $f_n(x) = \sin^n(x)$ on $[0,\pi]$
- $f_n(x) = \dfrac{1}{n}\sin(nx)$ on $\mathbb{R}$
- $f_n(x) = x^2\exp\left(-\sin\dfrac{x}{n}\right)$ on $\mathbb{R}$
- $f_n(x) = \dfrac{\sin(nx)}{n\sqrt{x}}$ on $\mathbb{R}_+$
- $f_n(x) = e^{-x}\displaystyle\sum_{k=0}^{n}\frac{x^k}{k!}$ on $\mathbb{R}_+$

2. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions on $I=[a,b]$. Suppose that $f_n \rightrightarrows f$ sur I, i.e. f_n converges uniformly to a function f on I. Are the following assertions correct? Justify your answer.

(1) If f is discontinuous at $a\in I$, then $\forall n\in\mathbb{N}$, f_n is discontinuous at a.

(2) If f is discontinuous at $a\in I$, then $\exists n_0\in\mathbb{N}$ such that, $\forall n\geq n_0$, the function f_n is discontinuous at a.

(3) If f is continuous at $a\in I$, then $\forall n\in\mathbb{N}$, f_n is continuous at a.

(4) If f is continuous at $a\in I$, then $\exists n_0\in\mathbb{N}$ such that, $\forall n\geq n_0$, the function f_n is continuous at a.

(5) If f_n is continuous and F_n is a primitive of f_n, then F_n converges uniformly to a primitive of f.

(6) If $f_n: I\to J$ and $\phi: J\to\mathbb{R}$ is continuous, then $\phi\circ f_n \rightrightarrows \phi\circ f$ on I.

(7) If $f_n: I\to J$ and $\phi: J\to\mathbb{R}$ is uniformly continuous, then $\phi\circ f_n \rightrightarrows \phi\circ f$ on I.

3. Let $f_n(x)=\begin{cases}0 & \text{if } x<\frac{1}{n+1},\\ \sin^2\frac{\pi}{x} & \text{if } \frac{1}{n+1}\leq x\leq\frac{1}{n},\\ 0 & \text{if } \frac{1}{n}<x.\end{cases}$

Show that (f_n) converges, but not uniformly, to a continuous function.

Use the series $\sum f_n$ to show that absolute convergence, even at each point of $\mathbb{R}$, does not necessarily imply uniform convergence on $\mathbb{R}$.

4. (Continuous but differentiable nowhere)**.** Put

$$\varphi(x)=|x|, \qquad -1\leq x\leq 1$$

and extend the definition of $\varphi(x)$ to all $x\in\mathbb{R}$ by defining

$$\varphi(x+2)=\varphi(x)\,.$$

Let

$$f(x)=\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\varphi(4^n x)\,.$$

Show that f is continuous on $\mathbb{R}$ but not differentiable at any point of $\mathbb{R}$.

5. (Dini Theorems)**.**

(1) First Theorem: let $f_n:[a,b]\to\mathbb{R}$ be a monotone sequence of continuous functions (i.e. $f_{n+1}\geq f_n$ or $f_{n+1}\leq f_n$) which converge pointwise to a function f. If f is continuous, show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

(2) Application: let $P_0(x)=0$ and $P_{n+1}(x)=P_n(x)+\frac{x^2-P_n^2(x)}{2}$ for $n\in\mathbb{N}$. Show that $(P_n)_{n\in\mathbb{N}}$ converges uniformly to the function $|x|$ on $[-1,1]$.

(3) Second Theorem: let $f_n:[a,b]\to\mathbb{R}$ be a sequence of increasing functions (i.e. $x\leq y$ implies $f_n(x)\leq f_n(y)$) which converge pointwise to a function f. If f is continuous, show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

(4) Application: let $f_n(x)=(1+\frac{x}{n})^n$. Show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the function e^x on each interval $[-a,a]$, where $a>0$.

6. Let $f_n(x)=\frac{x}{1+nx^2}$ for $x\in\mathbb{R}$. Show that (f_n) converges uniformly on $\mathbb{R}$ to a function f, and that the equation $f'(x)=\lim_{n\to\infty}f_n'(x)$ is true for $x\neq 0$, but false for $x=0$.

7. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real-valued differentiable functions on $[a,b]$. Suppose that f_n converges pointwise to a function f and that there exists $M>0$ such that $\forall n\in\mathbb{N}$, $\forall x\in\mathbb{R}$, $|f_n'(x)|\le M$. Show that the convergence is uniform on $[a,b]$. Is the function f necessarily differentiable on $[a,b]$?

8. Show that the series $\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$ converges uniformly on any bounded interval of $\mathbb{R}$, but does not converge absolutely for any value of x.

9. Let $f_n(x)=\frac{1}{x^2+n^2}$ for $x\in\mathbb{R}$, $n\in\mathbb{N}$.

(1) Show that $\sum_{n\in\mathbb{N}}f_n(x)$ converges normally on $\mathbb{R}$. Denote its sum by $S(x)$.

(2) Show that $S(x)$ is a continuous function on $\mathbb{R}$.

(3) Show that $\lim_{x\to\infty}S(x)=\lim_{x\to-\infty}S(x)=0$.

10. Let $f_n(x)=\frac{x}{(1+x^2)^n}$ for $x\in\mathbb{R}$, $n\in\mathbb{N}$.

(1) Show that $\sum_{n\in\mathbb{N}}f_n(x)$ converges pointwise on $\mathbb{R}$. Denote its sum by $S(x)$.

(2) Show that $\sum_{n\in\mathbb{N}}f_n(x)$ converges normally on $[a,+\infty[$ and $]-\infty,-a]$ for any $a>0$. Deduce that the function $S(x)$ is continuous on $]-\infty,0[\,\cup\,]0,\infty[$.

(3) Calculate $S(x)$ for each $x\in\mathbb{R}$. Is the function $S(x)$ continuous on $\mathbb{R}$?

11. Consider $\sum f_n$, where f_n is defined below. In each case,

(1) show that the series converges pointwise on I ;

(2) determine the intervals on which the series converges normally ;

(3) show that the limit function is continuous on I ;

(4) study the series of derivatives ;

(5) is the sum differentiable ?

- $f_n(x)=e^{-n(x^2+1)}$, $I=\mathbb{R}$
- $f_n(x)=\dfrac{x}{n^2+x}$, $I=\mathbb{R}_+$
- $f_n(x)=x^2e^{-nx}$, $I=\mathbb{R}_+^*$
- $f_n(x)=\sin\left(\dfrac{x}{2^n}\right)$, $I=\mathbb{R}$
- $f_n(x)=\dfrac{1}{n^2(x+n)}$, $I=\mathbb{R}_+$
- $f_n(x)=\dfrac{xe^{-n^2x}}{1+n^2x}$, $I=\mathbb{R}_+$

12. By considering the series of functions $S(x)=\sum_{n\ge0}(\frac{x}{2})^n$, calculate $\sum_{n\ge0}\frac{n}{2^n}$.

13. Put $f_n(x)=nxe^{-nx}$ for $x\in\mathbb{R}_+$ et $n\in\mathbb{N}$. Show that $\sum f_n$ converges pointwise on $\mathbb{R}_+$. Study the uniform convergence on $\mathbb{R}_+$ and $[\varepsilon,+\infty[$, where $\varepsilon>0$.

14. Let $f_n(x)=\frac{(-1)^n}{n+x^2}$ for $x\in\mathbb{R}$, $n\in\mathbb{N}$.

(1) Show that $\sum f_n$ converges uniformly on $\mathbb{R}$. We put $S(x)=\sum_{n=1}^{\infty}f_n(x)$.

(2) Deduce that $\lim_{x\to\infty}S(x)=0$.

(3) Show that $\sum f_n'$ converges normally on any interval $[-a,a]$, $a>0$.

(4) Deduce that S is of class C^1 on $\mathbb{R}$.

15. Are the following assertions correct ? Justify your answer.

(1) If $f:\mathbb{R}_+^*\to\mathbb{R}$ is continuous on $[\varepsilon,+\infty[$ for any $\varepsilon>0$, then f is continuous on $\mathbb{R}_+^*$.

(2) If $h:\mathbb{R}_+^*\to\mathbb{R}$ is bounded on $[\varepsilon,+\infty[$ for any $\varepsilon>0$, then h is bounded on $\mathbb{R}_+^*$.

(3) If $\sum f_n$ converges normally on $[\varepsilon,+\infty[$ for any $\varepsilon>0$, then $\sum f_n$ converges normally on $\mathbb{R}_+^*$.

16. Let $f_n(x)=\frac{1}{n(1+nx)}$ for $x\in\mathbb{R}_+$ and $n\ge1$.

(1) Show that $\sum f_n$ converges pointwise on $\mathbb{R}_+^*$. Does it converge at 0?
For $x > 0$, we put $S(x) = \sum_{n=1}^{\infty} f_n(x)$ and $S_n(x) = \sum_{k=1}^{n} f_k(x)$.

(2) Show that $\sum f_n$ is not uniformly convergent on $\mathbb{R}_+^*$.

(3) Show that $S(x)$ is nonnegative, decreasing, and $\forall n \geq 1$, $x \in \mathbb{R}_+^*$, $S_n(x) \leq S(x)$.

(4) Show that if $H_n = \sum_{k=1}^{n} \frac{1}{k}$, we have $S_n(\frac{1}{n}) \geq \frac{1}{2} H_n$.

(5) Deduce that $\lim_{n\to\infty} S(\frac{1}{n}) = +\infty$. Deduce that $\lim_{x\to 0} S(x) = +\infty$.

Thus, we see that we can sometimes interchange the limit with the summation, even if we cannot apply the general theorem directly.

17. Study the pointwise, normal and uniform convergence of $\sum f_n$ in the following cases:

(1) $f_n(x) = \frac{xe^{-nx}}{\ln n}$ on $\mathbb{R}_+$.

(2) $f_n(x) = \frac{1}{n+n^2(x-n)^2}$ on $\mathbb{R}$.

(3) $f_n(x) = \frac{(-1)^n}{nx+(-1)^n}$ on $[1, +\infty[$ (*Hint*: $f_n(x) = \big(f_n(x) - \frac{(-1)^n}{nx}\big) + \frac{(-1)^n}{nx}$).

In each case, study the limit as $x \to +\infty$.

18. Determine the domain of definition $\mathcal{D}$ of $S(x) := \sum_{n=1}^{\infty}(-1)^n \ln\big(1+\frac{x}{n}\big)$. Is the function of class $\mathcal{C}^1$ on $\mathcal{D}$?

19. Let (a_n) be nonnegative. Show that $\sum a_n$ converges iff $\sum a_n \cos(nx)$ converges uniformly on $]0, 2\pi[$.

Deduce that if $0 < p \leq 1$, then $\sum \frac{\cos(nx)}{n^p}$ does not converge uniformly on $]0, 2\pi[$. However, it converges pointwise on $]0, 2\pi[$ by Abel's test.

20. Show that for any $a > 0$, $\int_0^1 \frac{1}{1+x^a}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{1+na}$.

21. Show that the function $f(x) = \sum_{n=1}^{\infty} \frac{\sin(2^n x)}{2^n}$ is not differentiable at 0. (*Hint*: near 0, we may find $C > 0$ such that $\sin y \geq Cy$).

22. (Sophomore's dream)**.** Show that $\int_0^1 x^{-x}\,\mathrm{d}x = \sum_{n=1}^{\infty} n^{-n}$ and $\int_0^1 x^x\,\mathrm{d}x = \sum_{n=1}^{\infty}(-1)^{n+1} n^{-n}$.

Chapter 4

Power Series

4.1 Power series and radius of convergence

Definition 4.1. A *power series* is a series of functions of the form $\sum a_n z^n$, where z is a variable in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and (a_n) is a numerical sequence.

Definition 4.2. The *radius of convergence* of the power series $\sum a_n z^n$ is the value

$$R := \sup I_a, \quad \text{where} \quad I_a = \{r \in \mathbb{R}_+ : (a_n r^n) \text{ is bounded }\}.$$

Note that $I_a \neq \emptyset$ because $0 \in I_a$. So R exists; it is either finite or $+\infty$. Moreover, $I_a = [0, R[$ or $[0, R]$. Indeed, if $r \in I_a$, then $0 \leq r \leq R$, hence $I_a \subseteq [0, R]$. Conversely, if $0 \leq r < R$, then r is not an upper bound for I_a, so $\exists\, \tilde{r} \in I_a$ such that $r < \tilde{r}$, so $(a_n \tilde{r}^n)$ is bounded, hence $(a_n r^n)$ is bounded and $r \in I_a$. Hence, $[0, R[\subseteq I_a$.

Example 4.3. 1. The power series $\sum a_n z^n$, $\sum |a_n| z^n$ and $\sum \lambda a_n z^n$ with $\lambda \in \mathbb{C}^*$ have the same radius of convergence.

2. The series $\sum \frac{z^n}{n!}$ has $R = +\infty$, since $\frac{r^n}{n!} \to 0$ for any $r \in \mathbb{R}_+$.
3. The series $\sum n^n z^n$ has $R = 0$, since $n^n r^n$ is unbounded for any $r \neq 0$.
4. The series $\sum \frac{z^n}{n}$ has $R = 1$, since $\frac{r^n}{n}$ is bounded for $0 \leq r \leq 1$ and unbounded for $r > 1$.
5. The series $\sum n^{(-1)^n} z^n$ has $R = 1$, because $|n^{(-1)^n} r^n| \leq n r^n \to 0$ if $0 \leq r < 1$, and for $r = 1$, $n^{(-1)^n}$ is unbounded. Hence, $R = \sup [0, 1[= 1$.

Lemma 4.4. *If R is the radius of convergence of the power series $\sum a_n z^n$, then*

- ◇ *$\sum a_n z^n$ is absolutely convergent for any z such that $|z| < R$,*
- ◇ *$\sum a_n z^n$ diverges for any z such that $|z| > R$.*

Proof. ◇ If $|z| < R$, choose r such that $|z| < r < R$. Since $[0, R[\subseteq I_a$, we have $r \in I_a$, so $(a_n r^n)$ is bounded, say $|a_n r^n| \leq M$. Hence,

$$|a_n z^n| = |a_n r^n| \Big(\frac{|z|}{r}\Big)^n \leq M \Big(\frac{|z|}{r}\Big)^n.$$

But $\sum M \big(\frac{|z|}{r}\big)^n = \frac{M}{1 - |z|/r}$ converges, hence $\sum |a_n z^n|$ converges.

◇ If $|z| > R$, then $|z| \notin I_a$, so $(a_n |z|^n)$ is unbounded. Hence, $a_n z^n \nrightarrow 0$ and $\sum a_n z^n$ diverges. □

Remark 4.5. The radius of convergence R of $\sum a_n z^n$ is thus the *unique* $R \in \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ such that

- ◇ if $|z| < R$, then $\sum |a_n z^n|$ converges,
- ◇ if $|z| > R$, then $\sum |a_n z^n|$ diverges.

(R satisfies this by Lemma 4.4. If there were two $R_1 < R_2$ satisfying this, then for $R_1 < |z| < R_2$, we would have $\sum |a_n z^n|$ convergent and divergent, a contradiction).

Remark 4.6. The behavior of the series is unpredictable when $|z| = R < +\infty$. For example, the series $S_0(z) = \sum z^n$, $S_1(z) = \sum \frac{z^n}{n}$ and $S_2(z) = \sum \frac{z^n}{n^2}$ have all a radius of convergence equal to 1. But $S_0(z)$ diverges at all points with $|z| = 1$, $S_1(z)$ diverges at $z = 1$ and converges at $z = -1$, and $S_2(z)$ converges at all points with $|z| = 1$.

Lemma 4.7. *Let $\sum a_n z^n$ be a power series with radius of convergence R. If*

$$\sqrt[n]{|a_n|} \to L \in \overline{\mathbb{R}}_+ \quad \text{or} \quad \left|\frac{a_{n+1}}{a_n}\right| \to L \in \overline{\mathbb{R}}_+\,,$$

then $R = \frac{1}{L}$. Here, $\frac{1}{L} = 0$ if $L = +\infty$ and $\frac{1}{L} = +\infty$ if $L = 0$.

Proof. Fix $z \in \mathbb{K}$ and put $u_n = a_n z^n$. Then $\sqrt[n]{|u_n|} = \sqrt[n]{|a_n|} \cdot |z| \to L\,|z|$. If $L = 0$, then $\sum |a_n z^n|$ converges $\forall z$ by the root test, so $R = +\infty$. If $L = +\infty$, then $\sum |a_n z^n|$ diverges $\forall z \neq 0$, so $R = 0$. If $0 < L < +\infty$, then $\sum |a_n z^n|$ converges if $|z| < \frac{1}{L}$ and diverges if $|z| > \frac{1}{L}$. Hence, $R = \frac{1}{L}$. The second case is proved similarly using the ratio test. □

Example 4.8. The series $\sum \frac{n}{3^n} z^n$ has a radius of convergence equal to 3, since $\sqrt[n]{\frac{n}{3^n}} \to \frac{1}{3}$.

Lemma 4.9. *Let $\sum a_n z^n$ and $\sum b_n z^n$ be two power series with radii of convergence R_a and R_b, respectively.*

1. *If $a_n = \mathrm{O}(b_n)$, then $R_a \geq R_b$.*
2. *If $a_n = \mathrm{o}(b_n)$, then $R_a \geq R_b$.*
3. *If $a_n \sim b_n$, then $R_a = R_b$.*
4. *If R_{a+b} is the radius of convergence of $\sum (a_n + b_n) z^n$, then $R_{a+b} \geq \min(R_a, R_b)$. If moreover $R_a \neq R_b$, then $R_{a+b} = \min(R_a, R_b)$.*
5. *If $c_n = \sum_{k=0}^n a_k b_{n-k}$, then $\sum c_n z^n = \big(\sum a_n z^n\big)\big(\sum b_n z^n\big)$ for any $|z| < \min(R_a, R_b)$. If R_{ab} is the radius of convergence of $\sum c_n z^n$, then $R_{ab} \geq \min(R_a, R_b)$.*

Proof. 1. Let $I_a = \{r \in \mathbb{R}_+ : (a_n r^n) \text{ is bounded }\}$, $I_b = \{r \in \mathbb{R}_+ : (b_n r^n) \text{ is bounded }\}$ and $r \in I_b$. Then $(b_n r^n)$ is bounded, say $|b_n r^n| \leq M$. But $\exists n_0 \in \mathbb{N}$, $C > 0$ such that $|a_n| \leq C |b_n|$ $\forall n \geq n_0$. Hence, $|a_n r^n| \leq M' := \max\{|a_0|, |a_1 r|, \ldots, |a_{n_0-1} r^{n_0-1}|, CM\}$. Hence, $(a_n r^n)$ is bounded and $r \in I_a$. Thus, $I_b \subseteq I_a$ and $R_b \leq R_a$.

2. If $a_n = \mathrm{o}(b_n)$ then $a_n = \mathrm{O}(b_n)$.
3. If $a_n \sim b_n$, then $a_n = \mathrm{O}(b_n)$ and $b_n = \mathrm{O}(a_n)$.
4. If $|z| < \min(R_a, R_b)$, then $\sum a_n z^n$ and $\sum b_n z^n$ converge absolutely, so $\sum (a_n + b_n) z^n$ converge absolutely, so $|z| \leq R_{a+b}$. Thus, $\min(R_a, R_b) = \sup_{|z| < \min(R_a, R_b)} |z| \leq R_{a+b}$.
   If moreover $R_a \neq R_b$, say $R_a < R_b$, then for z satisfying $R_a < |z| < R_b$, the series $\sum (a_n + b_n) z^n$ diverges since it is a sum of a divergent series and a convergent series. Hence, $|z| \geq R_{a+b}$. Hence, $R_a = \inf_{|z| > R_a} |z| \geq R_{a+b}$ and thus $\min(R_a, R_b) = R_a \geq R_{a+b}$.
5. If $|z| < \min(R_a, R_b)$, then $\sum a_n z^n$ and $\sum b_n z^n$ converge absolutely. Let $u_n = a_n z^n$ and $v_n = b_n z^n$. The Cauchy product $\sum w_n$, where $w_n = \sum_{k=0}^n u_k v_{n-k}$ converges absolutely and equals $\big(\sum u_n\big)\big(\sum v_n\big)$. But $w_n = \sum_{k=0}^n a_k z^k b_{n-k} z^{n-k} = z^n c_n$, so $\sum c_n z^n$ converges absolutely and equals $\big(\sum a_n z^n\big)\big(\sum b_n z^n\big)$. In particular, $|z| \leq R_{ab}$ and thus $\min(R_a, R_b) = \sup_{|z| < \min(R_a, R_b)} |z| \leq R_{ab}$. □

Example 4.10. The power series $\sum_{n\geq 1}(1+\frac{1}{2}+\ldots+\frac{1}{n})t^n$ has a radius of convergence equal to 1. Indeed, $1 \leq (1+\frac{1}{2}+\ldots+\frac{1}{n}) \leq n$, so $\sqrt[n]{1} \leq \sqrt[n]{(1+\frac{1}{2}+\ldots+\frac{1}{n})} \leq \sqrt[n]{n}$ and thus $\sqrt[n]{(1+\frac{1}{2}+\ldots+\frac{1}{n})} \to 1$ by the Sandwich theorem. Actually, this is just the Cauchy product of the series $\sum a_n t^n$ and $\sum t^n$, where $a_0 = 0$ and $a_n = \frac{1}{n}$ for $n \geq 1$. Hence, for $t \in \,]-1, 1[$ we have

$$\sum_{n\geq 1}\Big(1+\frac{1}{2}+\ldots+\frac{1}{n}\Big)t^n = \Big(\sum_{n\geq 0} t^n\Big)\Big(\sum_{n\geq 1}\frac{1}{n}t^n\Big) = \frac{\ln(1-t)}{t-1}\,,$$

where we used the Taylor series of $\ln(1-t)$ in the last equality.

Theorem 4.11. *The power series $\sum_{n\geq 0} a_n z^n$ and the series of derivatives $\sum_{n\geq 1} n a_n z^{n-1}$ have the same radius of convergence.*

Proof. Let R and R' be the radii of convergence of $\sum_{n\geq 0} a_n z^n$ and $\sum_{n\geq 1} n a_n z^{n-1}$ and put $I = \{r \in \mathbb{R}_+ : (a_n r^n) \text{ is bounded }\}$ and $I' = \{r \in \mathbb{R}_+ : (n a_n r^{n-1}) \text{ is bounded }\}$.

If $r \in I'$, then for $n \geq 1$, $|a_n r^n| \leq n|a_n r^n| = r|n a_n r^{n-1}| \leq rM$ for some M. Hence, $|a_n r^n| \leq M'\ \forall n$, where $M' = \max(|a_0|, rM)$. Hence, $r \in I$ and $I' \subseteq I$. Thus, $R' \leq R$.

Suppose $R' < R$ and let r_1, r_2 such that $R' < r_1 < r_2 < R$. Then $|n a_n r_1^{n-1}| = (\frac{n r_1^{n-1}}{r_2^n})|a_n r_2^n| \leq (\frac{n r_1^{n-1}}{r_2^n})M'$ for some M', since $r_2 < R$. But $\frac{n r_1^{n-1}}{r_2^n} = \frac{n}{r_1}(\frac{r_1}{r_2})^n \to 0$ since $r_1 < r_2$. Hence, $\frac{n r_1^{n-1}}{r_2^n}$ is bounded by an M''. Thus, $|n a_n r_1^{n-1}| \leq M'M''$, which is absurd since $r_1 > R'$. Thus, $R' = R$. $\square$

4.2 Properties of the sum function

We shall suppose in this section that $\sum a_n z^n$ is a power series with radius of convergence $R \neq 0$ and sum function $S(z)$.

Definition 4.12. We define the *disk of convergence* of $\sum a_n z^n$ as the open ball $D = B_R(0) = \{z \in \mathbb{K} : |z| < R\}$.

If the variable z is real, this disk is just the interval $]-R, R[$.

Theorem 4.13. *A power series converges normally, hence uniformly, on any closed ball with center* 0 *contained in the open disk of convergence.*

Proof. Let $\overline{B}_r(0) \subseteq D$. Then $r < R$. But $\sup_{z\in\overline{B}_r(0)} |a_n z^n| = |a_n| r^n$ and $\sum |a_n r^n|$ converges because $r < R$. Hence, $\sum a_n r^n$ converges normally on $\overline{B}_r(0)$. $\square$

Remark 4.14. 1. In general, the series does not converge uniformly on D. For example, $\sum z^n$ has radius of convergence 1 and does not converge uniformly on its disk, since z^n does not converge uniformly to 0 on this disk.

2. Actually, if the series converges uniformly on D, then it converges uniformly on $\overline{D} = \{z \in \mathbb{K} : |z| \leq R\}$. Indeed, using the Cauchy criterion, uniform convergence on D means that

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N} : n \geq n_0, p \geq 0 \implies \forall z \in D : |a_n z^n + \ldots + a_{n+p} z^{n+p}| \leq \varepsilon\,.$$

Fix n, p and z_0 on the boundary of D, i.e. $|z_0| = R$. By continuity of the map $z \mapsto |a_n z^n + \ldots + a_{n+p} z^{n+p}|$, by letting $z \to z_0$, we obtain the same inequality at z_0. Hence, uniform convergence holds on all $\overline{D}$.

3. If there exists z_0 with $|z_0| = R$ at which the series is absolutely convergent, then we have normal convergence on $\overline{D}$. Indeed, $\sup_{z\in\overline{D}} |a_n z^n| = |a_n||z_0|^n$ and $\sum |a_n z_0^n|$ converges by hypothesis.

Theorem 4.15. *The sum function $S(z)$ is continuous on its disk of convergence.*

Proof. Let $z_0 \in D$ and $r = \frac{R+|z_0|}{2}$. Then $|z_0| < r < R$, so $z_0 \in \overline{B}_r(0) \subseteq D$. But the series converges uniformly on $\overline{B}_r(0)$. Since each function $a_n z^n$ is continuous, it follows that S is continuous on $\overline{B}_r(0)$, in particular at z_0. Since this holds for arbitrary $z_0 \in D$, the function S is continuous on D. □

Corollary 4.16. *For all $p \geq 0$, the sum function $S(z)$ admits an expansion of order p near 0, which takes the form $S(z) = a_0 + a_1 z + \ldots + a_p z^p + \mathrm{O}(z^{p+1})$.*

Proof. We have $S(z) = a_0 + a_1 z + \ldots + a_p z^p + \sum_{n\geq p+1} a_n z^n$. But $\sum_{n\geq p+1} a_n z^n = z^{p+1} \sum_{n\geq p+1} a_n z^{n-p-1}$ and $\sum_{n\geq p+1} a_n z^{n-p-1} = a_{p+1} + a_{p+2} z + \ldots$ is a power series, so it is continuous on its disk of convergence. Hence, there exists a neighborhood of 0 in which $|\sum_{n\geq p+1} a_n z^n| \leq C|z^{p+1}|$, where $C = |a_{p+1}| + 1$. Hence, $\sum_{n\geq p+1} a_n z^n = \mathrm{O}(z^{p+1})$. □

For differentiation and integration, we suppose from now on that we have a power series $S(x) = \sum a_n x^n$ of a **real variable**, i.e. $\mathbb{K} = \mathbb{R}$, with radius of convergence $R > 0$.

Theorem 4.17. *If $[a, b] \subset]-R, R[$, then*

$$\int_a^b S(t)\,\mathrm{d}t = \sum_{n\geq 0} a_n \int_a^b t^n\,\mathrm{d}t, \ .$$

In particular, for any $0 < x < R$,

$$\int_0^x S(t)\,\mathrm{d}t = \sum_{n\geq 0} \frac{a_n}{n+1} x^{n+1} \, .$$

Proof. $\sum a_n x^n$ converges uniformly on $[a, b]$ and each $a_n x^n$ is continuous, so we may integrate term by term by a theorem in Chapter 3. □

Example 4.18. We may thus recover the Taylor series of ln and arctan :

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,\mathrm{d}t = \int_0^x \sum_{n\geq 0} (-1)^n t^n\,\mathrm{d}t = \sum_{n\geq 0} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n \, ,$$
$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\,\mathrm{d}t = \int_0^x \sum_{n\geq 0} (-1)^n t^{2n}\,\mathrm{d}t = \sum_{n\geq 0} \frac{(-1)^n}{2n+1} x^{2n+1} \, ,$$

both expansions are valid on $]-1, 1[$.

Theorem 4.19. *The sum function $S :]-R, R[\to \mathbb{C}$ is of class C^∞, and its derivatives are given by*

$$S^{(k)}(x) = \sum_{n\geq k} n(n-1)\ldots(n-k+1) a_n x^{n-k} \, .$$

In particular,

$$S^{(k)}(0) = k! a_k \, .$$

Proof. Let $f_n(x) = a_n x^n$. The series $\sum f_n$ converges pointwise on $]-R, R[$, each f_n is of class C^∞ and $\sum f_n^{(k)}$, which also has radius of convergence R by Theorem 4.11, converges uniformly on any $[-r, r] \subseteq]-R, R[$. Let $x_0 \in]-R, R[$. Then $x_0 \in [-r, r]$ for $r = \frac{R+|x_0|}{2}$. Hence, S is of class C^∞ on $[-r, r]$ by the theorem. Since any point in $]-R, R[$ has a neighborhood $[-r, r]$ on which S is of class C^∞, it follows that S is of class C^∞ on $]-R, R[$ with the given formula for $S^{(k)}$.

Finally, we have $S^{(k)}(x) = k!a_k + \frac{(k+1)!}{1!}a_{k+1}x + \frac{(k+2)!}{2!}a_{k+2}x^2 + \ldots$, so $S^{(k)}(0) = k!a_k$. $\square$

4.3 Problems on the boundary

Theorem 4.20 (Abel). *If $\sum_{n\geq 0} a_n$ converges, then the function S defined by*

$$S(x) = \sum_{n\geq 0} a_n x^n, \qquad -1 < x < 1\,,$$

satisfies

$$\lim_{x\to 1^-} S(x) = \sum_{n\geq 0} a_n\,.$$

This result is immediate if $a_n \geq 0$ for all n, since in this case $\sum a_n x^n$ converges normally on $[-1, 1]$. In the general case, the proof goes as follows:

Proof. Put $S_n = \sum_{k=0}^n a_k$ if $n \geq 0$ and $S_{-1} = 0$. Then

$$\begin{aligned}\sum_{n=0}^m a_n x^n &= \sum_{n=0}^m (S_n - S_{n-1})x^n = \sum_{n=0}^m S_n x^n - \sum_{n=1}^m S_{n-1}x^n \\ &= \sum_{n=0}^m S_n x^n - \sum_{n=0}^{m-1} S_n x^{n+1} = S_m x^m + (1-x)\sum_{n=0}^{m-1} S_n x^n\,.\end{aligned}$$

Taking $m \to +\infty$ we get

$$S(x) = (1-x)\sum_{n\geq 0} S_n x^n \qquad (\star)$$

because S_m converges and $x^m \to 0$ for $|x| < 1$. Let $\varepsilon > 0$ and put $S = \sum_{n\geq 0} a_n$. There exists n_0 such that $|S - S_n| \leq \varepsilon/2$ for all $n \geq n_0$. Hence, noting that $(1-x)\sum_{n\geq 0} x^n = 1$ for $|x| < 1$, we obtain by $(\star)$ that for any $0 \leq x < 1$,

$$\begin{aligned}|S(x) - S| &= \left|(1-x)\sum_{n\geq 0}(S_n - S)x^n\right| \leq (1-x)\sum_{n=0}^{n_0-1}|S_n - S|x^n + (1-x)\sum_{n\geq n_0}\frac{\varepsilon}{2}x^n \\ &\leq (1-x)\sum_{n=0}^{n_0-1}|S_n - S|x^n + \frac{\varepsilon}{2}(1-x)\sum_{n\geq 0}x^n = (1-x)\sum_{n=0}^{n_0-1}|S_n - S|x^n + \frac{\varepsilon}{2}\,.\end{aligned}$$

As $x \to 1^-$, this tends to $\varepsilon/2$, so $\exists\delta > 0 : x \in]1-\delta, 1[\implies |S(x) - S| \leq \varepsilon$. $\square$

Corollary 4.21. *If $S(x) = \sum a_n x^n$ has radius of convergence R, and if $\sum a_n R^n$ converges, then S is continuous on $]-R, R]$.*

Proof. Let $b_n = a_n R^n$ and $f(t) = \sum b_n t^n$. By Theorem 4.20, $\lim_{t\to 1^-} f(t) = \sum b_n$. Let $\varepsilon > 0$. There is thus $\delta > 0$ such that

$$t \in]1-\delta, 1[\implies \left|\sum b_n t^n - \sum b_n\right| \le \varepsilon \implies \left|\sum a_n (Rt)^n - \sum a_n R^n\right| \le \varepsilon\,.$$

Put $x = Rt$. Then

$$x \in]R - R\delta, R[\implies t \in]1-\delta, 1[\implies \left|\sum a_n x^n - \sum a_n R^n\right| \le \varepsilon\,.$$

We thus showed that $\lim_{x\to R^-} S(x) = S(R)$, so S is continuous at R. Since we already know it is continuous on $]-R, R[$, the corollary is proved. $\square$

4.4 Usual functions

Definition 4.22. The power series $\sum_{n\ge 0} \frac{z^n}{n!}$ has radius of convergence $R = +\infty$. Its sum is called the *complex exponential function*, and denoted $\exp : z \mapsto e^z$.

Theorem 4.23. *Starting from the series $e^z = \sum \frac{z^n}{n!}$, we may recover the following identities, without using any geometric arguments :*

(1) $\forall z_1, z_2 \in \mathbb{C}$, $e^{z_1+z_2} = e^{z_1} e^{z_2}$.

(2) $e^z \neq 0$ *and* $\frac{1}{e^z} = e^{-z}$.

(3) $\overline{e^z} = e^{\overline{z}}$ *and* $|e^z| = e^{\mathrm{Re}(z)}$.

(4) $\frac{d}{dz} e^z = e^z$.

(5) *The restriction of* exp *to* $\mathbb{R}$ *is positive, increasing and satisfies* $e^x \to \infty$ *as* $x \to \infty$ *and* $e^x \to 0$ *as* $x \to -\infty$.

(6) *There exists a number* π *such that* $e^{\pi i/2} = i$ *and* $e^z = 1$ *iff* $\frac{z}{2\pi i} \in \mathbb{Z}$.

(7) exp *is periodic, with period* $2\pi i$.

(8) *Let* S^1 *be the unit circle. The map* $f : \mathbb{R} \to S^1$ *defined by* $f(t) = e^{it}$ *is surjective.*

(9) *The map* $\exp : \mathbb{C} \to \mathbb{C}^*$ *defined by* $\exp(z) = e^z$ *is surjective.*

(10) *There exists a unique continuous function* L *defined on* $\mathbb{C}\setminus\mathbb{R}_-$ *such that* $L(1) = 0$ *and* $\forall z \in \mathbb{C}\setminus\mathbb{R}_-$, $e^{L(z)} = z$. *This function is called the* principal value of the logarithm *and coincides with the function* ln *on* $\mathbb{R}_+^*$.

Proof. Admitted, see the prologue of [15]. For (10), see [2]. $\square$

Definition 4.24. We call *complex cosine* and *sine*, denoted respectively by cos and sin, the maps from $\mathbb{C}$ to $\mathbb{C}$ defined by

$$\forall z \in \mathbb{C},\ \cos z = \frac{e^{iz} + e^{-iz}}{2} = \sum_{n\ge 0} (-1)^n \frac{z^{2n}}{(2n)!}\,,$$

$$\forall z \in \mathbb{C},\ \sin z = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n\ge 0} (-1)^n \frac{z^{2n+1}}{(2n+1)!}\,.$$

We call *complex hyperbolic cosine* and *sine*, denoted respectively by cosh and sinh (or ch and sh), the maps from $\mathbb{C}$ to $\mathbb{C}$ defined by

$$\forall z \in \mathbb{C},\ \cosh z = \frac{e^{z} + e^{-z}}{2} = \sum_{n\ge 0} \frac{z^{2n}}{(2n)!}\,,$$

$$\forall z \in \mathbb{C},\ \sinh z = \frac{e^{z} - e^{-z}}{2} = \sum_{n\ge 0} \frac{z^{2n+1}}{(2n+1)!}\,.$$

We call *complex tangent* and *cotangent*, denoted respectively by tan and cot, the maps defined by

$$\forall z \notin \frac{\pi}{2} + \pi\mathbb{Z},\ \tan z = \frac{\sin z}{\cos z}, \qquad \forall z \notin \pi\mathbb{Z},\ \cot z = \frac{\cos z}{\sin z} = \tan\left(\frac{\pi}{2} - z\right).$$

Lemma 4.25. *For all $z, a, b \in \mathbb{C}$ we have*

$$\begin{aligned}
&\cos(iz) = \cosh z && \cosh(iz) = \cos z \\
&\sin(iz) = i\sinh z && \sinh(iz) = i\sin z \\
&e^{iz} = \cos z + i\sin z && e^{z} = \cosh z + \sinh z \\
&e^{-iz} = \cos z - i\sin z && e^{-z} = \cosh z - \sinh z \\
&\cos^2 z + \sin^2 z = 1 && \cosh^2 z - \sinh^2 z = 1 \\
&\cos(a+b) = \cos a\cos b - \sin a\sin b && \cosh(a+b) = \cosh a\cosh b + \sinh a\sinh b \\
&\sin(a+b) = \sin a\cos b + \cos a\sin b && \sinh(a+b) = \sinh a\cosh b + \cosh a\sinh b
\end{aligned}$$

Proof. Exercise. □

Lemma 4.26. (1) cos *is even, periodic, of period 2π. Moreover, $\forall z \in \mathbb{C}$, $\cos(z+\pi) = -\cos z$ and $\cos(\frac{\pi}{2} - z) = \sin z$.*

(2) sin *is odd, periodic, of period 2π. Moreover, $\forall z \in \mathbb{C}$, $\sin(z+\pi) = -\sin z$ and $\sin(\frac{\pi}{2} - z) = \cos z$.*

(3) cosh *is even, periodic, of period $2\pi i$. Moreover, $\forall z \in \mathbb{C}$, $\cosh(z+\pi i) = -\cosh z$ and $\cosh(z + i\frac{\pi}{2}) = i\sinh z$.*

(4) sinh *is odd, periodic, of period $2\pi i$. Moreover, $\forall z \in \mathbb{C}$, $\sinh(z+\pi i) = -\sinh z$ and $\sinh(z + i\frac{\pi}{2}) = i\cosh z$.*

Proof. Exercise. Use Theorem 4.23. □

4.5 Analytic functions

Definition 4.27. Let $X \subseteq \mathbb{K}$. We say that $f : X \to \mathbb{C}$ is *analytic near* 0 if there exists a power series $\sum a_n z^n$ with radius of convergence $R > 0$ and an open ball $B_r(0) \subseteq X$ with radius $0 < r \leq R$ such that $f(z) = \sum a_n z^n$ on $B_r(0)$.

We say that f is analytic near $z_0 \in \mathbb{C}$ if the function $g(z) = f(z + z_0)$ is analytic near 0.

We say that f is analytic on X if f is analytic near any point in X.

For example, the function $f(z) = \frac{1}{1-z}$, defined on $\mathbb{C} \setminus \{1\}$, is analytic near 0 because $f(z) = \sum z^n$ on $B_1(0)$. More generally, we have

Lemma 4.28. *The function $f(z) = \frac{1}{(1-z)^{k+1}}$ is analytic near 0 for any $k \in \mathbb{N}$. Moreover,*

$$\frac{1}{(1-z)^{k+1}} = \sum_{n \geq 0} \binom{n+k}{k} z^n \qquad \text{on } B_1(0).$$

This lemma is easy to prove if the variable is real, by differentiating $\frac{1}{1-x} = \sum x^n$ and using Theorem 4.19. For complex variables, we argue by induction.

Proof. The coefficient $\binom{n+k}{k}$ is a polynomial of degree k in n, so $\binom{n+k}{k}r^n \to 0$ if $0 \leq r < 1$, and $\binom{n+k}{k}r^n \to \infty$ if $r \geq 1$. Hence, the radius of convergence of the above series is 1; it is thus well defined on $B_1(0)$.

We already know the result for $k = 0$; this is just a geometric series. Suppose the result is true for k. Write $1 - z = \sum a_n z^n$ with $a_0 = 1$, $a_1 = -1$ and $a_k = 0$ for $k \geq 2$, and put $b_n = \binom{n+k+1}{k+1}$. Then the Cauchy product of $\sum a_n z^n$ with $\sum b_n z^n$ has general term $c_n = a_0 b_n + a_1 b_{n-1} = \binom{n+k+1}{k+1} - \binom{n+k}{k+1} = \frac{(n+k+1)!}{(k+1)!n!} - \frac{(n+k)!}{(k+1)!(n-1)!} = \frac{(n+k+1)!-n(n+k)!}{(k+1)!n!} = \frac{(n+k+1)(n+k)!-n(n+k)!}{(k+1)!n!} = \frac{(n+k)!}{k!n!} = \binom{n+k}{k}$. Hence, $(1-z)\sum\binom{n+k+1}{k+1}z^{n+1} = \sum\binom{n+k}{k}z^n = \frac{1}{(1-z)^{k+1}}$, which proves the result for $k+1$. □

For the rest of this section, we suppose that f is a function of a **real variable**, i.e. $X \subseteq \mathbb{R}$.

Lemma 4.29. *If f is analytic near x_0, then f is of class C^∞ on an interval $]x_0 - r, x_0 + r[$. Moreover, the expansion of f near x_0 is given by $\sum \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$.*

We call this the *Taylor series* of f at x_0.

Proof. Let $g(x) = f(x + x_0)$. By hypothesis, there exists a power series $\sum a_n x^n$ with radius $R > 0$ and an interval $]-r, r[$ with $0 < r \leq R$ such that $g(x) = \sum a_n x^n$ on $]-r, r[$. But $x \in]x_0 - r, x_0 + r[$ iff $x - x_0 \in]-r, r[$. Hence, $f(x) = g(x - x_0) = \sum a_n(x - x_0)^n$ on $]x_0 - r, x_0 + r[$. By Theorem 4.19, g is of class C^∞ on $]-r, r[$, so f is of class C^∞ on $]x_0 - r, x_0 + r[$. Moreover, $a_n = \frac{g^{(n)}(0)}{n!} = \frac{f^{(n)}(x_0)}{n!}$. □

This lemma ensures the uniqueness of the series expansion : if $f(x) = \sum a_n(x-x_0)^n = \sum b_n(x - x_0)^n$, then $a_n = b_n = \frac{f^{(n)}(x_0)}{n!}$. Here is an application of this :

Lemma 4.30. *Let f be analytic near 0 and $\sum a_n x^n$ its power series.*
- ◇ *If f is even, then $\forall p \in \mathbb{N}$, $a_{2p+1} = 0$.*
- ◇ *If f is odd, then $\forall p \in \mathbb{N}$, $a_{2p} = 0$.*

Proof. Suppose that $f(x) = \sum a_n x^n$ on $]-r, r[$. Then $g(x) = f(-x)$ satisfies $g(x) = \sum a_n(-x)^n = \sum(-1)^n a_n x^n$. If f is even, then $g(x) = f(x)$, so by the uniqueness of the expansion, $a_n = (-1)^n a_n$ and thus $a_{2p+1} = 0$ $\forall p$. Similarly, if f is odd, $g(x) = -f(x)$, so $a_n = (-1)^{n+1}a_n$ and thus $a_{2p} = 0$ $\forall p$. □

Note that the converse of Lemma 4.29 is incorrect : we shall see in Exercise 8 some functions of class C^∞ which are not analytic near 0. The following results give us criteria that ensure the converse is true.

Lemma 4.31. *Let $f :]-a, a[\to \mathbb{C}$ be of class C^∞. Let $S_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!}x^k$ and $R_n(x) = f(x) - S_n(x)$. Then f is analytic near 0 iff there exists an interval $]-r, r[$ on which R_n converges pointwise to the zero function.*

Proof. If f is analytic near 0, then there exists an interval $]-r, r[$ on which $f(x) = \sum_{k\geq 0}\frac{f^{(k)}(0)}{k!}x^k$, so for any $x \in]-r, r[$, $S_n(x) \to f(x)$, hence $R_n(x) \to f(x) - f(x) = 0$.

Conversely, suppose R_n converges pointwise to the zero function on $]-r, r[$ and let $x \in]-r, r[$. Then $f(x) = \lim_{n\to\infty}(f(x) - R_n(x)) = \lim_{n\to\infty} S_n(x) = \sum_{k=0}^\infty \frac{f^{(k)}(0)}{k!}x^k$, so f is analytic near 0. □

Corollary 4.32. *If $f :]-a, a[\to \mathbb{C}$ is of class C^∞ and if there exist $C \geq 0$ and $\omega > 0$ such that for all $x \in]-a, a[$ and $n \in \mathbb{N}$ we have $|f^{(n)}(t)| \leq Cn!\,\omega^n$, then f is analytic near 0.*

Proof. By the Taylor-Lagrange formula, we have $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \sup_{\theta\in[0,x]} |f^{(n+1)}(\theta)| \leq C|x|^{n+1}\omega^{n+1}$. Hence, R_n converges pointwise to 0 on $]-r, r[$, where $r = \min(a, \omega^{-1})$. □

Remark 4.33. The hypothesis of the corollary is satisfied in particular if the successive derivatives of f are uniformly bounded, i.e. bounded independently of x.

Remark 4.34. The converse of the corollary is also true; see Exercise 11.

4.6 Power series and differential equations

To show that a function f is analytic near a point, and to determine the expansion, it is often useful to use differential equations. Here is the general argument :

1. Find a differential equation that f satisfies, and indicate initial conditions.
2. Suppose $y(x) = \sum a_n x^n$ is a solution of the differential equation and deduce the a_n and the radius R. If $R \neq 0$, we will have $f(x) = y(x) = \sum a_n x^n$ on $]-R, R[$ by uniqueness of the solution of the differential equation.

Before we give an example to clarify this method, let us state a general result on differential equations.

Theorem 4.35 (Cauchy-Lipschitz Theorem)**.** *Let $a_1, \ldots, a_n, b$ be continuous functions on an interval I with values in $\mathbb{C}$. Let $x_0 \in I$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Then there exists a unique solution f, defined on I, to the differential equation*

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = b(x)$$

which satisfies

$$f(x_0) = \alpha_1, \quad f'(x_0) = \alpha_2, \quad \ldots, \quad f^{(n-1)}(x_0) = \alpha_n\,.$$

Proof. Admitted. See [1] for a proof. □

Actually, we shall only need the cases $n = 1$ and $n = 2$, and we only use the uniqueness of the solution. Let us now give an example to illustrate the method described above.

Lemma 4.36. *The function $f(x) = (1+x)^\alpha$ is analytic near 0 for any $\alpha \in \mathbb{R}$. Moreover,*

$$(1+x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n \qquad \text{on }]-1, 1[\ ,$$

where $\binom{\alpha}{0} = 1$ and $\binom{\alpha}{n} = \alpha(\alpha-1)\ldots(\alpha-n+1)/n!$ for $n \geq 1$.

Proof. We note that f satisfies on $]-1, 1[$ the differential equation

$$(E) \qquad \begin{cases} (1+x)y' = \alpha y, \\ y(0) = 1. \end{cases}$$

Suppose (E) has a solution $y(x)=\sum_{n\geq 0} a_n x^n$. Then $y'(x)=\sum_{n\geq 1} n a_n x^{n-1}=\sum_{n\geq 0}(n+1)a_{n+1}x^n$. Hence,

$$0=(1+x)y'-\alpha y=\sum_{n\geq 0}(n+1)a_{n+1}x^n+\sum_{n\geq 0}(n+1)a_{n+1}x^{n+1}-\alpha\sum_{n\geq 0}a_n x^n$$

But $\sum_{n\geq 0}(n+1)a_{n+1}x^{n+1}=\sum_{n\geq 1} n a_n x^n=\sum_{n\geq 0} n a_n x^n$. Hence,

$$\sum_{n\geq 0}\big((n+1)a_{n+1}+(n-\alpha)a_n\big)x^n=0\,.$$

By the uniqueness of the expansion, we thus have $\big((n+1)a_{n+1}+(n-\alpha)a_n\big)=0$ for all n, in other words,

$$a_{n+1}=\frac{\alpha-n}{n+1}a_n \qquad n\geq 0\,.$$

By a simple calculation we deduce that $a_n=a_0\binom{\alpha}{n}$. But $a_0=y(0)=1$, so $a_n=\binom{\alpha}{n}$. Hence, $y(x)=\sum\binom{\alpha}{n}x^n$. If $\alpha\in\mathbb{N}$, then $\binom{\alpha}{n}=0$ for all $n\geq\alpha+1$, so the radius of convergence of this series is $R=+\infty$. Otherwise, $\frac{|a_{n+1}|}{|a_n|}=\frac{|\alpha-n|}{n+1}\to 1$, so $R=1$. In any case, $y(x)=\sum\binom{\alpha}{n}x^n$ is a solution of (E) which is well defined on $]-1,1[$. Since f is also a solution of (E), we have $f(x)=y(x)=\sum\binom{\alpha}{n}x^n$ on $]-1,1[$ by Theorem 4.35. □

To conclude, note that conversely, given a differential equation, we can try to find its solution by supposing that it takes the form of a power series $\sum a_n x^n$, then calculate the a_n and the radius R.

4.7 Exercises

1. Let P and Q be nonzero polynomials and let $\alpha\in\mathbb{R}$. Calculate the radius of convergence of the power series $\sum_{n\geq 2} a_n z^n$, for

- $a_n=\alpha^n$
- $a_n=100\cos(n)$
- $a_n=e^{-2n}$
- $a_n=\dfrac{\cosh n}{n}$
- $a_n=\left(1+\dfrac{1}{2n}\right)^n$
- $a_n=\left(\dfrac{2n-1}{n+1}\right)^{2n}$
- $a_n=\dfrac{1}{n^\alpha}$
- $a_n=\dfrac{(-1)^n}{n\pi^n}$
- $a_n=\sin\left(e^{-(n!)}\right)$
- $a_n=\dfrac{1}{n+e^{-n}}$
- $a_n=e^{-\sqrt{n}}$
- $a_n=\ln n$
- $a_n=e^{n^2}$
- $a_n=\dfrac{e^{\sqrt{n}}}{n^3}$
- $a_n=\dfrac{n}{\ln n}$
- $a_n=n!$
- $a_n=\dfrac{n!}{(2n)!}$
- $a_n=\dfrac{n!}{n^n}$
- $a_n=\dfrac{1}{3^{4n-1}}$
- $a_n=\dfrac{-n^3-n+2}{n^2+5}$
- $a_n=\dfrac{\ln n+e^{-n}}{n^3+\ln(1+\sin(3^{-n}))}$
- $a_n=\dfrac{\ln n}{2^{3n-2}}$
- $a_n=\dfrac{P(n)}{Q(n)}$
- $a_{2k}=2^k$ et $a_{2k+1}=0$

2. Let $a\in\mathbb{R}$ and $k\in\mathbb{N}^*$. Calculate the radius of convergence of the following power series :

- $\displaystyle\sum_{n\geq 0}25^n z^n$
- $\displaystyle\sum_{n\geq 0}25^n z^{2n}$
- $\displaystyle\sum_{n\geq 0}\frac{z^{3n}}{27^n}$
- $\displaystyle\sum_{n\geq 0}a^n z^{kn}$
- $\displaystyle\sum_{n\geq 0}a_n z^n$,

where

1. a_n is the sum of the squares of the divisors of n.
2. $a_n = 1$ if n is prime, $a_n = 0$ otherwise.
3. a_n is the number of couples $(x, y) \in \mathbb{Z}^2$ such that $x^2 + y^2 \leq n^2$.

3. Let $\sum a_n z^n$ be a power series with radius of convergence R and let $k \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}$. Calculate the radius of convergence of the power series with general term:

- $a_n z^{kn}$
- $\dfrac{n}{n^2+n+2} a_n z^n$
- $n^\alpha a_n z^n$
- $a_n^k z^n$
- $a_n e^{\sqrt{n}} z^n$
- $a_n z^{n^2}$

(*Hint*: For the last series, first consider the case where R is finite, nonzero, then study the cases $R = +\infty$ and $R = 0$).

4. Let a_n be a real sequence, $a_n \to a$. Calculate the radius of convergence of $f(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n$, then calculate $\lim_{x\to\infty} e^{-x} f(x)$.

5. Determine the radius of convergence and the sum of the following real power series:

- $\displaystyle\sum_{n\geq 0} n^2 x^n$
- $\displaystyle\sum_{n\geq 1} \frac{(n+2)^2}{(n+2)!} x^n$
- $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{2n-1} x^{2n}$
- $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{2n} x^{2n}$
- $\displaystyle\sum_{n\geq 1} \frac{x^n}{1+\ldots+n}$
- $\displaystyle\sum_{n\geq 0} \frac{\sin(n\alpha)}{n!} x^n$
- $\displaystyle\sum_{n\geq 0} (-1)^{n+1} n x^{2n+1}$
- $\displaystyle\sum_{n\geq 0} (\cosh n) x^n$
- $\displaystyle\sum_{n\geq 1} \frac{x^n}{n(n+2)}$
- $\displaystyle\sum_{n\geq 0} \Big(\frac{1}{n!} \sum_{k=0}^{n} k \cdot k!\Big) x^n$
- $\displaystyle\sum_{n\geq 0} \Big(\sum_{k=0}^{n} \frac{1}{k!}\Big) x^n$
- $\displaystyle\sum_{n\geq 1} n^{(-1)^n} x^n$

6. Show that if $\sum a_n$, $\sum b_n$ and $\sum c_n$ converge to A, B and C, and if $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $C = AB$. (*Hint*: use Abel's continuity theorem).

Thus, we do not need $\sum a_n$ and $\sum b_n$ to converge absolutely if we already know that their Cauchy product converges.

7. Give a power series expansion near 0 for the following functions:

- $\dfrac{1}{(x-1)(x-2)}$
- $\ln(x+a), a > 0$
- $\dfrac{1+x}{(1-x)^3}$
- $\dfrac{1}{(1+x^2)(1-x)}$
- $\displaystyle\int_0^x \frac{\arctan(t^2)}{t}\,\mathrm{d}t$
- $\arctan\Big(\dfrac{1-x^2}{1+x^2}\Big)$
- $e^x \cos x$
- $\ln(x^2 - 5x + 6)$
- $\cos^3 x$
- $\dfrac{1}{\sqrt{1-x}}$
- $\arcsin x$
- $\ln(1 + x + x^2)$

8. In this exercise we show that a function of class C^∞ is not necessarily equal to its Taylor series.

1. Give an example of a function f of class C^∞ on a domain $\mathcal{D}$ such that the equality $f(x) = \sum \frac{f^{(n)}(0)}{n!} x^n$ is only true on an interval which is strictly smaller than $\mathcal{D}$.
2. Show that if $f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$, then $\sum \frac{f^{(n)}(0)}{n!} x^n$ converges, but $f(x) \neq \sum \frac{f^{(n)}(0)}{n!} x^n$ for any $x \neq 0$.

3. Show that if $f(x) = \sum e^{-n} \sin n^2 x$, then f is of class C^∞ but $\sum \frac{f^{(n)}(0)}{n!} x^n$ diverges for all $x \neq 0$.

9. (Bernstein Theorem)**.** Show that if f is of class C^∞ on $]-a, a[$, and if f and all its derivatives are nonnegative on $]-a, a[$, then f is analytic near 0.

Deduce a new proof of the fact that for $-1 < x < 1$ we have $(1+x)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} x^n$ for any $\alpha \in \mathbb{R}$, where $\binom{\alpha}{0} = 1$ and $\binom{\alpha}{n} = \alpha(\alpha-1)\ldots(\alpha-n+1)/n!$

(*Hint*: Consider $g(y) = (1-y)^{-\beta}$ with $\beta > 0$, then deduce the general case).

10. Calculate $\sum_{n=1}^\infty \frac{1}{2^n n}$, $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}$ and $\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}$. (*Hint*: Use the standard Taylor series. The 2nd and 3rd series will need some justifications).

11. (Converse of Corollary 4.32)**.** Show that if f is analytic near 0, then there exist strictly positive reals C, ω and r such that for all $n \in \mathbb{N}$ and $x \in]-r, r[$, $|f^{(n)}(x)| \leq C n! \, \omega^n$.

12. Expand $e^{x^2 - 2x}$ as a power series near 1.

13. Let $f(x) = \sum a_n x^n$ be a power series defined for $|x| < R$. Show that f is analytic on $]-R, R[$, i.e. f has a power series expansion near any $a \in]-R, R[$.

14. Let $\alpha \in \mathbb{R}$ and $f(x) = \cos(\alpha \arcsin x)$.

1) Show that f is the unique solution on $]-1, 1[$ of the differential equation

$$(E) \qquad \begin{cases} (1-x^2)y'' - xy' + \alpha^2 y = 0, \\ y(0) = 1, \; y'(0) = 0 \end{cases}$$

2) Show that if $y(x) = \sum a_n x^n$ is a solution of (E) on $]-R, R[$, $0 < R \leq 1$, then

$$\forall n \in \mathbb{N} : a_{n+2} = \frac{n^2 - \alpha^2}{(n+1)(n+2)} \, a_n \, .$$

Deduce the value of a_n.

3) Calculate the radius of convergence of $\sum a_n x^n$. Deduce a power series expansion for f.

15. Show that the differential equation

$$4xy'' + 2y' + y = 0$$

has, up to a multiplicative constant, a unique solution which is analytic near 0. Give this solution explicitly.

16. Show that the differential equation

$$(1+x^2)y'' - 2y = 0$$

with the conditions $y(0) = 0$ and $y'(0) = 1$ has a unique solution f which is analytic near 0. Calculate the coefficients and the radius of convergence of the power series, then make it explicit as much as possible.

17. (Tauberian theorem)**.** Let $f(x) = \sum a_n x^n$ for $-1 < x < 1$ and suppose $n a_n \to 0$. Show that if $f(x) \to \ell$ as $x \to 1^-$, then $\sum a_n$ converges, and is equal to ℓ.

Show that this partial converse of Abel's theorem is false without the condition $n a_n \to 0$ (*consider* $a_n = (-1)^n$.)

18. Calculate, for $x \in \mathbb{R}$ and in case of convergence, the sum $\sum_{n=1}^\infty \frac{x^n}{n(n+1)}$.

Chapter 5

Improper Integrals

5.1 Generalities

Reminder 5.1. If f is a bounded function on $[a, b]$ and if $X = (x_k)_{0 \le k \le n}$ is a subdivision of $[a, b]$, we put

$$c_k(f) = \inf_{x \in [x_k, x_{k+1}]} f(x), \qquad C_k(f) = \sup_{x \in [x_k, x_{k+1}]} f(x)\,,$$

define the *Darboux sums*

$$s(X, f) = \sum_{k=0}^{n-1} (x_{k+1} - x_k) c_k(f), \qquad S(X, f) = \sum_{k=0}^{n-1} (x_{k+1} - x_k) C_k(f)\,,$$

and denote $s = \sup_X s(X, f)$ and $S = \inf_X S(X, f)$. Then

- ◇ A bounded function on $[a, b]$ is said to be *Riemann integrable on* $[a, b]$ if $s = S$. We then denote $s = S = \int_a^b f(t)\,\mathrm{d}t$.
- ◇ Any continuous function on $[a, b]$, and any monotone function on $[a, b]$ is Riemann integrable on $[a, b]$.
- ◇ If f is continuous on $[a, b]$, it admits a primitive F, and $\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$.

Definition 5.2. We say that a function f is *locally integrable* on an interval I if it is Riemann integrable on any $[c, d] \subseteq I$.

Remark 5.3. If f is continuous or monotone on $[a, b[$, then it is locally integrable on $[a, b[$ by Reminder 5.1.

Definition 5.4. 1. Let f be locally integrable on $[a, b[$. If $\lim_{x \to b^-} \int_a^x f(t)\,\mathrm{d}t$ exists, we denote it by $\int_a^b f(t)\,\mathrm{d}t$, and say that the *improper integral* $\int_a^b f(t)\,\mathrm{d}t$ is convergent.

2. Let f be locally integrable on $]a, b]$. If $\lim_{x \to a^+} \int_x^b f(t)\,\mathrm{d}t$ exists, we denote it by $\int_a^b f(t)\,\mathrm{d}t$, and say that the *improper integral* $\int_a^b f(t)\,\mathrm{d}t$ is convergent.
3. Let f be locally integrable on $]a, b[$ and let $c \in\,]a, b[$. If the two improper integrals $\int_a^c f(t)\,\mathrm{d}t$ and $\int_c^b f(t)\,\mathrm{d}t$ are convergent, we denote their sum by $\int_a^b f(t)\,\mathrm{d}t$, and say that the *improper integral* $\int_a^b f(t)\,\mathrm{d}t$ is convergent.

Remark 5.5. The last definition does not depend on c. Indeed, if $c < c' \in\,]a, b[$, then $\int_c^x f(t)\,\mathrm{d}t = \int_c^{c'} f(t)\,\mathrm{d}t + \int_{c'}^x f(t)\,\mathrm{d}t$, where both integrals are well-defined Riemann integrals. Hence, $\int_c^x f(t)\,\mathrm{d}t$ has a limit as $x \to b^-$ iff $\int_{c'}^x f(t)\,\mathrm{d}t$ has a limit as $x \to b^-$. Moreover, if this limit exists, we have $\int_c^b = \int_c^{c'} + \int_{c'}^b$. The same can be done on the left side, so the improper integrals $\int_a^c f(t)\,\mathrm{d}t$ and $\int_c^b f(t)\,\mathrm{d}t$ converge iff $\int_a^{c'} f(t)\,\mathrm{d}t$ and $\int_{c'}^b f(t)\,\mathrm{d}t$ converge, and in this case $\int_a^c + \int_c^b = \int_a^{c'} - \int_c^{c'} + \int_c^{c'} + \int_{c'}^b = \int_a^{c'} + \int_{c'}^b$.

Example 5.6. (1) $\int_0^x e^{-t}\,\mathrm{d}t = -e^{-t}|_0^x = 1 - e^{-x} \to 1$ as $x \to +\infty$. Hence, $\int_0^{+\infty} e^{-t}\,\mathrm{d}t$ converges and equals 1.

(2) If $x > 1$, then $\int_1^x \frac{1}{t}\,\mathrm{d}t = \ln x \to +\infty$ as $x \to +\infty$. Hence, $\int_1^{+\infty} \frac{1}{t}\,\mathrm{d}t$ diverges.

(3) If $x \in [0,1[$, then $\int_0^x \frac{1}{\sqrt{1-t}}\,\mathrm{d}t = -2\sqrt{1-t}|_0^x = 2(1-\sqrt{1-x}) \to 2$ as $x \to 1^-$. Hence, $\int_0^1 \frac{1}{\sqrt{1-t}}\,\mathrm{d}t$ converges and equals 2.

(4) Arguing as in (1), $\int_{-\infty}^0 e^t\,\mathrm{d}t = 1$. Hence, $\int_{-\infty}^{+\infty} e^{-|t|}\,\mathrm{d}t$ converges and equals 2.

Remark 5.7. If f is defined on $]a,b[$, the two sides must be treated independently. For example, $\int_{-x}^x t\,\mathrm{d}t = 0$ for any x, but $\int_{-\infty}^{+\infty} t\,\mathrm{d}t$ diverges because $\int_0^{+\infty} t\,\mathrm{d}t$ diverges.

Theorem 5.8 (Riemann's rule)**.** *Let $\alpha \in \mathbb{R}$.*

1. $\int_0^1 \frac{1}{t^\alpha}\,\mathrm{d}t$ *converges iff* $\alpha < 1$.
2. $\int_1^{+\infty} \frac{1}{t^\alpha}\,\mathrm{d}t$ *converges iff* $\alpha > 1$.

Proof. If $\alpha = 1$, then $\int_\epsilon^1 \frac{1}{t}\,\mathrm{d}t = -\ln\epsilon \to +\infty$ as $\epsilon \to 0$. Also, $\int_1^x \frac{1}{t}\,\mathrm{d}t = \ln x \to +\infty$ as $x \to +\infty$. Hence, $\int_0^1 \frac{1}{t}\,\mathrm{d}t$ and $\int_1^{+\infty} \frac{1}{t}\,\mathrm{d}t$ diverge.

If $\alpha \neq 1$, then $\int_\epsilon^1 \frac{1}{t^\alpha}\,\mathrm{d}t = \frac{1}{1-\alpha}(1 - \frac{1}{\epsilon^{\alpha-1}})$ has a finite limit as $\epsilon \to 0$ iff $\alpha < 1$. Also, $\int_1^x \frac{1}{t^\alpha}\,\mathrm{d}t = \frac{1}{1-\alpha}(\frac{1}{x^{\alpha-1}} - 1)$ has a finite limit as $x \to +\infty$ iff $\alpha > 1$. □

Hence, $\int_0^{+\infty} \frac{1}{t^\alpha}\,\mathrm{d}t$ diverges for all $\alpha \in \mathbb{R}$.

5.2 Basic properties of improper integrals

Theorem 5.9. 1. (Linearity)**.** *Let f, g be locally integrable on $]a,b[$ and let $\lambda, \mu \in \mathbb{R}$. If $\int_a^b f(t)\,\mathrm{d}t$ and $\int_a^b g(t)\,\mathrm{d}t$ converge, then $\int_a^b (\lambda f(t) + \mu g(t))\,\mathrm{d}t$ converges and equals $\lambda\int_a^b f(t)\,\mathrm{d}t + \mu\int_a^b g(t)\,\mathrm{d}t$.*

2. (Integration by parts)**.** *Let $f, g \in C^1(\,]a,b[\,)$. Suppose fg has finite limits l_a at a and l_b at b. Then $\int_a^b f(t)g'(t)\,\mathrm{d}t$ converges iff $\int_a^b g(t)f'(t)\,\mathrm{d}t$ converges. In this case,*

$$\int_a^b f(t)g'(t)\,\mathrm{d}t = l_b - l_a - \int_a^b g(t)f'(t)\,\mathrm{d}t\,.$$

3. (Change of variables)**.** *Let f be continuous on $]a,b[$ and $\phi :]c,d[\to]a,b[$ be C^1, bijective, with a continuous inverse. Then $\int_a^b f(t)\,\mathrm{d}t$ converges iff $\int_c^d f(\phi(s))|\phi'(s)|\,\mathrm{d}s$ converges. In this case,*

$$\int_a^b f(t)\,\mathrm{d}t = \int_c^d f(\phi(s))|\phi'(s)|\,\mathrm{d}s\,.$$

Proof. In all properties we assume a is finite and f, g are locally integrable on $[a,b[$. The case $]a,b]$ is similar, and the case $]a,b[$ is deduced by considering $]a,c]$ and $[c,b[$.

1. Given $x \in [a,b[$, by the linearity of the Riemann integral, $\int_a^x (\lambda f(t) + \mu g(t))\,\mathrm{d}t = \lambda\int_a^x f(t)\,\mathrm{d}t + \mu\int_a^x g(t)\,\mathrm{d}t$. If each integral on the RHS has a limit when $x \to b^-$, then the LHS has a limit as asserted.
2. Given $x \in [a,b[$, integrate by parts on $[a,x]$ to get $\int_a^x f(t)g'(t)\,\mathrm{d}t = f(x)g(x) - f(a)g(a) - \int_a^x g(t)f'(t)\,\mathrm{d}t$. Assuming $\lim_{x\to b} f(x)g(x) = l_b$, then $\lim_{x\to b}\int_a^x f(t)g'(t)\,\mathrm{d}t$ exists iff $\lim_{x\to b}\int_a^x g(t)f'(t)\,\mathrm{d}t$ exists, and we have the stated equality.

3. Since ϕ is a continuous injection, it is strictly monotone [1]. Suppose ϕ is strictly increasing. Then $\phi(c) = a$, $\lim_{y\to d^-} \phi(y) = b$ and $\lim_{x\to b^-} \phi^{-1}(x) = d$.

   Let $y \in [c, d[$. By a change of variables in $[a, \phi(y)]$, we have $\int_c^y f(\phi(s))\phi'(s)\,\mathrm{d}s = \int_a^{\phi(y)} f(t)\,\mathrm{d}t$. Hence, $\lim_{y\to d^-} \int_c^y f(\phi(s))\phi'(s)\,\mathrm{d}s$ exists iff $\lim_{y\to d^-} \int_a^{\phi(y)} f(t)\,\mathrm{d}t$ exists iff $\lim_{x\to b^-} \int_a^x f(t)\,\mathrm{d}t$ exists, in which case we have the stated equality.

   If ϕ is strictly decreasing, then $\phi(c) = b$, $\lim_{y\to d^-} \phi(y) = a$ and the same argument applies (the limits of the integral now have to be inverted, so ϕ' becomes $|\phi'|$). □

Example 5.10. $\int_0^1 \ln t\,\mathrm{d}t$ converges. Indeed, $f(t) = t$ and $g(t) = \ln t$ are C^1 on $]0, 1]$ with $f(1)g(1) = 0$ and $\lim_{t\to 0} f(t)g(t) = 0$. Integrating by parts and noting that $\int_0^1 1\,\mathrm{d}t = 1$ converges, we get that $\int_0^1 \ln t\,\mathrm{d}t$ converges and equals $0 - 0 - 1 = -1$.

5.3 Integrals of nonnegative functions

Lemma 5.11. *Let f be nonnegative, locally integrable on $[a, b[$. Then $\int_a^b f(t)\,\mathrm{d}t$ converges iff the function $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is bounded from above on $[a, b[$.*

Proof. Since f is nonnegative, $F(x)$ is increasing, so it has a limit [2] at b which is finite or $+\infty$. This limit is finite iff F is bounded from above on $[a, b[$. □

Corollary 5.12 (Comparison test)**.** *Let f, g be nonnegative and locally integrable on $[a, b[$.*

(1) *If $f = \mathrm{O}(g)$ near b, then the convergence of $\int_a^b g(t)\,\mathrm{d}t$ implies the convergence of $\int_a^b f(t)\,\mathrm{d}t$, and the divergence of $\int_a^b f(t)\,\mathrm{d}t$ implies the divergence of $\int_a^b g(t)\,\mathrm{d}t$.*

(2) *If $f = \mathrm{o}(g)$ near b, then the convergence of $\int_a^b g(t)\,\mathrm{d}t$ implies the convergence of $\int_a^b f(t)\,\mathrm{d}t$, and the divergence of $\int_a^b f(t)\,\mathrm{d}t$ implies the divergence of $\int_a^b g(t)\,\mathrm{d}t$.*

(3) *If $f \sim g$ near b, then $\int_a^b g(t)\,\mathrm{d}t$ converges iff $\int_a^b f(t)\,\mathrm{d}t$ converges.*

Proof. (1) By hypothesis, $\exists M, \delta > 0$ such that $0 \le f(x) \le Mg(x)$ for all $x \in]b - \delta, b[$. Integrating we get

$$\forall x \in]b-\delta, b[\,: \int_a^x f(t)\,\mathrm{d}t = \int_a^{b-\delta} f(t)\,\mathrm{d}t + \int_{b-\delta}^x f(t)\,\mathrm{d}t \le \int_a^{b-\delta} f(t)\,\mathrm{d}t + M\int_{b-\delta}^x g(t)\,\mathrm{d}t\,.$$

If $\int_a^b g(t)\,\mathrm{d}t$ converges, then $\int_{b-\delta}^x g(t)\,\mathrm{d}t$ has a finite limit as $x \to b$. Hence, the RHS is bounded by a $C > 0$ independent of x, so $F(x) = \int_a^x f(t)\,\mathrm{d}t$ converges by Lemma 5.11.

If $\int_a^b f(t)\,\mathrm{d}t$ does not converge, then $\int_a^b g(t)\,\mathrm{d}t$ does not converge by contraposition.

(2) If $f = \mathrm{o}(g)$, then $f = \mathrm{O}(g)$, so this follows from (1).

(3) If $f \sim g$, then $f = \mathrm{O}(g)$ and $g = \mathrm{O}(f)$, so this follows from (1). □

Remark 5.13. Analogous results hold for functions f, g locally integrable on $]a, b[$. The results are also true if f and g are negative functions, by considering $-f$ and $-g$. The restriction is that g cannot change sign.

1. If ϕ is not strictly monotone, there exist $t_1 < t_2 < t_3$ such that $\phi(t_1) < \phi(t_2) > \phi(t_3)$ or $\phi(t_1) > \phi(t_2) < \phi(t_3)$. Suppose the first case holds; the other case is similar. The first case has two sub-cases :

If $\phi(t_1) < \phi(t_3) < \phi(t_2)$, then by the intermediate value theorem there exists $c \in]t_1, t_2[$ such that $\phi(c) = \phi(t_3)$, so ϕ is not injective, a contradiction.

If $\phi(t_3) < \phi(t_1) < \phi(t_2)$, there exists $c \in]t_2, t_3[$ with $\phi(c) = \phi(t_1)$, again a contradiction.

2. This is proved exactly like sequences; we only indicate the bounded case. Let $\varepsilon > 0$ and $\ell = \sup_{[a,b[} F$. Then $\ell - \varepsilon$ is not an upper bound, so there exists x_0 such that $\ell - \varepsilon < F(x_0)$. But F is increasing, so $x_0 \le x < b \implies \ell - \varepsilon < F(x_0) \le F(x) \le \ell$. Hence, $|F(x) - \ell| \le \varepsilon$ and F converges to ℓ.

Example 5.14. Consider $I = \int_{\frac{2}{\pi}}^{+\infty} \ln(\cos(\frac{1}{x}))\,\mathrm{d}x$. If $x > \frac{2}{\pi}$, then $0 < \cos(\frac{1}{x}) < 1$, so $x \mapsto \ln(\cos(\frac{1}{x}))$ is continuous and strictly negative on $\left]\frac{2}{\pi}, +\infty\right[$. To study the convergence near $\frac{2}{\pi}$, put $x = \frac{2}{\pi} + h$. Then

$$\frac{1}{x} = \frac{1}{\frac{2}{\pi}(1+\frac{\pi h}{2})} = \frac{\pi}{2}\Big(1 - \frac{\pi h}{2} + \mathrm{o}(h)\Big) = \frac{\pi}{2} - \frac{\pi^2 h}{4} + \mathrm{o}(h)\,.$$

Hence, $\cos(\frac{1}{x}) = \sin(\frac{\pi^2 h}{4} + \mathrm{o}(h)) = \frac{\pi^2 h}{4} + \mathrm{o}(h) = \frac{\pi^2 h}{4}(1+\mathrm{o}(1))$ and thus $\ln(\cos(\frac{1}{x})) = \ln(\frac{\pi^2 h}{4}) + \mathrm{o}(1) = \ln(\frac{\pi^2}{4}) + \ln(x - \frac{2}{\pi}) + \mathrm{o}(1)$. But $\int_0^{1-\frac{2}{\pi}} \ln x\,\mathrm{d}x$ converges, so $\int_{\frac{2}{\pi}}^{1} \ln(x-\frac{2}{\pi})\,\mathrm{d}x$ converges and thus $\int_{\frac{2}{\pi}}^{1} \ln(\cos(\frac{1}{x}))\,\mathrm{d}x$ converges.

Near $+\infty$, $\cos(\frac{1}{x}) = 1 - \frac{1}{2x^2} + \mathrm{o}(\frac{1}{x^2})$, so $\ln(\cos(\frac{1}{x})) = \ln\big(1 + (\cos(\frac{1}{x}) - 1)\big) \sim -\frac{1}{2x^2}$. Since $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$ converges, then $\int_1^{\infty} \ln(\cos(\frac{1}{x}))\,\mathrm{d}x$ converges. Conclusion : I converges.

5.4 Absolute convergence

Definition 5.15. Let f be locally integrable on $]a,b[$. We say that $\int_a^b f(t)\,\mathrm{d}t$ is absolutely convergent if $\int_a^b |f(t)|\,\mathrm{d}t$ is convergent.

Lemma 5.16. *Let f be locally integrable on $]a,b[$. If $\int_a^b f(t)\,\mathrm{d}t$ is absolutely convergent, then it is convergent.*

Proof. Let $f_+ = \frac{1}{2}(|f| + f)$ and $f_- = \frac{1}{2}(|f| - f)$. Then $0 \le f_+ \le |f|$ and $0 \le f_- \le |f|$. But $\int_a^b |f(t)|\,\mathrm{d}t$ converges, so $\int_a^b f_+(t)\,\mathrm{d}t$ and $\int_a^b f_-(t)\,\mathrm{d}t$ converge by the comparison test. But $f = f_+ - f_-$, so $\int_a^b f(t)\,\mathrm{d}t$ converges. □

Corollary 5.17 (Bounded interval criterion)**.** *Let $[a,b] \subset \mathbb{R}$ be a bounded interval and suppose f is locally integrable and bounded on $]a,b[$. Then $\int_a^b f(t)\mathrm{d}t$ converges.*

In particular, if f is locally integrable on $[a,b[$ and has a limit at b, then $\int_a^b f(t)\,\mathrm{d}t$ converges.

Proof. Assume $|f(t)| \le M$ for all $t \in]a,b[$. Then $\int_a^b M\,\mathrm{d}t = M(b-a)$ converges, hence $\int_a^b |f(t)|\,\mathrm{d}t$ converges by the comparison test, so $\int_a^b f(t)\,\mathrm{d}t$ converges. □

Example 5.18. (a) $\int_0^1 \cos(\frac{1}{t})\,\mathrm{d}t$ converges by Corollary 5.17, since $\cos(\frac{1}{t})$ is continuous and bounded by 1 on $]0,1]$.

(b) Making the change of variables $t = \frac{1}{s}$ in the previous integral, we deduce that $\int_1^{+\infty} \frac{\cos(s)}{s^2}\,\mathrm{d}s$ converges.

(c) Consider $\int_0^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$. The function $\frac{\sin t}{t}$ is continuous on $]0,+\infty[$ and tends to 1 as $t \to 0$. Hence, $\int_0^1 \frac{\sin t}{t}\,\mathrm{d}t$ converges by the bounded interval criterion. We now study $\int_1^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$. For this, we integrate by parts, taking $f(t) = \frac{1}{t}$ and $g(t) = -\cos t$, which are C^1 on $[1,+\infty[$ and satisfy $f(1)g(1) = -\cos(1)$ and $\lim_{t\to+\infty} f(t)g(t) = 0$. Hence, $\int_1^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$ and $\int_1^{+\infty} \frac{\cos t}{t^2}\,\mathrm{d}t$ have the same nature. We just saw that the latter converges, hence $\int_1^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$ converges. Conclusion : $\int_0^{+\infty} \frac{\sin t}{t}\,\mathrm{d}t$ converges.

(d) Let us show that $\int_0^{+\infty} \frac{|\sin t|}{t}\,\mathrm{d}t$ diverges. Consider $u_n = \int_0^{n\pi} \frac{|\sin t|}{t}\,\mathrm{d}t$.

We have $u_n = \sum_{k=0}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin t|}{t}\,\mathrm{d}t \ge \sum_{k=0}^{n-1} \int_{k\pi+\frac{\pi}{3}}^{k\pi+\frac{2\pi}{3}} \frac{|\sin t|}{t}\,\mathrm{d}t$.

But if $t \in \left[k\pi + \frac{\pi}{3}, k\pi + \frac{2\pi}{3}\right]$, then $|\sin t| \geq \frac{1}{2}$ and $\frac{1}{t} \geq \frac{1}{(k+1)\pi}$. Hence, $u_n \geq \sum_{k=0}^{n-1} \frac{1}{2(k+1)\pi} \cdot \frac{\pi}{3} = \frac{1}{6}\sum_{k=1}^{n} \frac{1}{k} \to +\infty$ as $n \to \infty$. Thus, $u_n \to +\infty$ and $\int_0^{+\infty} \frac{|\sin t|}{t}\,\mathrm{d}t$ diverges.

Example 5.19 (Bertrand integrals)**.** Given $\alpha, \beta \in \mathbb{R}$, we prove that

$$\begin{aligned}
I_{\alpha,\beta} &:= \int_2^{\infty} \frac{1}{x^\alpha (\ln x)^\beta}\,\mathrm{d}x \quad \text{converges} \iff \alpha > 1 \text{ or } (\alpha = 1 \text{ and } \beta > 1)\,,\\
J_{\alpha,\beta} &:= \int_0^{1/2} \frac{1}{x^\alpha (|\ln x|)^\beta}\,\mathrm{d}x \text{ converges} \iff \alpha < 1 \text{ or } (\alpha = 1 \text{ and } \beta > 1)\,,\\
K_{\alpha,\beta} &:= \int_{1/2}^{2} \frac{1}{x^\alpha (|\ln x|)^\beta}\,\mathrm{d}x \quad \text{converges} \iff \beta < 1\,.
\end{aligned}$$

Let $f_{\alpha,\beta}(x) = \frac{1}{x^\alpha(\ln x)^\beta}$. Then $f_{\alpha,\beta}$ is continuous and nonnegative on $]2, +\infty[$.

If $\alpha < 1$, then $\frac{1/x}{f_{\alpha,\beta}} = x^{\alpha-1}(\ln x)^\beta = \frac{(\ln x)^\beta}{x^{1-\alpha}} \to 0$ as $x \to \infty$, so $\frac{1}{x} = \mathrm{o}(f_{\alpha,\beta})$ near ∞. But $\int_2^\infty \frac{1}{x}\,\mathrm{d}x$ diverges, so $I_{\alpha,\beta}$ diverges.

If $\alpha > 1$, take $\gamma = \frac{1+\alpha}{2}$. Then $\frac{f_{\alpha,\beta}}{1/x^\gamma} = \frac{1}{x^{\alpha-\gamma}(\ln x)^\beta} \to 0$ as $x \to \infty$ since $\alpha - \gamma > 0$. Hence, $f_{\alpha,\beta} = \mathrm{o}(\frac{1}{x^\gamma})$ near ∞. Since $\gamma > 1$, $\int_2^\infty \frac{1}{x^\gamma}\,\mathrm{d}x$ converges, hence $I_{\alpha,\beta}$ converges.

If $\alpha = 1$, making the change of variables $u = \ln x$, we get that $I_{1,\beta}$ converges $\iff$ $\int_{\ln 2}^\infty \frac{1}{u^\beta}\,\mathrm{d}u$ converges $\iff \beta > 1$.

To study $J_{\alpha,\beta}$, consider the change of variables $u = \frac{1}{x}$. Then $J_{\alpha,\beta}$ converges $\iff$ $\int_2^\infty \frac{u^\alpha}{(\ln u)^\beta}\frac{\mathrm{d}u}{u^2} = I_{2-\alpha,\beta}$ converges $\iff 2 - \alpha > 1$ or $(2 - \alpha = 1$ and $\beta > 1)$, i.e. $\alpha < 1$ or $(\alpha = 1$ and $\beta > 1)$.

To study $K_{\alpha,\beta}$, let $L_{\alpha,\beta} = \int_{1/2}^1 \frac{1}{x^\alpha(|\ln x|)^\beta}\,\mathrm{d}x$. Near 1 we have $\ln x = \ln(1 + (x-1)) \sim x - 1$, so $\frac{1}{x^\alpha(|\ln x|)^\beta} \sim \frac{1}{|x-1|^\beta}$. But $\int_{1/2}^1 \frac{1}{|x-1|^\beta}\,\mathrm{d}x$ converges $\iff$ $\int_0^{1/2} \frac{1}{u^\beta}\,\mathrm{d}u$ converges $\iff \beta < 1$. Hence, $L_{\alpha,\beta}$ converges iff $\beta < 1$. Similarly, $M_{\alpha,\beta} = \int_1^2 f_{\alpha,\beta}(x)\,\mathrm{d}x$ converges iff $\beta < 1$, so $K_{\alpha,\beta} = L_{\alpha,\beta} + M_{\alpha,\beta}$ converges iff $\beta < 1$.

5.5 Cauchy criterion

Theorem 5.20 (The continuous Cauchy criterion)**.** *Let $A = B_r(z_0) \subseteq \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let a be a limit point of A. Then f has a limit at a iff for any $\varepsilon > 0$ there exists a neighborhood V of a such that $x, y \in V \cap A \implies |f(x) - f(y)| \leq \varepsilon$.*

Proof. Suppose $\lim_{x\to a} f(x) = \ell$. Then given $\varepsilon > 0$, there exists a neighborhood V of a such that $x \in V \cap A \implies |f(x) - \ell| \leq \frac{\varepsilon}{2}$. Hence, $x, y \in V \cap A \implies |f(x) - f(y)| \leq |f(x) - \ell| + |f(y) - \ell| \leq \varepsilon$.

Conversely, suppose f satisfies the continuous Cauchy condition. Let $\varepsilon > 0$ and let $\{a_n\} \subseteq A$ with $a_n \to a$. Then there exists $N \in \mathbb{N}$ such that $a_n \in V \cap A$ for all $n \geq N$. So by Cauchy's condition, $|f(a_n) - f(a_m)| \leq \varepsilon$ for any $n, m \geq N$. Hence, $(f(a_n))$ is Cauchy, so it converges.

Suppose (a_n) and (b_n) are two sequences converging to a. We showed that $(f(a_n))$ and $(f(b_n))$ converge. The sequence (c_n) defined by $c_{2n} = a_n$ and $c_{2n+1} = b_n$ also converges to a, so $(f(c_n))$ converges and thus the subsequences $(f(a_n))$ and $(f(b_n))$ converge to the same limit.

We showed that there exists ℓ such that for any sequence $\{a_n\} \subseteq A$, $a_n \to a$, the sequence $f(a_n) \to \ell$. Hence, $\lim_{x\to a} f(x) = \ell$. ☐

Corollary 5.21 (Cauchy criterion for integrals)**.** 1. *Let f be locally integrable on $[a,b[$, $b \in \mathbb{R}$. Then $\int_a^b f(t)\,\mathrm{d}t$ converges iff*

$$\forall \varepsilon > 0\ \exists \delta > 0 : \forall x, x' \in [a,b[\,,\ x, x' \in]b-\delta, b[\ \Longrightarrow\ \Big|\int_x^{x'} f(t)\,\mathrm{d}t\Big| \le \varepsilon\,.$$

2. *Let f be locally integrable on $[a,+\infty[$. Then $\int_a^{+\infty} f(t)\,\mathrm{d}t$ converges iff*

$$\forall \varepsilon > 0\ \exists A \in \mathbb{R} : \forall x, x' \in [a,+\infty[\,,\ x, x' > A\ \Longrightarrow\ \Big|\int_x^{x'} f(t)\,\mathrm{d}t\Big| \le \varepsilon\,.$$

Proof. Apply the continuous Cauchy criterion to $g(x) = \int_a^x f(t)\,\mathrm{d}t$, noting that neighborhoods of b have the form $]b-\delta, b[$, while neighborhoods of $+\infty$ have the form $]A, +\infty[$. □

A similar criterion holds of course for integrals which are improper on the left.

5.6 Exercises

1. Let $\alpha < \beta < \gamma \in \mathbb{R}$. Calculate the following integrals, then deduce the convergence and the value of the improper integrals when $\alpha \to -\infty$, $\beta \to 0$ and $\gamma \to +\infty$

- $\displaystyle\int_0^\gamma \frac{\mathrm{d}x}{(x+1)(x+2)(x+3)}$
- $\displaystyle\int_1^\gamma \frac{\ln x}{x^a}\,\mathrm{d}x,\ a > 1$
- $\displaystyle\int_0^\gamma e^{-\sqrt{x}}\,\mathrm{d}x$
- $\displaystyle\int_\alpha^\gamma \frac{\mathrm{d}x}{x^2+2x+3}$
- $\displaystyle\int_\beta^\gamma \frac{\ln x}{(x+1)^2}\,\mathrm{d}x$
- $\displaystyle\int_\beta^\gamma \Big(1 - x\arctan\frac{1}{x}\Big)\,\mathrm{d}x$
- $\displaystyle\int_0^\gamma \frac{\mathrm{d}x}{\sqrt{1+e^x}}$
- $\displaystyle\int_0^\gamma \frac{\mathrm{d}x}{x^3+1}$
- $\displaystyle\int_0^\gamma \frac{x\arctan x}{(1+x^2)^2}\,\mathrm{d}x$

2. (Gamma Function)**.** We define the Γ function on $]0,+\infty[$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t\,.$$

1) Show that Γ is well defined, i.e. the integral converges for all $x > 0$.

2) Show that for all $x > 0$, $\Gamma(x+1) = x\Gamma(x)$ and for all $n \in \mathbb{N}$, $\Gamma(n+1) = n!$.

3. (Gauss integral)**.**

1) Let $f(x) = \int_0^1 \frac{e^{-x(1+t^2)}}{1+t^2}\,\mathrm{d}t$. Show that f is differentiable on $\mathbb{R}$ and that $f'(x) = -\int_0^1 e^{-x(1+t^2)}\,\mathrm{d}t$.

2) Put $g(x) = f(x^2)$. Show that for all $x \in \mathbb{R}$,

$$g'(x) = -2e^{-x^2}\int_0^x e^{-t^2}\,\mathrm{d}t, \qquad \text{then that} \qquad g(x) + \Big(\int_0^x e^{-t^2}\,\mathrm{d}t\Big)^2 = \frac{\pi}{4}\,.$$

3) Deduce the value of $\int_0^\infty e^{-t^2}\,\mathrm{d}t$, then that of $\Gamma(\frac{1}{2})$.

4. Study the convergence of the following improper integrals:

- $\displaystyle\int_0^\infty (x+2)-\sqrt{x^2+4x+1}\,\mathrm{d}x$
- $\displaystyle\int_1^\infty e^{-x^{1/3}+\sin x}\,\mathrm{d}x$
- $\displaystyle\int_0^\infty x^a e^{-x}\,\mathrm{d}x,\ a\in\mathbb{R}$
- $\displaystyle\int_1^\infty e^{-\sqrt{\ln x}}\,\mathrm{d}x$
- $\displaystyle\int_1^\infty \frac{e^{-\sqrt{\ln x}}}{x}\,\mathrm{d}x$
- $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x^{\cos(1/x)}}$
- $\displaystyle\int_2^\infty \frac{x^{\ln x}}{(\ln x)^x}\,\mathrm{d}x$
- $\displaystyle\int_0^\infty e^{-(\ln x)^2}\,\mathrm{d}x$
- $\displaystyle\int_0^\infty \frac{\ln x}{x^2-1}\,\mathrm{d}x$
- $\displaystyle\int_0^1 \arctan\Big(\frac{1}{x}\sin(\frac{1}{x})\Big)\,\mathrm{d}x$
- $\displaystyle\int_0^1 \frac{\arctan x}{x^2}\,\mathrm{d}x$
- $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{x^a+x^b},\ a,b\in\mathbb{R}$
- $\displaystyle\int_1^\infty x^a(1-e^{-\frac{1}{\sqrt{x}}})\,\mathrm{d}x,\ a\in\mathbb{R}$
- $\displaystyle\int_0^\infty \frac{x-1}{(x+2)^{7/3}}\,\mathrm{d}x$
- $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(x+3)^2\sqrt{|x-2|}}$
- $\displaystyle\int_0^1 |1-x^a|^b\,\mathrm{d}x,\ a,b\in\mathbb{R}$
- $\displaystyle\int_0^1 \frac{x^a-1}{\ln x}\,\mathrm{d}x,\ a\in\mathbb{R}$
- $\displaystyle\int_{-\infty}^\infty \frac{\mathrm{d}x}{e^x+x^2e^{-x}}$
- $\displaystyle\int_1^e \frac{\mathrm{d}x}{\ln x}$
- $\displaystyle\int_e^\infty \frac{\mathrm{d}x}{\ln x}$
- $\displaystyle\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,\mathrm{d}x$
- $\displaystyle\int_0^\infty \frac{\sin(5x)-\sin(3x)}{x^{5/3}}\,\mathrm{d}x$
- $\displaystyle\int_0^\infty \frac{x^a}{1+x^b}\,\mathrm{d}x,\ a,b\in\mathbb{R}$
- $\displaystyle\int_0^\infty (\sqrt[3]{x+1}-\sqrt[3]{x})^{\sqrt{x}}\mathrm{d}x$

5. (Improper integrals and limits)**.** We learned that if a numerical series $\sum a_n$ converges, then $a_n\to 0$.

1. Find a function f which is nonnegative, continuous and such that $\int_0^\infty f(x)\,\mathrm{d}x$ converges, but $\lim\limits_{x\to\infty} f(x)$ does not exist.
2. Show that if f is locally integrable on $[a,+\infty[$, if $\int_a^\infty f(x)\,\mathrm{d}x$ converges and if $\lim\limits_{x\to\infty} f(x)$ exists, then $\lim\limits_{x\to\infty} f(x)=0$.
3. Show that if f is locally integrable on $[a,+\infty[$, if $\int_a^\infty f(x)\,\mathrm{d}x$ converges and if f is uniformly continuous, then $\lim\limits_{x\to\infty} f(x)=0$.
4. Application: study the convergence of $\int_0^\infty \sin(\sin x)\,\mathrm{d}x$.

6. Study the convergence of the following improper integrals:

- $\displaystyle\int_0^\infty \sin(x)\,\mathrm{d}x$
- $\displaystyle\int_0^1 \frac{\sin x}{x^a}\,\mathrm{d}x,\ a>0$
- $\displaystyle\int_1^\infty \frac{\sin x}{x^a}\,\mathrm{d}x,\ a>0$
- $\displaystyle\int_0^\infty \sin(e^x)\,\mathrm{d}x$
- $\displaystyle\int_0^1 \frac{1}{x^2}\sin(\frac{1}{x^2})\,\mathrm{d}x$
- $\displaystyle\int_1^\infty \frac{\sin^2 x}{x}\,\mathrm{d}x$
- $\displaystyle\int_1^\infty \frac{\sin x}{\sqrt{x}+\sin x}\,\mathrm{d}x$

7. (Abel's criterion)**.** Let f and g be two functions on $[a,b[$, $b\in\mathbb{R}\cup\{+\infty\}$, such that
◇ f is of class C^1, monotone and tends to 0 at b,
◇ g is continuous and $\exists M>0:\forall c\in[a,b[,\ |\int_a^c g(x)\,\mathrm{d}x|\le M$.
Show that $\int_a^b f(x)g(x)\,\mathrm{d}x$ converges [3].
Application: study the convergence of $\int_0^\infty (\sqrt{x+\cos x}-\sqrt{x})\,\mathrm{d}x$.

8. Show that $\int_0^{\pi/2}\ln(\sin x)\,\mathrm{d}x=\int_0^{\pi/2}\ln(\cos x)\,\mathrm{d}x=-\frac{\pi\ln 2}{2}$.

9. Show that $\int_0^\infty \frac{1}{1+x^4}\,\mathrm{d}x=\int_0^\infty \frac{x^2}{1+x^4}\,\mathrm{d}x=\frac{\pi}{2\sqrt{2}}$.

3. The criterion remains true if f is not of class C^1 and if g is not continuous, but the proof becomes more complicated.

10. Let $a, b > 0$ and let f be a continuous function on $\mathbb{R}_+$ such that $\int_1^\infty \frac{f(x)}{x}\,\mathrm{d}x$ converges.

1. Show that for $\alpha > 0$, we have

$$\int_\alpha^\infty \frac{f(bx) - f(ax)}{x}\,\mathrm{d}x = \int_{b\alpha}^{a\alpha} \frac{f(x)}{x}\,\mathrm{d}x\,.$$

2. Deduce that the integral $\int_0^\infty \frac{f(bx)-f(ax)}{x}\,\mathrm{d}x$ converges and is equal to $f(0)\ln\frac{a}{b}$.
3. Show the existence of $\int_0^\infty \frac{e^{-bx}-e^{-ax}}{x}\,\mathrm{d}x$ and give its value.
4. Deduce from the previous question the convergence and value of $\int_0^1 \frac{x-1}{\ln x}\,\mathrm{d}x$.
5. Calculate $\int_0^\infty \frac{\sin^3 x}{x^2}\,\mathrm{d}x$ using the relation $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$.

11. Calculate the following integrals after establishing their convergence ($n \in \mathbb{N}^*$):

- $\displaystyle\int_0^\infty \frac{1}{(x^2+1)^n}\,\mathrm{d}x$
- $\displaystyle\int_{-\infty}^\infty \frac{1}{(x^2+2x+3)^n}\,\mathrm{d}x\,.$

Bibliography

[1] E. A. Coddington. *An Introduction to Ordinary Differential Equations.* Dover Publications, 1989.

[2] C. Deschamps and A. Warusfel. *Mathématiques tout-en-un [2e année] MP, PC, PSI.* Dunod, 2001.

[3] C. Deschamps and A. Warusfel. *Mathématiques tout-en-un MPSI-PCSI.* Dunod, 2ème edition, 2003.

[4] C. Deschamps and A. Warusfel. *Mathématiques tout-en-un [2e année] ECS.* Dunod, nouvelle edition, 2008.

[5] E. Hairer and G. Wanner. *Analysis by Its History.* Springer, 2008.

[6] M. Hindry. Cours de Mathématiques L1. *Polycopié, Université Paris 7.*

[7] T. Joly. Algèbre et Analyse fondamentales MI3. *Polycopié, Université Paris 7*, 2008.

[8] J-P. Marco and L. Lazzarini. *Mathématiques L1 : Cours complet avec 1000 tests et exercices corrigés.* Pearson, 2007.

[9] J-P. Marco, P. Thieullen, and J-A. Weil. *Mathématiques L2 : Cours complet avec 700 tests et exercices corrigés.* Pearson, 2007.

[10] D. Monasse. *Cours de Mathématiques MP-MP*.* Spartacus, 2012.

[11] M. Mureşan. *A Concrete Approach to Classical Analysis.* Springer, 2009.

[12] D. Newman and T. D. Parsons. On monotone subsequences. *Amer. Math. Monthly*, 95:44–45, 1988.

[13] R. Palais. A simple proof of the Banach contraction principle. *J. fixed point theory appl.*, 2:221–223, 2007.

[14] W. Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, third edition, 1976.

[15] W. Rudin. *Real and Complex Analysis.* McGraw-Hill, third edition, 1986.