THE COLORED BENJAMINI-SCHRAMM TOPOLOGY

MOSTAFA SABRI

Abstract. These are lecture notes on the Benjamini-Schramm topology. We discuss basic notions, examples and implications for the spectral measures.

1. Introduction

In these notes we discuss the basic properties of the Benjamini-Schramm topology, carry out some examples, and prove that convergence in the sense of Benjamini-Schramm implies convergence of the associated empirical measures. Nothing presented here is new; our purpose is simply to give complete proofs for newcomers[1].

What Benjamini and Schramm introduced in [8] is a notion of convergence for a sequence of finite graphs (G_n). An important aspect is that if this sequence has a uniformly bounded degree, then there always exists a limit object up to passing to a subsequence. Moreover, it gives a "correct" notion of convergence from the point of view of spectral theory. For example it has been known at least since [2] that if one is interested in the spectral properties of the $(q+1)$-regular tree $\mathcal{T}_q$, then studying what happens on a sequence of growing balls around some fixed origin is not a good idea, because the spectral behavior of the limit object is completely different from those of $\mathcal{T}_q$. And indeed, such tree balls do not Benjamini-Schramm converge to $\mathcal{T}_q$. On the other hand, a sequence of $(q+1)$-regular graphs with few short cycles do converge to $\mathcal{T}_q$ in the sense of Benjamini-Schramm, and it is known that the mean spectral measures of the corresponding adjacency matrices converge to the density of states of the adjacency matrix on $\mathcal{T}_q$. This is known as the law of *Kesten-McKay*, and it turns out to be a general phenomenon.

As previously mentioned, an important advantage of this notion of convergence is the existence of a limit object. A very broad question is whether specific information about the limit object implies something on the convergent sequence when n gets large. It is one such question that is considered in [6], where we show that if (G_n, W_n) is a sequence of colored graphs of uniformly bounded degree and coloring, and if (G_n, W_n) has a local weak limit ρ supported on colored trees, then roughly speaking, AC spectrum of the limit "Schrödinger" operator implies quantum ergodicity for the sequence. A lot of different spectral questions have also been studied in the literature, and relations have been found with conjectures in group theory [3].

2. Basic definitions

A *colored rooted graph* is a triple (G, o, W), where $G = (V, E)$ is a graph, o is a marked vertex in G called the *root*, and W is a map from $V \to \mathbb{R}$ which we see as a "coloring"; it can also be regarded as a potential on $\ell^2(V)$. This is a special case of what is called a *network* in [3], but the discussion of this note applies to this general setting as well. All graphs are assumed to be *locally finite*, i.e. each vertex has a finite degree.

If G is connected, we denote by $B_G(x, r)$ the *r-ball* $\{y \in V : d_G(x, y) \leq r\}$, where d_G is the length of the shortest path between x and y in G. We say there is a *rooted isomorphism*

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[1] Thanks to Nalini Anantharaman for clarifications !

φ between two balls $B_{G_1}(x_1,r)$ and $B_{G_2}(x_2,r)$, and denote

$$\varphi : B_{G_1}(x_1,r) \xrightarrow{\sim} B_{G_2}(x_2,r)$$

if $\varphi : B_{G_1}(x_1,r) \to B_{G_2}(x_2,r)$ is a graph isomorphism with $\varphi(x_1) = x_2$.

We define a distance between colored connected graphs by

$$d_{loc}[(G_1,o_1,W_1),(G_2,o_2,W_2)] = \frac{1}{1+\alpha_{1,2}}\,, \tag{2.1}$$

where

$$\begin{aligned}\alpha_{1,2} = \sup\bigl\{ r>0 : \exists\, \varphi : B_{G_1}(o_1,[r]) \xrightarrow{\sim} B_{G_2}(o_2,[r]) \text{ with} \\ |W_2(\varphi(v)) - W_1(v)| < 1/r \;\forall v \in B_{G_1}(o_1,[r]) \bigr\}\,.\end{aligned}$$

Lemma 2.1. *The function d_{loc} is a pseudo-metric on the set of colored graphs.*

Proof. We prove the triangle inequality.

Denote $\|W_j \circ \varphi - W_i\| = \sup_{x\in B_{G_i}(o_i,t)} |W_j(\varphi(v)) - W_i(v)|$, where the radius t of the ball will be clear from the context. We need to show that $\frac{1}{1+\alpha_{1,2}} \le \frac{1}{1+\alpha_{1,3}} + \frac{1}{1+\alpha_{2,3}}$.

Assume $\alpha_{1,2} = r$ and let $s > r$. Then there is no $\varphi : B_{G_1}(o_1,[s]) \xrightarrow{\sim} B_{G_2}(o_2,[s])$ having $\|W_2 \circ \varphi - W_1\| < 1/s$.

We have two cases : if there is no $\psi : B_{G_1}(o_1,[s]) \xrightarrow{\sim} B_{G_3}(o_3,[s])$, then $\alpha_{1,3} < s$. In this case, $d(G_1,G_2) = \frac{1}{1+r} = \sup_{s>r} \frac{1}{1+s} \le \frac{1}{1+\alpha_{1,3}} = d(G_1,G_3)$. Similarly, if there is no $\psi : B_{G_2}(o_2,[s]) \xrightarrow{\sim} B_{G_3}(o_3,[s])$, then $\alpha_{2,3} < s$, implying $d(G_1,G_2) \le d(G_2,G_3)$. In either case we get $d(G_1,G_2) \le d(G_1,G_3) + d(G_2,G_3)$.

So assume now that there exists $\psi_1 : B_{G_1}(o_1,[s]) \xrightarrow{\sim} B_{G_3}(o_3,[s])$ and $\psi_2 : B_{G_2}(o_2,[s]) \xrightarrow{\sim} B_{G_3}(o_3,[s])$. Then $\psi_2^{-1} \circ \psi_1 : B_{G_1}(o_1,[s]) \xrightarrow{\sim} B_{G_2}(o_2,[s])$. So by assumption, we must have $\|W_2 \circ \psi_2^{-1} \circ \psi_1 - W_1\| \ge 1/s$. Hence,

$$\begin{aligned}\frac{1}{s} &\le \|W_1 - W_2 \circ \psi_2^{-1} \circ \psi_1\| \le \|W_1 - W_3 \circ \psi_1\| + \|W_3 \circ \psi_1 - W_2 \circ \psi_2^{-1} \circ \psi_1\| \\ &= \|W_1 - W_3 \circ \psi_1\| + \|W_3 - W_2 \circ \psi_2^{-1}\| = \|W_3 \circ \psi_1 - W_1\| + \|W_3 \circ \psi_2 - W_2\|\,.\end{aligned}$$

So $\frac{1}{\|W_3\circ\psi_1 - W_1\| + \|W_3\circ\psi_2 - W_2\|} \le s$ and thus

$$\begin{aligned}\frac{1}{1+s} &\le \frac{1}{1+\frac{1}{\|W_3\circ\psi_1 - W_1\| + \|W_3\circ\psi_2 - W_2\|}} = \frac{\|W_3 \circ \psi_1 - W_1\| + \|W_3 \circ \psi_2 - W_2\|}{1 + \|W_3 \circ \psi_1 - W_1\| + \|W_3 \circ \psi_2 - W_2\|} \\ &\le \frac{\|W_3 \circ \psi_1 - W_1\|}{1 + \|W_3 \circ \psi_1 - W_1\|} + \frac{\|W_3 \circ \psi_2 - W_2\|}{1 + \|W_3 \circ \psi_2 - W_2\|} \\ &= \frac{1}{1+\frac{1}{\|W_3\circ\psi_1 - W_1\|}} + \frac{1}{1+\frac{1}{\|W_3\circ\psi_2 - W_1\|}}\,.\end{aligned}$$

Finally, we have $\alpha_{1,3} \le \frac{1}{\|W_3\circ\psi_1 - W_1\|}$ and $\alpha_{2,3} \le \frac{1}{\|W_3\circ\psi_2 - W_1\|}$ by definition (because the r must satisfy $r < \frac{1}{\|W_j\circ\varphi - W_i\|}$). We thus showed that $\frac{1}{1+s} \le \frac{1}{1+\alpha_{1,3}} + \frac{1}{1+\alpha_{2,3}}$, and this completes the proof as before. $\square$

We say that two colored graphs (G_1,o_1,W_1) and (G_2,o_2,W_2) are *equivalent* if there is a graph isomorphism $\varphi : G_1 \to G_2$ such that $\varphi(o_1) = o_2$ and $W_2 \circ \varphi = W_1$. We denote the equivalence class of (G,o,W) by $[G,o,W]$.

We denote by $\mathcal{G}_*$ the set of equivalence classes of connected colored rooted graphs.

Lemma 2.2. *The function d_{loc} induces a metric on $\mathcal{G}_*$.*

Proof. The value of $\alpha_{1,2}$ is independent of the choice of the representative in the equivalence class. Suppose $d_{loc}[(G,o,W),(G',o',W')]=0$. We show that $[G,o,W]=[G',o',W']$.

For any $r\in\mathbb{N}$, there is $\varphi_r : B_G(o,r)\xrightarrow{\sim} B_{G'}(o',r)$ with $\|W'\circ\varphi_r - W\|_{B_{G'}(o',r)} < 1/r$.

Let $\varphi_r^{(n)} = \varphi_r|_{B_G(o,n)}$ for $r\geq n$. Then $\varphi_r^{(n)} : B_G(o,n)\to B_{G'}(o,r)$ for $r\geq n$. Actually, $\varphi_r^{(n)} : B_G(o,n)\to B_{G'}(o,n)$ for all $r\geq n$, because φ_r is a graph isomorphism and thus preserves neighbors. So $\varphi_r^{(n)} : B_G(o,n)\to \operatorname{Ran}\varphi_r^{(n)}$ is a graph isomorphism. So $|\operatorname{Ran}\varphi_r^{(n)}| = |B_G(o,n)| = |B_{G'}(o',n)|$, where the last equality holds because φ_n is a graph isomorphism. It follows that $\operatorname{Ran}\varphi_r^{(n)} = B_{G'}(o,n)$. Hence, $\varphi_r^{(n)} : B_G(o,n)\xrightarrow{\sim} B_{G'}(o,n)$ for all $r\geq n$. Since G and G' are locally finite, $\varphi_r^{(n)}$ has a convergent (in fact stationary) subsequence $\varphi_{r_j}^{(n)}$. Denote its limit by $\varphi^{(n)}$. Then $\varphi^{(n)} : B_G(o,n)\xrightarrow{\sim} B_{G'}(o',n)$.

Now $\varphi^{(n+1)}|_{B_G(o,n)} = \lim\varphi_{r_j}^{(n+1)}|_{B_G(o,n)} = \lim\varphi_{r_j}^{(n)} = \varphi^{(n)}$. So $\varphi^{(m)}|_{B_G(o,n)} = \varphi^{(n)}$ for all $m\geq n$. Hence, if for $v\in G$, say $v\in B_G(o,n)$ for some n, we put $\varphi(v) := \varphi^{(n)}(v)$, then φ is well-defined. Moreover, φ is bijective. In fact, if $w\in G'$, then $w=\varphi^{(n)}(v)$ for some n and v, so $w=\varphi(v)$; injectivity is similar. Finally φ is a graph isomorphism since given $v\sim v'$, say both in $B_G(o,n)$, we have $\{\varphi(v),\varphi(v')\} = \{\varphi^{(n)}(v),\varphi^{(n)}(v')\}$ is an edge in G'. We thus showed that $\varphi : (G,o)\xrightarrow{\sim}(G',o')$. It remains to check the coloring. Note that for any n and $r\geq n$, we have $\|W'\circ\varphi_r^{(n)} - W\|_{B_{G'}(o',n)} < 1/r$, since φ_r satisfies this on the bigger ball $B_{G'}(o,r)$. Since $\varphi^{(n)}$ is the limit of $\varphi_{r_j}^{(n)}$, we get $\|W'\circ\varphi^{(n)} - W\|_{B_{G'}(o',n)} = 0$. This shows that $W'\circ\varphi = W$. Hence, $[G,o,W]=[G',o',W']$ as required. □

Lemma 2.3. *The metric space $(\mathcal{G}_*, d_{loc})$ is a* Polish space*, i.e. separable and complete.*

Proof. Consider the family $\mathcal{C}_n = ([n],1,W)$, where $[n]=\{1,\dots,n\}$ and W takes values in $\mathbb{Q}$. This is a countable family, since the set of maps W is just $\mathbb{Q}^{[n]}$. We denote $\mathcal{C}=\cup_n\mathcal{C}_n$. Given $[G,o,W]\in\mathcal{G}_*$ and $\varepsilon>0$, choose $r\in\mathbb{N}$ such that $\frac{1}{1+r}<\varepsilon$. Since G is locally finite, the number of vertices in $B_G(o,r)$ is some $n=n(r)\in\mathbb{N}$. If $\varphi : B_G(o,r)\xrightarrow{\sim}([n],1)$ is a rooted graph isomorphism, we may choose $W_n : [n]\to\mathbb{Q}$ such that $\|W_n\circ\varphi - W\| < 1/r$. Then $d_{loc}[(G,o,W),([n],1,W_n)]\leq\frac{1}{1+r}$, so $\mathcal{C}$ is dense in $\mathcal{G}_*$.

Next, suppose $([G_n,o_n,W_n])$ is Cauchy in $\mathcal{G}_*$. Assuming $|V(G_n)|=V_n$, we consider equivalently the sequence $([[V_n],1,W_n'])$, where $W_n' = W_n\circ\varphi_n$ for some $\varphi_n : G_n\xrightarrow{\sim}[V_n]$. We may check that for any $r\in\mathbb{N}$ there exists n_r and $\psi_n^m : B_{[V_n]}(1,r)\xrightarrow{\sim} B_{[V_m]}(1,r)$ such that $\|W_m'\circ\psi_n^m - W_n'\| < 1/r$ for all $n,m\geq n_r$. For any $v\in[V_{n_r}]$, the sequence $\{a_1,\dots,a_{n_r-1},W_{n_r}'(v),W_{n_r+1}'\circ\psi_{n_r}^{n_r+1}(v),W_{n_r+2}'\circ\psi_{n_r}^{n_r+2}(v),\dots\}$, $a_j=1$, is thus Cauchy in $\mathbb{R}$ and converge to some A_v. Since $[B_{[V_n]}(1,r)]$ is stationary in n for any r, it converges to some $[B_{[V]}(1,r)]$, possibly $[V]=\mathbb{N}$. If we define $A:[V]\to\mathbb{R}$ by $A(v)=A_v$, it follows that $d_{loc}[([V_n],1,W_n'),([V],1,A)]\leq\frac{1}{1+r}$ for $n\geq n_r'$. As $r\in\mathbb{N}$ is arbitrary, the sequence converges to $[[V],1,A]$. □

We next define $\mathcal{G}_*^{D,A}$ to be the subset of equivalence classes $[G,o,W]$ such that G has degree uniformly bounded by D and W takes values in $[-A,A]$.

Lemma 2.4. *The metric space $(\mathcal{G}_*^{D,A}, d_{loc})$ is compact.*

Proof. Clearly, $\mathcal{G}_*^{D,A}$ is closed in $\mathcal{G}_*$ and thus complete. So it suffices to show that it is totally bounded. Suppose this is not true. Then for some $\varepsilon>0$, there is no finite ε-net. So we may construct a sequence $([G_n,o_n,W_n])$ such that $d_{loc}[(G_n,o_n,W_n),(G_m,o_m,W_m)]\geq\varepsilon$ for all $n\neq m$. Hence, there exists $r\in\mathbb{N}$ such that $\alpha_{n,m}<r$ for all $n\neq m$, where $\alpha_{n,m}$ is as in (2.1). Now observe that there are only finitely many equivalence classes of rooted balls $B_{G_n}(o_n,r)$, since the degree is uniformly bounded by D. So if $\alpha_{n,m}<r$ for all $n\neq m$, there must be a sequence of isomorphic balls $B_{G_{n_j}}(o_{n_j},r)$ on which the coloring is distant,

say $\|W_{n_p}\circ\psi_q^p - W_{n_q}\| \ge 1/r$ for all $p\neq q$, where $\psi_q^p : B_{G_{n_q}}(o_{n_q},r)\xrightarrow{\sim} B_{G_{n_p}}(o_p,r)$ is a rooted isomorphism. So the sequence $\{W_{n_1}, W_{n_2}\circ\psi_1^2, W_{n_3}\circ\psi_1^3,\dots\}$ on $B_{G_{n_1}}(o_{n_1},r)$ has no convergent subsequence. But this is a sequence in $[-A,A]^{B_{G_{n_1}}(o_{n_1},r)}$, which is compact for the product topology, which coincides with the topology of pointwise convergence, i.e. the topology endowed by $\|W\| = \sup_{v\in B_{G_{n_1}(o_{n_1},r)}} |W(v)|$. This contradiction completes the proof. $\square$

So far we have defined a metric on connected colored rooted graphs. We now introduce a notion of convergence for unrooted graphs (G_n,W_n), which are not necessarily connected. The idea is to consider the convergence of the law of G_n under uniform rooting.

Since $\mathcal{G}_*$ is a Polish space, we may consider the set of probability measures on $\mathcal{G}_*$, denoted by $\mathcal{P}(\mathcal{G}_*)$. The latter is also a Polish space.

If (G,W) is a finite colored graph, $G=(V,E)$, we denote $(G(v),W)$ the subgraph spanned by the vertices in the connected component on v. We then define $U_{(G,W)}\in\mathcal{P}(\mathcal{G}_*)$ by

$$U_{(G,W)} = \frac{1}{|V|}\sum_{v\in V}\delta_{[G(v),v,W]}\,.$$

This captures the idea of choosing the root v uniformly at random in V.

If (G_n,W_n) is a sequence of finite colored graphs, we say that $\rho\in\mathcal{P}(\mathcal{G}_*)$ is the *local weak limit* of (G_n,W_n) if $U_{(G_n,W_n)}$ converges weakly to ρ in $\mathcal{P}(\mathcal{G}_*)$. This notion of convergence was first introduced in [8] and later generalized in [3]. It is often called the *Benjamini-Schramm convergence.*

Let $C(\mathcal{G}_*^{D,A})$ be the set of continuous functions $f:\mathcal{G}_*^{D,A}\to\mathbb{R}$. Recall that a linear subspace $\mathscr{A}\subset C(\mathcal{G}_*^{D,A})$ is called an *algebra* if is is closed under multiplication and contains the constant function 1. We say that $\mathscr{A}$ separates points if for any $[G,o,W]\neq[G',o',W']\in\mathcal{G}_*^{D,A}$, there is some $f\in\mathscr{A}$ such that $f([G,o,W])\neq f([G',o',W'])$.

Lemma 2.5. *Let (G_n,W_n) be a sequence of finite colored graphs, $G_n=(V_n,E_n)$, with degree uniformly bounded by D and coloring $W_n : V_n\to[-A,A]$ for all n. Then*

1. *(G_n,W_n) has a subsequence which converges in the sense of Benjamini-Schramm, i.e. $U_{(G_{n_j},W_{n_j})}$ converges weakly to some $\mu\in\mathcal{P}(\mathcal{G}_*^{D,A})$.*
2. *(G_n,W_n) has a local weak limit ρ iff there is an algebra $\mathscr{A}\subset C(\mathcal{G}_*^{D,A})$ which separates points, such that for all $f\in\mathscr{A}$,*

$$\lim_{n\to\infty}\frac{1}{|V_n|}\sum_{v\in V_n} f\left([G_n(v),v,W_n]\right) = \int_{\mathcal{G}_*^{D,A}} f\left([G,o,W]\right)\mathrm{d}\rho\left([G,o,W]\right)\,.$$

Proof. Both items follow from the compactness of $\mathcal{G}_*^{D,A}$, see [12, Chapter 13]. $\square$

The previous lemma gives a convenient criterion to prove that (G_n,W_n) has a local weak limit. However, it may not be very clear how a continuous function on $\mathcal{G}_*^{D,A}$ looks like. We start with the special case where $A=0$, i.e. without coloring.

Lemma 2.6. *Let (G_n) be a sequence of finite graphs, $G_n=(V_n,E_n)$, with degree uniformly bounded by D. Then (G_n) has a local weak limit ρ iff for any r-ball $B_F(o,r)$,*

$$\lim_{n\to\infty}\frac{\#\{x : B_{G_n(x)}(x,r)\cong B_F(o,r)\}}{|V_n|} = \rho\left(\{[H,x] : B_H(x,r)\cong B_F(o,r)\}\right)\,.$$

Here, $B_{G_n(x)}(x,r)\cong B_F(o,r)$ means there exists $\varphi : B_{G_n(x)}(x,r)\xrightarrow{\sim} B_F(o,r)$.

Proof. Let $\mathcal{C}_{[F_r,o]} = \{[H,x] : B_H(x,r)\cong B_F(o,r)\}$. We first note that $\mathcal{C}_{[F_r,o]}$ is a clopen subset of $\mathcal{G}_*^{D,0}$. Indeed, given $[H_1,x_1]\in\mathcal{C}_{[F_r,o]}$ and $[H_2,x_2]\in\mathcal{G}_*^{D,0}$, denote $\alpha_{1,2}=\sup\{s\in\mathbb{N} : B_{H_1}(x_1,s)\cong B_{H_2}(x_2,s)\}$. Then for any $[H_2,x_2]$ with $\alpha_{1,2}\ge r$, we have $[H_2,x_2]\in$

$\mathcal{C}_{[F_r,o]}$. Hence, $d_{loc}([H_1,x_1],[H_2,x_2]) \le \frac{1}{1+r}$ implies $[H_2,x_2] \in \mathcal{C}_{[F_r,o]}$, so $\mathcal{C}_{[F_r,o]}$ is open. Next, suppose $([H_n,x_n]) \subset \mathcal{C}_{[F_r,o]}$ converges to some $[H,x]$. Then if $\alpha_n = \sup\{s \in \mathbb{N} : B_{H_n}(x_n,s) \cong B_H(x,s)\}$, we may find n_r such that $\alpha_n \ge r$ for all $n \ge n_r$. In particular, $B_H(x,r) \cong B_{H_{n_r}}(x_{n_r},r) \cong B_F(o,r)$, so $[H,x] \in \mathcal{C}_{[F_r,o]}$. Hence $\mathcal{C}_{[F_r,o]}$ is closed.

Note that $U_{G_n}(\mathcal{C}_{[F_r,o]}) = \frac{\#\{x : B_{G_n(x)}(x,r) \cong B_F(o,r)\}}{|V_n|}$. Since $\mathcal{C}_{[F_r,o]}$ is clopen, $U_{G_n}(\partial\mathcal{C}_{[F_r,o]}) = 0$, so if U_{G_n} converges weakly to ρ, then $U_{G_n}(\mathcal{C}_{[F_r,o]}) \to \rho(\mathcal{C}_{[F_r,o]})$ for any $B_F(o,r)$.

For the converse, note that since $\mathcal{C}_{[F_r,o]}$ is clopen, its indicator function $\chi_{\mathcal{C}_{[F_r,o]}}$ is continuous. Next, $\mathcal{C}_{[F_{r_1},o]} \cap \mathcal{C}_{[F'_{r_2},o']} = \mathcal{C}_{[F_{r_1},o]}$ if $B_F(o,r_1) \cong B_{F'}(o',r_2)$ and is empty otherwise, so $\chi_{\mathcal{C}_{[F_{r_1},o]}}\chi_{\mathcal{C}_{[F'_{r_2},o']}} = \chi_{\mathcal{C}_{[F_{r_1},o]}}$ or 0. Now if the limit in the lemma holds for any $B_F(o,r)$, this means that in Lemma 2.5(2), the limit is true for any $\chi_{\mathcal{C}_{[F_r,o]}}$. This implies is also holds for linear combinations thereof. Finally, it trivially holds for the constant functions 1 and 0. So the limit holds for the algebra of functions $\mathscr{A} = \{\alpha\chi_{\mathcal{C}_{[F_{r_1},o]}} + \beta\chi_{\mathcal{C}_{[F'_{r_2},o']}}\} \cup \{0,1\}$. Note that $\mathscr{A}$ separates points: if $[G,o] \neq [G',o']$, then $d_{loc}([G,o],[G',o']) > 0$, so we may find r such that $B_G(o,r)$ is not isomorphic to $B_{G'}(o,r)$ as rooted graphs. Taking $B_F(o,r) = B_G(o,r)$, we get $\chi_{\mathcal{C}_{[F_r,o]}}([G,o]) = 1$ but $\chi_{\mathcal{C}_{[F_r,o]}}([G',o']) = 0$. It now follows from by Lemma 2.5(2) that U_{G_n} converges weakly to ρ. ☐

We now discuss the general case. We first have a partial analogy with Lemma 2.6.

Lemma 2.7. *Let (G_n,W_n) be a sequence of finite colored graphs, $G_n = (V_n,E_n)$, with degree uniformly bounded by D and coloring $W_n : V_n \to [-A,A]$ for all n. If (G_n,W_n) has a local weak limit ρ, then for any $r \in \mathbb{N}$ and any r-ball $(B_F(o,r),o,W_F)$,*

$$\lim_{n\to\infty} \frac{\#\{x : \exists\varphi_n^x : B_{G_n(x)}(x,r) \xrightarrow{\sim} B_F(o,r) \text{ with } \|W_F\circ\varphi_n^x - W_n\|_{B_{G_n(x)}(x,r)} < 1/r\}}{|V_n|}$$
$$= \rho\left(\{[H,x,W] : \exists\varphi : B_H(x,r) \xrightarrow{\sim} B_F(o,r) \text{ with } \|W_F\circ\varphi - W\|_{B_H(x,r)} < 1/r\}\right).$$

Proof. Given an r-ball $(B_F(o,r),o,W_F)$ with $r \in \mathbb{N}$, let

$$\mathcal{C}_F = \{[H,x,W] : \exists\varphi : B_H(x,r) \xrightarrow{\sim} B_F(o,r) \text{ with } \|W_F\circ\varphi - W\|_{B_H(x,r)} < 1/r\}.$$

Then

$$\mathcal{C}_F = \left\{[H,x,W] : d_{loc}([F,o,W_F],[H,x,W]) \le \frac{1}{1+r}\right\}.$$

Hence, $\mathcal{C}_F$ is closed. It is also open: if $[H,x,W] \in \mathcal{C}_F$, then there exists $\varphi : B_F(o,r) \xrightarrow{\sim} B_H(x,r)$ with $\|W\circ\varphi - W_F\|_{B_F(o,r)} < 1/r$. Choose $s \in \mathbb{N}$, $s > r$, such that $0 < \frac{1}{s} < \frac{1}{r} - \|W\circ\varphi - W_F\|_{B_F(o,r)}$. If $d_{loc}([H,x,W],[H',x',W']) < \frac{1}{1+s}$, there exists $\psi : B_H(x,s) \xrightarrow{\sim} B_{H'}(x',s)$ with $\|W'\circ\psi - W\|_{B_H(x,r)} < 1/s$. As $\|W'\circ\psi - W\|_{B_H(x,r)} = \|W'\circ\psi\circ\varphi - W\circ\varphi\|_{B_F(o,r)}$, it follows that $\|W'\circ\psi\circ\varphi - W_F\|_{B_F(o,r)} < 1/s + \|W\circ\varphi - W_F\|_{B_F(o,r)} < 1/r$. But $\psi\circ\varphi : B_F(o,r) \xrightarrow{\sim} B_{H'}(x',r)$ since $s > r$. Hence, $[H',x',W'] \in \mathcal{C}_F$ and $\mathcal{C}_F$ is open.

Note that $U_{(G_n,W_n)}(\mathcal{C}_F) = \frac{\#\{x : [B_{G_n(x)}(x,r),x,W] \in \mathcal{C}_F\}}{|V_n|}$. Since $\mathcal{C}_F$ is clopen, $U_{(G_n,W_n)}(\partial\mathcal{C}_F) = 0$, so if $U_{(G_n,W_n)}$ converges weakly to ρ, then $U_{G_n}(\mathcal{C}_F) \to \rho(\mathcal{C}_F)$. ☐

To obtain a converse, one needs to assume moreover that the limit holds for all elements of the form $\mathcal{C}_{F_1} \cap \mathcal{C}_{F_2}$, in order to argue as before.

Under the hypotheses of the lemma, it is also true that

$$\lim_{n\to\infty} \frac{\#\{x : B_{G_n(x)}(x,r) \cong B_F(o,r)\}}{|V_n|} = \rho\left(\{[H,x,W] : B_H(x,r) \cong B_F(o,r)\}\right),$$

since the sets on the RHS are still clopen. So as expected, if (G_n,W_n) converge as colored graphs, they converge in particular as graphs without coloring.

3. Examples

We start with simple examples without coloring.

3.1. **Cycle graphs.** The cycle graph with n vertices C_n converges to $\mathbb{Z}$ in the sense of Benjamini-Schramm. More precisely, C_n has the local weak limit $\delta_{[\mathbb{Z},o]}$, where $o \in \mathbb{Z}$ is an arbitrary root. Indeed, given $r \in \mathbb{N}$, if $B_F(o,r)$ is isomorphic to an r-ball in $\mathbb{Z}$, then $\frac{\#\{x : B_{C_n}(x,r) \cong B_F(o,r)\}}{n} = 1$ for all $n > r$. If $B_F(o,r)$ is not isomorphic to an r-ball in $\mathbb{Z}$, then $\frac{\#\{x : B_{C_n}(x,r) \cong B_F(o,r)\}}{n} = 0$ for all $n > r$. So the limit of $\frac{\#\{x : B_{C_n}(x,r) \cong B_F(o,r)\}}{n}$ is 1 (or 0) if $B_F(o,r)$ is isomorphic to an r-ball in $\mathbb{Z}$ (or not). Since $\delta_{[\mathbb{Z},o]}(\{[H,x] : B_H(x,r) \cong B_F(o,r)\})$ has the same values then the claim follows from Lemma 2.6.

3.2. **Lattice cubes.** The cubes $\Lambda_n = \{1,\dots,n\}^d$ converge to $\mathbb{Z}^d$ in the sense of Benjamini-Schramm. Indeed, if $B_F(o,r)$ is isomorphic to an r-ball in $\mathbb{Z}^d$, then $\frac{\#\{x : B_{\Lambda_n}(x,r) \cong B_F(o,r)\}}{n^d} = \frac{(n-2r)^d}{n^d} \to 1$. Otherwise, $\frac{\#\{x : B_{\Lambda_n}(x,r) \cong B_F(o,r)\}}{n^d} \le \frac{(2r)^d}{n^d} \to 0$. It follows as before that U_{Λ_n} has the local weak limit $\delta_{[\mathbb{Z}^d,o]}$, where $o \in \mathbb{Z}^d$ is arbitrary.

3.3. **Regular graphs with few cycles.** Let $G_N = (V_N, E_N)$ be a sequence of $(q+1)$-regular connected graphs with $|V_N| = N$. As in [4, 5], we define the property

(BST) For all $R > 0$,

$$\lim_{N\to\infty} \frac{|\{x \in V_N : \rho_{G_N}(x) < R\}|}{N} = 0\,,$$

where $\rho_{G_N}(x)$ is the *injectivity radius* at x, i.e. the largest ρ such that $B_{G_N}(x,\rho)$ is a tree. This property holds in particular if the girth of G_N grows to infinity.

We claim that (G_N) satisfies **(BST)** iff (G_N) converges to the $(q+1)$-regular tree $\mathcal{T}_q$ in the sense of Bejamini-Schramm, i.e. iff (G_N) has the local weak limit $\delta_{[\mathcal{T}_q,o]}$, where $o \in \mathcal{T}_q$ is an arbitrary root.

Indeed, let $B_F(o,r)$ be an r-ball and assume (G_N) satisfies **(BST)**. If $B_F(o,r)$ is isomorphic to a ball in $\mathcal{T}_q$, then $\frac{\#\{x : B_{G_N}(x,r) \cong B_F(o,r)\}}{|V_N|} = \frac{\#\{x : \rho_{G_N}(x) \ge r\}}{N} \to 1$. If $B_F(o,r)$ is not isomorphic to a ball in $\mathcal{T}_q$, then $\frac{\#\{x : B_{G_N}(x,r) \cong B_F(o,r)\}}{|V_N|} \le \frac{\#\{x : \rho_{G_N}(x) < r\}}{N} \to 0$. It follows as before that (G_N) has the local weak limit $\delta_{[\mathcal{T}_q,o]}$.

Conversely, if (G_N) has the local weak limit $\delta_{[\mathcal{T}_q,o]}$, given $R > 0$, pick a ball $B_F(o,R)$ in $\mathcal{T}_q$, Then $\frac{\#\{x : \rho_{G_N}(x) \ge R\}}{N} = \frac{\#\{x : B_{G_N}(x,R) \cong B_F(o,R)\}}{|V_N|} \to 1$, so **(BST)** follows.

3.4. **Graphs with bounded degree.** If we assume that $G_N = (V_N, E_N)$ is a sequence of graphs, $|V_N| = N$, with degree uniformly bounded by D, then **(BST)** no longer guarantees convergence. For instance if G_{2N} are 3-regular and G_{2N+1} are 4-regular, and if (G_N) satisfies **(BST)**, then G_{2N} will converge to $\mathcal{T}_2$ while G_{2N+1} will converge to $\mathcal{T}_3$.

3.5. **Tree balls.** Fix a root o in the $(q+1)$-regular tree $\mathcal{T}_q$ and let $G_N = B_{\mathcal{T}_q}(o,N)$. Then (G_N) does not converge to $\mathcal{T}_q$ in the sense of Benjamini-Schramm. Indeed, if $B_F(o,r)$ is isomorphic to a ball in $\mathcal{T}_q$, then $\frac{\#\{x : B_{G_N}(x,r) \cong B_F(o,r)\}}{|V_N|} = \frac{|V_{N-r}|}{|V_N|} \to \frac{1}{q^r}$, since $|V_n| = 1 + (q+1)\sum_{j=1}^n q^{j-1} = 1 + (q+1)\frac{1-q^n}{1-q}$, so that $\frac{|V_{N-r}|}{|V_N|} = \frac{(q-1)q^{-N} + (q+1)(q^{-r} - q^{-N})}{(q-1)q^{-N} + (q+1)(1-q^{-N})} \to q^{-r}$. This already shows the local weak limit cannot be $\mathcal{T}_q$. For $B_F(o,r)$ which are not isomorphic to an r-ball in $\mathcal{T}_q$, the value of $\frac{\#\{x : B_{G_N}(x,r) \cong B_F(o,r)\}}{|V_N|}$ is 0 if $B_F(o,r)$ is also not isomorphic to any $B_{G_N}(x,r)$ with $x \in S_n$, $N - r + 1 \le n \le N$, where S_n is the n-th sphere. If $B_F(o,r)$ is isomorphic to $B_{G_N}(x,r)$ with $x \in S_{N-j+1}$, then $\frac{\#\{x : B_{G_N}(x,r) \cong B_F(o,r)\}}{|V_N|} = \frac{|S_{N-j+1}|}{|V_N|} = \frac{(q+1)q^{N-j}}{1 + \frac{q+1}{q-1}(q^N - 1)} \to \frac{q-1}{q^j}$. Based on this information, we construct a random rooted tree (T_q^*, o) called the *canopy tree,* cf. [2] and [11, Chapter 14] :

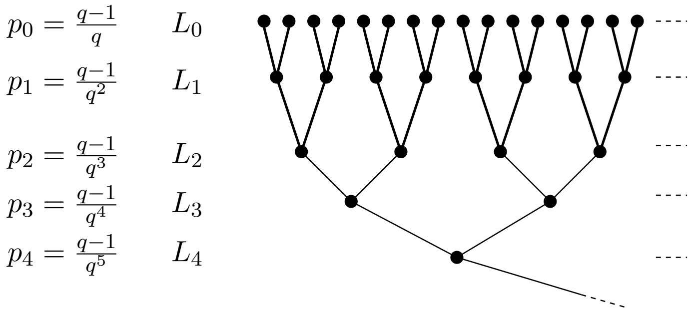

FIGURE 1. The canopy tree, as introduced in [2].

This is a fixed infinite tree T_q^*; the randomness comes from the root. More precisely, this tree is not transitive, so the position of the root matters. We divide the tree into infinite levels $(L_j)_{j=0}^{\infty}$ as in Figure 1. For fixed j, all trees (T_q^*, o) with $o \in L_j$ are equivalent. By some abuse of notation we let $[T_q^*, o] = L_j$ in this case. We then define $\rho \in \mathcal{P}(\mathcal{G}_*^{D,A})$ by $\rho = \sum_{j=0}^{\infty} \frac{q-1}{q^{j+1}} \delta_{L_j}$. This is indeed a probability measure, since $\sum_{j=0}^{\infty} \frac{q-1}{q^{j+1}} = 1$. By construction, $B_{T_q^*}(o, r)$ is isomorphic to an r-ball $B_F(o, r)$ in $\mathcal{T}_q$ precisely when o is in any level L_j with $j \geq r$. In other words, $\rho(\{[H, x, W] : B_H(x, r) \cong B_F(o, r)\}) = \sum_{j=r}^{\infty} \frac{q-1}{q^{j+1}} = \frac{q-1}{q^{r+1}} \sum_{j=0}^{\infty} q^{-j} = \frac{q-1}{q^{r+1}} \frac{q}{q-1} = \frac{1}{q^r}$, which is the limiting value we obtained along G_N. Similarly, if $B_F(o, r)$ is not isomorphic to any $B_{G_N}(x)$ with $x \in S_n$, $N - r + 1 \leq n \leq N$, we find $\rho(\{[H, x, W] : B_H(x, r) \cong B_F(o, r)\}) = 0$, and if $B_F(o, r)$ is isomorphic to $B_{G_N}(x, r)$ with $x \in S_{N-j+1}$, then $\rho(\{[H, x, W] : B_H(x, r) \cong B_F(o, r)\}) = \frac{q-1}{q^j} \delta_{L_{j-1}}(L_{j-1}) = \frac{q-1}{q^j}$. This completes the proof that ρ is the local weak limit of (G_N).

We now turn to colored graphs.

3.6. **Colored graphs with few cycles.** Let (G_N, W_N) be a sequence of colored connected graphs $G_N = (V_N, E_N)$ with degree uniformly bounded by D, coloring $W_N : V_N \to [-A, A]$ for all N and $|V_N| = N$. Let $\mathcal{T}_*^{D,A}$ be the subset of colored rooted trees in $\mathcal{G}_*^{D,A}$.

We show that if (G_N, W_N) has a local weak limit ρ which is concentrated on $\mathcal{T}_*^{D,A}$, then (G_N) satisfies **(BST)**. Conversely, if (G_N) satisfies **(BST)**, and if (G_{N_j}, W_{N_j}) is a subsequence with a local weak limit ρ, then ρ is concentrated on $\mathcal{T}_*^{D,A}$.

Indeed, we may assume the R in **(BST)** is in $\mathbb{N}^*$. We observe that $\frac{\#\{x : \rho_{G_N}(x) < R\}}{N} = \frac{\#\{x : B_{G_N}(x, R) \text{ is not a tree }\}}{N} = U_{(G_N, W_N)}(\{[H, x, W] : B_H(x, R) \text{ is not a tree }\})$. Let $\mathscr{C}_R = \{[H, x, W] : B_H(x, R) \text{ is not a tree }\}$. The $\mathscr{C}_R$ is clopen. Indeed, if $[H, x, W] \in \mathscr{C}_R$ and $d_{loc}[(H, x, W), (H', x', W')] < \frac{1}{1+R}$, then $B_{H'}(x', R) \cong B_H(x, R)$, so $[H', x', W'] \in \mathscr{C}_R$. Hence, $\mathscr{C}_R$ is open. If $[H_n, x_n, W_n] \subset \mathscr{C}_R$ converges to some $[H, x, W]$, then $B_H(x, R) \cong B_{H_n}(x_n, R)$ for all $n \geq n_R$ and thus $[H, x, W] \in \mathscr{C}_R$. So $\mathscr{C}_R$ is closed.

If (G_N, W_N) has a local weak limit ρ concentrated on $\mathcal{T}_*^{D,A}$, then $U_{(G_N, W_N)}(\mathscr{C}_R) \to \rho(\mathscr{C}_R) = 0$. Hence (G_N) satisfies **(BST)**. Conversely, if (G_N) satisfies **(BST)** and ρ is the local weak limit of a subsequence, then we need to show that for any $\mathscr{M} \subset \mathcal{G}_*^{D,A}$, we have $\rho(\mathscr{M}) = \rho(\mathscr{M} \cap \mathcal{T}_*^{D,A})$. For this, note that $(\mathcal{T}_*^{D,A})^c = \cup_{R \in \mathbb{N}} \mathscr{C}_R$, with $\mathscr{C}_1 \subseteq \mathscr{C}_2 \subseteq \dots$. Hence, $\rho[(\mathcal{T}_*^{D,A})^c] = \lim_{R \to \infty} \rho(\mathscr{C}_R)$. But by hypothesis, $\rho(\mathscr{C}_R) = \lim_{j \to \infty} U_{(G_{N_j}, W_{N_j})}(\mathscr{C}_R) = \lim_{j \to \infty} \frac{\#\{x : \rho_{N_j}(x) < R\}}{N_j} = 0$. So ρ is concentrated on $\mathcal{T}_*^{D,A}$.

3.7. **The Anderson model.** In this example we closely follow our presentation in [7].

Let $\Omega = [-A,A]^{\mathbb{Z}^d}$ and define $\mathbb{P}$ on Ω by $\mathbb{P} = \otimes_{v\in\mathbb{Z}^d}\nu$ for some probability measure ν on $[-A,A]$. Given $\omega = (\omega_v) \in \Omega$, define $W^\omega(v) = \omega_v$ for $v \in \mathbb{Z}^d$. Then the $\{\omega_v\}_{v\in\mathbb{Z}^d}$ are i.i.d. random variables with common distribution ν.

Let $\Lambda_n = \{1,\ldots,n\}^d$. Given $\omega\in\Omega$, we define $W_n^\omega(v) = \omega_v$ for $v \in \Lambda_n$.

We will show that for $\mathbb{P}$-a.e. ω, the sequence of graphs (Λ_n, W_n^ω) has a local weak limit ρ which is concentrated on $\{[\mathbb{Z}^d, 0, W^\omega] : \omega\in\Omega\}$, and acts by taking the expectation w.r.t. $\mathbb{P}$. More precisely, denoting $D = 2d$,

$$\int_{\mathcal{G}_*^{D,A}} f([G,o,W])\,\mathrm{d}\rho([G,o,W]) = \mathbb{E}[f([\mathbb{Z}^d,0,W^\omega])]\,.$$

Let $\mathscr{A} = \cup_{r\in\mathbb{N}}\mathscr{A}_r$, where

$$\begin{aligned}\mathscr{A}_r = \big\{f \in C(\mathcal{G}_*^{D,A}) : f([G,o,W]) = f([G',o',W'])&\\ \text{if } [B_G(o,r),o,W] = [B_{G'}(o',r),o',W']\big\}&\,.\end{aligned}$$

Then $\mathscr{A}$ is an algebra of continuous functions containing 1. To see that it separates points, let $[G,o,W] \neq [G',o',W']$. Then we may find $r\in\mathbb{N}$ with $d_{loc}([G,o,W],[G',o',W']) > \frac{1}{1+r}$. Define $\mathcal{C}_G = \{[H,x,V] : d_{loc}([G,o,W],[H,x,V]) \le \frac{1}{1+r}\}$. We showed in Lemma 2.7 that $\chi_{\mathcal{C}_G}$ is continuous. Moreover, $\chi_{\mathcal{C}_G}([H,x,V]) = \chi_{\mathcal{C}_G}([H',x',V'])$ if $[B_H(o,r),o,V] = [B_{H'}(o',r),o',V']$. Indeed, if $\chi_{\mathcal{C}_G}([H,x,V]) = 1$, there is $\varphi : B_H(x,r) \xrightarrow{\sim} B_G(o,r)$ with $\|W\circ\varphi - V\|_{B_H(x,r)} < 1/r$. If $\psi : B_{H'}(x',r) \xrightarrow{\sim} B_H(o,r)$ with $V\circ\psi = V'$, then $\varphi\circ\psi : B_{H'}(x',r) \xrightarrow{\sim} B_G(o,r)$ has $\|W\circ\varphi\circ\psi - V'\| < 1/r$ and thus $\chi_{\mathcal{C}_G}([H',x',V']) = 1$. Similarly, if $\chi_{\mathcal{C}_G}([H,x,V]) = 0$, no such φ exists and $\chi_{\mathcal{C}_G}([H',x',V']) = 0$. Hence, $\chi_{\mathcal{C}_G} \in \mathscr{A}_r$. Finally, $\chi_{\mathcal{C}_G}([G,o,W]) = 1$ while $\chi_{\mathcal{C}_G}([G',o',W']) = 0$. We thus showed that $\mathscr{A}$ separates points.

Using Lemma 2.5, it now suffices to show that there exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for any $\omega\in\Omega_0$ and any $f\in\mathscr{A}$, we have

$$\lim_{n\to\infty}\frac{1}{n^d}\sum_{x\in\Lambda_n} f([\Lambda_n,x,W_n^\omega]) = \mathbb{E}[f([\mathbb{Z}^d,0,W^\omega])]\,. \tag{3.1}$$

For this, we first adapt the strong law of large numbers in [10, Theorem 2.3.5]. Given $f\in\mathscr{A}_r$, let

$$Y_x = Y_x^{(n)} = f([\Lambda_n,x,W_n^\omega]) - \mathbb{E}[f([\Lambda_n,x,W_n^\omega])] \quad\text{and}\quad S_n = \frac{1}{n^d}\sum_{x\in\Lambda_n} Y_v\,.$$

Then $\mathbb{E}[Y_x] = 0$. Moreover, Y_x only depends on $(\omega_z)_{z\in B_{\Lambda_n}(x,r)}$, since $f([\Lambda_n,x,W_n^\omega]) = f([\Lambda_n,x,\tilde{W}_n^\omega])$ if $W_n^\omega = \tilde{W}_n^\omega$ on $B_{\Lambda_n}(x,r)$. It follows that Y_x and Y_y are independent if $d_{\Lambda_n}(x,y) > 2r$. Now

$$\begin{aligned}\mathbb{E}\Big[\sum_{x\in\Lambda_n}Y_x\Big]^4 &= \sum_{x\in\Lambda_n}\mathbb{E}(Y_x^4) + 6\sum_{x,y\in\Lambda_n}\mathbb{E}(Y_x^2Y_y^2) + 4\sum_{x,y\in\Lambda_n}\mathbb{E}(Y_xY_y^3 + Y_yY_x^3)\\ &\quad + 12\sum_{x,y,z\in\Lambda_n}\mathbb{E}(Y_xY_yY_z^2 + Y_xY_y^2Y_z + Y_x^2Y_yY_z) + 24\sum_{x,y,z,t\in\Lambda_n}\mathbb{E}(Y_xY_yY_zY_t)\,.\end{aligned}$$

The first three sums are $O(n^d)$ and $O(n^{2d})$. For the fourth, note that $\mathbb{E}(Y_xY_yY_z^2) = 0$ if $d(x,y) > 4r$, since either $d(x,z) > 2r$ and Y_x is independent of the pair (Y_y,Y_z), or $d(y,z) > 2r$ and Y_y is independent of (Y_x,Y_z). Thus, we have either $\mathbb{E}(Y_xY_yY_z^2) = \mathbb{E}(Y_x)\,\mathbb{E}(Y_yY_z^2) = 0$ or $\mathbb{E}(Y_yY_xY_z^2) = \mathbb{E}(Y_y)\,\mathbb{E}(Y_xY_z^2) = 0$. Hence, $|\sum_{x,y,z\in\Lambda_n}\mathbb{E}(Y_xY_yY_z^2)| \le n^{2d}(4r)^d(2\|f\|_\infty)^4$. The other terms of this sum are treated similarly.

Finally, for $\mathbb{E}(Y_xY_yY_zY_t)$ to be non zero, each point must be at distance $\le 2r$ from one of the three others. Hence, we must have $[d(x,y)\le 2r$ and $d(z,t)\le 2r]$ (or a permutation thereof) or $[d(x,\bullet) \le 8r$ for $\bullet = y,z,t.]$ It follows that $\sum_{x,y,z,t\in\Lambda_n} |\mathbb{E}(Y_xY_yY_zY_t)| \le 3n^{2d}(2r)^{2d}(2\|f\|_\infty)^4 + n^d(8r)^{3d}(2\|f\|_\infty)^4$. In any case $\mathbb{E}(|S_n|^4) \le C_{r,f,d}n^{-2d}$.

By the Borel-Cantelli Lemma, if $A^\varepsilon_{n,f} = \{|S_n| > \varepsilon\}$, then $\mathbb{P}(A^\varepsilon_{n,f} \text{ i.o.}) = 0$. Thus, if

$$\Omega^\varepsilon_{0,f} = \{A^\varepsilon_{n,f} \text{ occurs finitely often}\}\,,$$

we have $\mathbb{P}(\Omega^\varepsilon_{0,f}) = 1$. Since $C(\mathcal{G}_*^{D,A})$ is separable, $\mathscr{A}$ is separable, and we may choose a countable dense subset $\{f_j\} \subset \mathscr{A}$. We then let $\Omega_0 = \cap_{\varepsilon\in\mathbb{Q}^+}\cap_{j\in\mathbb{N}}\Omega^\varepsilon_{0,f_j}$. Then $\mathbb{P}(\Omega_0) = 1$.

Let $\omega \in \Omega_0$. Given $j \in \mathbb{N}$ and $\varepsilon > 0$ let $0 < \varepsilon' < \varepsilon$, $\varepsilon' \in \mathbb{Q}^+$. Then $\omega \in \Omega^{\varepsilon'}_{0,f_j}$, so there is n_ω such that $|S_n| \le \varepsilon' < \varepsilon$ for any $n > n_\omega$. Hence, $S_n \to 0$ for any $\omega \in \Omega_0$.

Now if $f \in \mathscr{A}$, say $f \in \mathscr{A}_r$, we have

$$\begin{aligned}&\Big|\frac{1}{n^d}\sum_{x\in\Lambda_n} f([\Lambda_n,x,W^\omega_n]) - \mathbb{E}[f([\mathbb{Z}^d,0,W^\omega])]\Big|\\ &\qquad\le |S_n| + \Big|\frac{1}{n^d}\sum_{x\in\Lambda_n}\mathbb{E}[f([\Lambda_n,x,W^\omega_n])] - \mathbb{E}[f([\mathbb{Z}^d,0,W^\omega])]\Big|\,.\end{aligned}$$

Assume $n > r$. If $x \in \{r+1,\dots,n-r\}^d =: C^r_n$, there is $\varphi : B_{\Lambda_n}(x,r) \xrightarrow{\sim} B_{\mathbb{Z}^d}(0,r)$. In fact, we take $\varphi(v) = v - x$. Denoting $W^\omega_x(v) = W^\omega(v+x)$, we get $[B_{\Lambda_n}(x,r),x,W^\omega_n] = [B_{\mathbb{Z}^d}(0,r),0,W^\omega_x]$, so $f([\Lambda_n,x,W^\omega_n]) = f([\mathbb{Z}^d,0,W^\omega_x])$. By usual measure-preserving transformations, we check that $\mathbb{E}[f([\mathbb{Z}^d,0,W^\omega_x])] = \mathbb{E}[f([\mathbb{Z}^d,0,W^\omega])]$. Hence,

$$\begin{aligned}&\Big|\frac{1}{n^d}\sum_{x\in\Lambda_n} f([\Lambda_n,x,W^\omega_n]) - \mathbb{E}[f([\mathbb{Z}^d,0,W^\omega])]\Big|\\ &\qquad\le |S_n| + \frac{1}{n^d}\sum_{x\notin C^r_n}\Big|\mathbb{E}[f([\Lambda_n,x,W^\omega_n])] - \mathbb{E}[f([\mathbb{Z}^d,0,W^\omega])]\Big| \le |S_n| + \frac{(2r)^d}{n^d}(2\|f\|_\infty)\,.\end{aligned}$$

Taking $n \to \infty$, it follows that if $\omega \in \Omega_0$, then (3.1) is true for any $f \in \{f_j\}$, the dense subset of $\mathscr{A}$. Arguing as in [12, Corollary 15.3], the proof is complete.

4. Convergence of spectral measures

Our aim in this section is to show that the Benjamini-Schramm convergence implies the convergence of the *mean* spectral measures. This can be interpreted as the assertion that the integrated density of states of the limit operator has a finite-volume approximation. Though this can be proved directly, we will first prove a convergence result for *rooted* spectral measures as in [13, Chapter 2], which is of independent interest.

Let $[G,o,W] \in \mathcal{G}_*^{D,A}$, let $\mathcal{A}$ be the adjacency matrix on G and define the (Schrödinger) operator $H = \mathcal{A} + W$. This is a bounded self-adjoint operator. We sometimes denote $H = H_{(G,W)}$ to avoid confusion. We define the rooted spectral measure $\mu_o^{(G,W)}$ by

$$\mu_o^{(G,W)}(J) = \langle\delta_o, \chi_J(H)\delta_o\rangle \qquad \text{for Borel } J \subseteq \mathbb{R}\,.$$

Lemma 4.1. *Suppose* $[G_n,o_n,W_n] \subset \mathcal{G}_*^{D,A}$ *converges to* $[G,o,W]$ *in the metric topology of* $(\mathcal{G}_*^{D,A}, d_{loc})$. *Then* $\mu_{o_n}^{(G_n,W_n)}$ *converges weakly to* $\mu_o^{(G,W)}$. *So for any continuous* $\varphi : \mathbb{R} \to \mathbb{R}$, *we have*

$$\lim_{n\to\infty}\langle\delta_{o_n}, \varphi(H_{(G_n,W_n)})\delta_{o_n}\rangle = \langle\delta_o, \varphi(H_{(G,W)})\delta_o\rangle\,.$$

Proof. Since all operators $H_n = \mathcal{A}_n + W_n$ and $H = \mathcal{A} + W$ are uniformly bounded by some $A + D$, the supports of the spectral measures is compact, so it suffices to show that for any $k \in \mathbb{N}$, $\int t^k\mathrm{d}\mu_{o_n}(t) \to \int t^k\mathrm{d}\mu_o(t)$; see [12, Chapter 13].

Given $k \in \mathbb{N}$, choose an arbitrary integer $r \geq k$. Then we may find n_r such that for $n \geq n_r$, there exists $\varphi_r : B_{G_n}(o_n, r) \xrightarrow{\sim} B_G(o, r)$ with $\|W \circ \varphi_r - W_n\|_{B_{G_n}(o,r)} < 1/r$. Now

$$\int t^k \,\mathrm{d}\mu_{o_n}(t) = \langle \delta_{o_n}, H_n^k \delta_{o_n} \rangle = \sum_{u_0,\dots,u_{k-1}} H_n(o_n, u_0) H_n(u_0, u_1) \dots H_n(u_{k-1}, o_n)\,,$$

and $H_n(v, w) = (\mathcal{A}_n \delta_w)(v) + W_n(v)\delta_w(v)$. So the RHS only depends on $B_{G_n}(o_n, k)$ and its coloring. As $r \geq k$ and $\varphi_r : B_{G_n}(o_n, r) \xrightarrow{\sim} B_G(o, r)$, if we let $\mathcal{H}_n = \mathcal{A} + W_n \circ \varphi_r^{-1}$ on G, we get $\langle \delta_{o_n}, H_n^k \delta_{o_n} \rangle = \langle \delta_o, \mathcal{H}_n^k \delta_o \rangle$. So for $n \geq n_r$,

$$\begin{aligned} \left| \int t^k \,\mathrm{d}\mu_{o_n}(t) - \int t^k \,\mathrm{d}\mu_o(t) \right| &= |\langle \delta_o, (\mathcal{H}_n^k - H^k)\delta_o \rangle| = \left| \left\langle \delta_o, \sum_{i=1}^k \mathcal{H}_n^{k-i}(\mathcal{H}_n - H) H^{i-1} \delta_o \right\rangle \right| \\ &\leq C_{k,D,A} \|W_n \circ \varphi_r^{-1} - W\|_{B_G(o,r)} \leq \frac{C_{k,D,A}}{r}\,. \end{aligned}$$

Since $r \geq k$ is arbitrary, this completes the proof. □

If (G, W) if a finite colored graph, $G = (V, E)$, with degree uniformly bounded by D and coloring in $[-A, A]$, we define the mean spectral measure

$$\mu^{(G,W)} = \frac{1}{|V|} \sum_{x \in V} \mu_x^{(G,W)}\,.$$

Corollary 4.2. *Suppose (G_n, W_n) is a sequence of finite colored graphs with degrees uniformly bounded by D and coloring $W_n : V_n \to [-A, A]$ for all n. If (G_n, W_n) has a local weak limit ρ, then $\mu^{(G_n,W_n)}$ converges weakly to $\int_{\mathcal{G}_*^{D,A}} \mu_o^{(G,W)} \,\mathrm{d}\rho([G, o, W])$. So for any continuous $\varphi : \mathbb{R} \to \mathbb{R}$, we have*

$$\lim_{n\to\infty} \frac{1}{|V_n|} \operatorname{tr}[\varphi(H_{(G_n,W_n)})] = \int_{\mathcal{G}_*^{D,A}} \langle \delta_o, \varphi(H_{(G,W)})\delta_o \rangle \,\mathrm{d}\rho([G, o, W])\,.$$

Proof. Given continuous $\varphi : \mathbb{R} \to \mathbb{R}$, define the transform $\widehat{\varphi} : \mathcal{G}_*^{D,A} \to \mathbb{R}$ by $\widehat{\varphi}([G, o, W]) = \int \varphi(t) \,\mathrm{d}\mu_o^{(G,W)}(t)$. Then $\widehat{\varphi}$ is continuous on $\mathcal{G}_*^{D,A}$ by Lemma 4.1. It is also bounded since $\mathcal{G}_*^{D,A}$ is compact. By hypothesis, $U_{(G_n,W_n)}$ converges weakly to ρ. It follows that $\int \widehat{\varphi} \,\mathrm{d}U_{(G_n,W_n)} \to \int \widehat{\varphi} \,\mathrm{d}\rho$, i.e. $\frac{1}{|V_n|} \sum_{x \in V_n} \widehat{\varphi}([G_n, x, W_n]) \to \int \widehat{\varphi}([G, o, W]) \,\mathrm{d}\rho([G, o, W])$. Since $\widehat{\varphi}([G_n, x, W_n]) = \langle \delta_x, \varphi(H_{(G_n,W_n)})\delta_x \rangle$, the assertion follows. □

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IRMA, Université de Strasbourg, 7 rue René Descartes, 67084, Strasbourg, France.

Department of Mathematics, Faculty of Science, Cairo University, Cairo 12613, Egypt.
E-mail address: `sabri@math.unistra.fr`