

ORIGINAL ARTICLES

On the Transmuted Fréchet Distribution

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ABSTRACT

New parameters can be introduced to expand families of distributions for added flexibility or to construct covariate models and this could be done in various ways. In this paper, we will use the quadratic rank transmutation map (QRTM) in order to generate a flexible family of probability distribution taking Fréchet distribution as the base distribution. The new distribution is called the transmuted Fréchet distribution. Some properties of this distribution are studied. Also, Parameter estimation using maximum likelihood and Bayesian methods are discussed.

Key words: Fréchet distribution, quadratic rank transmutation map, moment generating function, maximum likelihood estimation and Bayesian estimation.

Introduction

Any statistical analysis depends greatly on the statistical model used to represent the phenomena under study. Hence, the larger the class of statistical models available to the statistician the easier it is to choose a model. A quick survey of the models in common use reveals the abundance of statistical models in the literature. However, data of many important and practical problems do not follow any of the probability models available. In such cases a non-parametric model may be recommended.

While a two parameter distribution may provide reasonably precision in fitting data, it may be still desirable to extend the flexibility of any distribution to allow for better description of data without having to resort to non-parametric models. The Fréchet distribution has applications ranging from accelerated life testing through earthquakes, floods, rainfall, horse racing, queues, sea currents, wind speeds and track race records. Kotz and Nadarajah (2000) give other applications in their book. In this paper, we introduce the transmuted Fréchet distribution which stems from the following idea. Shaw & Buckley (2009) considered composite maps of the following two forms: sample transmutation maps (STMs), $y = G^{-1}[F(x)]$, and rank transmutation maps (RTMs), $v = G[F^{-1}(w)]$, where F and G are cumulative distribution functions (CDFs). Gilchrist (2000) refers to STMs and RTMs as Q-transformation and P-transformation, respectively.

Shaw & Buckley (2009) focused on the RTM, which uses as a tool for the construction of new families of non-Gaussian distributions. They used it to modulate a given base distribution for the purposes of modifying the moments, in particular the skew and kurtosis. An attraction of the approach is that if the CDF and inverse CDF (or quantile function (QF)) are tractable for the base distribution, there is a good chance for the transmuted distribution to be so.

A random variable X is said to have transmuted distribution, according to the quadratic rank transmutation map (QRTM), if its cumulative distribution function is given by

$$F(x) = (1 + \lambda)G(x) - \lambda G(x)^2, \quad |\lambda| \leq 1 \quad (1)$$

where $G(x)$ is the cdf of the base distribution (see Shaw & Buckley (2009)). Note that when $\lambda=0$, the base distribution will be obtained. Aryal and Tsokos (2009, 2011) studied the transmuted extreme value distributions and provided the mathematical characterization of transmuted Gumbel and transmuted Weibull distributions and their applications to analyze real data set. Aryal (2013) proposed and studied the transmuted log-logistic distribution. He provided mathematical formulation of this distribution and some of its properties. Ashour and Eltehiwy (2013a, 2013b) studied the transmuted exponentiated modified Weibull and transmuted exponentiated Lomax distributions and discussed some properties of these families. Merovic (2013) introduced the transmuted Rayleigh distribution and provided some properties of this distribution.

2. Transmuted Fréchet Distribution:

A random variable X is said to have a Fréchet distribution with parameters $\mu > 0$ and $\sigma > 0$ if its pdf is given by

$$g(x) = \mu\sigma^\mu x^{-(\mu+1)} e^{-\left(\frac{\sigma}{x}\right)^\mu}, \quad x > 0 \tag{2}$$

The cdf of X is

$$G(x) = e^{-\left(\frac{\sigma}{x}\right)^\mu} \tag{3}$$

Now using (1) and (3), the cdf of transmuted Fréchet distribution with parameters μ, σ and λ takes the form

$$\begin{aligned} F(x) &= (1 + \lambda)e^{-\left(\frac{\sigma}{x}\right)^\mu} - \lambda \left[e^{-2\left(\frac{\sigma}{x}\right)^\mu} \right] \\ &= e^{-\left(\frac{\sigma}{x}\right)^\mu} \left[1 + \lambda - \lambda e^{-\left(\frac{\sigma}{x}\right)^\mu} \right], \quad |\lambda| \leq 1 \end{aligned} \tag{4}$$

and the pdf of transmuted Fréchet distribution with parameters μ, σ and λ becomes

$$f(x) = \frac{\mu}{\sigma} \left(\frac{\sigma}{x}\right)^{\mu+1} e^{-\left(\frac{\sigma}{x}\right)^\mu} \left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x}\right)^\mu} \right], \quad x > 0, \mu, \sigma > 0 \text{ and } |\lambda| \leq 1.$$

Note that: the Fréchet distribution is clearly a special case for $\lambda = 0$. The hazard rate function for the transmuted Fréchet random variables is given by

$$h(x) = \frac{\frac{\mu}{\sigma} \left(\frac{\sigma}{x}\right)^{\mu+1} e^{-\left(\frac{\sigma}{x}\right)^\mu} \left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x}\right)^\mu} \right]}{1 - e^{-\left(\frac{\sigma}{x}\right)^\mu} \left[1 + \lambda - \lambda e^{-\left(\frac{\sigma}{x}\right)^\mu} \right]}.$$

Figure1 illustrates some of the possible shapes of the pdf and hazard function of a transmuted Fréchet distribution for selected values of the parameters μ, σ and λ .

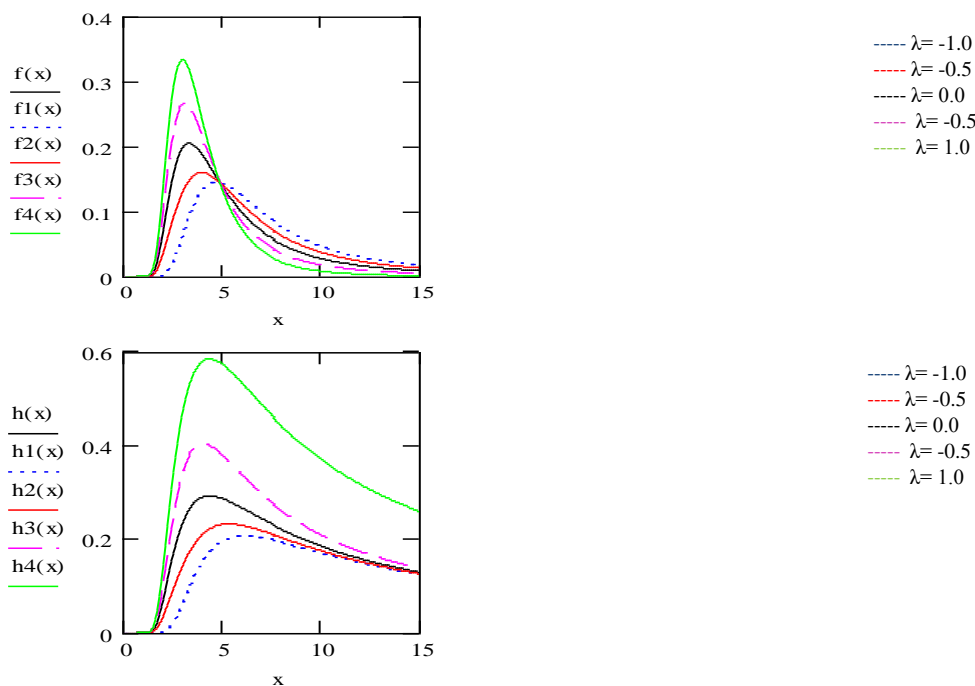


Fig.1: The pdf and hazard function of a transmuted Fréchet distribution for the parameters $\mu = 2, \sigma = 4$ and $\lambda = (-1, -0.5, 0, 0.5, 1)$.

3. Moment Generating Function:

The moment generating function of the transmuted Fréchet random variable is given by:

$$\begin{aligned}
 E(e^{tx}) &= \int_0^\infty e^{tx} \cdot f(x) dx \\
 &= \int_0^\infty e^{tx} \cdot \frac{\mu}{\sigma} \left(\frac{\sigma}{x}\right)^{\mu+1} e^{-\left(\frac{\sigma}{x}\right)^\mu} [1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x}\right)^\mu}] dx \\
 &= (1 + \lambda) \frac{\mu}{\sigma} \int_0^\infty e^{tx - \left(\frac{\sigma}{x}\right)^\mu} \left(\frac{\sigma}{x}\right)^{\mu+1} dx - 2\lambda \frac{\mu}{\sigma} \int_0^\infty e^{tx - 2\left(\frac{\sigma}{x}\right)^\mu} \left(\frac{\sigma}{x}\right)^{\mu+1} dx
 \end{aligned}$$

Using $e^{tx} = \sum_{i=0}^k \frac{t^i x^i}{i!}$
 the above integral become

$$E(e^{tx}) = (1 + \lambda) \frac{\mu}{\sigma} \sigma^{\mu+1} \int_0^\infty \sum_{i=0}^k \frac{t^i x^{i-\mu-1}}{i!} e^{-\left(\frac{\sigma}{x}\right)^\mu} dx - 2\lambda \frac{\mu}{\sigma} \sigma^{\mu+1} \int_0^\infty \sum_{i=0}^k \frac{t^i x^{i-\mu-1}}{i!} e^{-2\left(\frac{\sigma}{x}\right)^\mu} dx$$

Let $y = \left(\frac{\sigma}{x}\right)^\mu \Rightarrow x = \sigma y^{-\frac{1}{\mu}}, dx = -\frac{\sigma}{\mu} y^{-\frac{1}{\mu}-1} dy$.

Therefore

$$\begin{aligned}
 E(e^{tx}) &= (1 + \lambda) \mu \sigma^\mu \int_0^\infty \sum_{i=0}^k \frac{t^i}{i!} \left(\sigma y^{-\frac{1}{\mu}}\right)^{i-\mu-1} e^{-y} \left(\frac{\sigma}{\mu}\right) y^{-\frac{1}{\mu}-1} dy \\
 &\quad - 2\lambda \mu \sigma^\mu \int_0^\infty \sum_{i=0}^k \frac{t^i}{i!} \left(\sigma y^{-\frac{1}{\mu}}\right)^{i-\mu-1} e^{-2y} \left(\frac{\sigma}{\mu}\right) y^{-\frac{1}{\mu}-1} dy \\
 &= (1 + \lambda) \mu \sigma^\mu \sum_{i=0}^k \frac{t^i}{i!} \sigma^{i-\mu-1} \frac{\sigma}{\mu} \int_0^\infty y^{\frac{i}{\mu}-1} e^{-y} dy - 2\lambda \mu \sigma^\mu \sum_{i=0}^k \frac{t^i}{i!} \sigma^{i-\mu-1} \frac{\sigma}{\mu} \int_0^\infty y^{\frac{i}{\mu}-1} e^{-2y} dy \\
 &= (1 + \lambda) \sum_{i=0}^k \frac{(t\sigma)^i}{i!} \Gamma\left(1 - \frac{i}{\mu}\right) - 2\lambda \sum_{i=0}^k \frac{(t\sigma)^i}{i!} 2^{\frac{i}{\mu}} \Gamma\left(1 - \frac{i}{\mu}\right), \quad 1 - \frac{i}{\mu} > 0.
 \end{aligned}$$

From which

$$E(X^r) = \sigma^r \Gamma\left(1 - \frac{r}{\mu}\right) \left[(1 + \lambda) - \lambda 2^{\frac{r}{\mu}} \right], \quad 1 - \frac{r}{\mu} > 0 \tag{5}$$

Therefore, the mean and the variance of the transmuted Fréchet distribution are respectively

$$E(x) = \sigma \Gamma\left(1 - \frac{1}{\mu}\right) \left[1 + \lambda - \lambda 2^{\frac{1}{\mu}} \right], \quad \mu > 1$$

and

$$\text{var}(x) = \sigma^2 \left\{ \Gamma\left(1 - \frac{2}{\mu}\right) \left[1 + \lambda - \lambda 2^{\frac{2}{\mu}} \right] - \Gamma^2\left(1 - \frac{1}{\mu}\right) \left[1 + \lambda - \lambda 2^{\frac{1}{\mu}} \right]^2 \right\}, \quad \mu > 2.$$

Note that when $\lambda=0$ we have the mean and variance of Fréchet distribution. The skewness and kurtosis measures can be calculated using (5). Their variation for $\lambda=-1, -0.5, 0.5$ and 1 and $\sigma=1.5, \mu=5$ is illustrated in the following table.

	$\lambda=-1$	$\lambda=-0.5$	$\lambda=0$	$\lambda=0.5$	$\lambda=1$
mean	2.006	1.876	1.746	1.617	1.487
variance	0.397	0.366	0.301	0.202	0.070
skewness	3.535	3.402	3.535	3.854	1.502
kurtosis	45.092	42.442	45.092	54.692	5.103

4. Random Number Generation and Parameter Estimation (MLE):

For generation random numbers from the transmuted Fréchet distribution, one can use the method of inversion with the following formula

$$u = (1 + \lambda)e^{-\left(\frac{\sigma}{x}\right)^\mu} - \lambda e^{-2\left(\frac{\sigma}{x}\right)^\mu},$$

After simple calculation this yields

$$x = \sigma \left[-\ln \left(\frac{1 + \lambda - \sqrt{(1 + \lambda)^2 - 4\lambda u}}{2\lambda} \right) \right]^{\frac{1}{\mu}},$$

where u distributed as uniform distribution, $u \sim U[0, 1]$. The maximum likelihood estimates, MLEs, of the parameters for the transmuted Fréchet probability distribution function is given by the following:

Let X_1, X_2, \dots, X_n be a sample of size n from a transmuted Fréchet distribution. Then the likelihood function is given by

$$L = \left(\frac{\mu}{\sigma}\right)^n e^{-\sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\mu} \prod_{i=1}^n \left\{ \left(\frac{\sigma}{x_i}\right)^{\mu+1} \left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \right] \right\}$$

hence, the log-likelihood function $l = \ln L$ becomes

$$l = n \ln \mu - n \ln \sigma - \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\mu + (\mu + 1) \sum_{i=1}^n \ln \left(\frac{\sigma}{x_i}\right) + \sum_{i=1}^n \ln \left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \right] \quad (6)$$

Differentiate (6) with respect to μ , σ and λ . We have

$$\frac{\partial l}{\partial \mu} = \frac{n}{\mu} - \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\mu \ln \left(\frac{\sigma}{x_i}\right) + \sum_{i=1}^n \ln \left(\frac{\sigma}{x_i}\right) + \sum_{i=1}^n \frac{2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^\mu \ln \left(\frac{\sigma}{x_i}\right)}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \right]}, \quad (7)$$

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} - \sum_{i=1}^n \frac{\mu}{\sigma} \left(\frac{\sigma}{x_i}\right)^\mu + (\mu + 1) \frac{n}{\sigma} + \sum_{i=1}^n \frac{2\lambda \frac{\mu}{\sigma} \left(\frac{\sigma}{x_i}\right)^\mu e^{-\left(\frac{\sigma}{x_i}\right)^\mu}}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \right]}, \quad (8)$$

and

$$\frac{\partial l}{\partial \lambda} = \sum_{i=1}^n \frac{1 - 2e^{-\left(\frac{\sigma}{x_i}\right)^\mu}}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \right]}. \quad (9)$$

To obtain the maximum likelihood estimators $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\lambda}$, Equating the derivatives in (7), (8) and (9) to zero, then solve the three nonlinear equations numerically.

To obtain the elements of the information matrix we will get the second order derivative of logarithms of the likelihood functions as follow:

$$\begin{aligned} \frac{\partial^2 l}{\partial \mu^2} &= -\frac{n}{\mu^2} - \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\mu (\ln \left(\frac{\sigma}{x_i}\right))^2 + \sum_{i=1}^n 2\lambda \ln \left(\frac{\sigma}{x_i}\right) \left\{ \frac{e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^\mu \ln \left(\frac{\sigma}{x_i}\right) \left[1 - \left(\frac{\sigma}{x_i}\right)^\mu \right]}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \right]} - \frac{2\lambda \left(\frac{\sigma}{x_i}\right)^{2\mu} e^{-2\left(\frac{\sigma}{x_i}\right)^\mu} \ln \left(\frac{\sigma}{x_i}\right)}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \right]^2} \right\}, \\ \frac{\partial^2 l}{\partial \mu \partial \sigma} &= -\frac{\mu}{\sigma} \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\mu \ln \left(\frac{\sigma}{x_i}\right) - \frac{1}{\sigma} \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\mu + \frac{n}{\sigma} + 2\lambda \left\{ \sum_{i=1}^n \frac{\left\{ \frac{1}{\sigma} e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^\mu + \frac{\mu}{\sigma} e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^\mu \ln \left(\frac{\sigma}{x_i}\right) - \frac{\mu}{\sigma} e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^{2\mu} \ln \left(\frac{\sigma}{x_i}\right) \right\}}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \right]} - \right. \\ &\quad \left. \frac{2\lambda \frac{\mu}{\sigma} e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^{2\mu} \ln \left(\frac{\sigma}{x_i}\right)}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \right]^2} \right\}, \end{aligned}$$

$$\frac{\partial^2 l}{\partial \mu \partial \lambda} = 2 \sum_{i=1}^n \frac{e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^\mu \ln\left(\frac{\sigma}{x_i}\right)}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu}\right]} - \frac{\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^\mu \ln\left(\frac{\sigma}{x_i}\right) (1 - 2e^{-\left(\frac{\sigma}{x_i}\right)^\mu})}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu}\right]^2},$$

$$\frac{\partial^2 l}{\partial \sigma^2} = \frac{n}{\sigma^2} - \sum_{i=1}^n \frac{\mu(\mu-1) \left(\frac{\sigma}{x_i}\right)^\mu}{\sigma^2} - \frac{n(\mu+1)}{\sigma^2}$$

$$+ 2\lambda \sum_{i=1}^n \frac{\frac{\mu(\mu-1)}{\sigma^2} e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^\mu + \frac{\mu}{x_i} e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^{2(\mu-1)}}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu}\right]} - \frac{2\lambda \frac{\mu}{x_i} e^{-2\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^{2(\mu-1)}}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu}\right]^2},$$

$$\frac{\partial^2 l}{\partial \sigma \partial \lambda} = 2 \sum_{i=1}^n \frac{\frac{\mu}{\sigma} e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^\mu}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu}\right]} - \frac{\lambda \frac{\mu}{\sigma} e^{-\left(\frac{\sigma}{x_i}\right)^\mu} \left(\frac{\sigma}{x_i}\right)^\mu (1 - 2e^{-\left(\frac{\sigma}{x_i}\right)^\mu})}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu}\right]^2}$$

and

$$\frac{\partial^2 l}{\partial \lambda^2} = - \sum_{i=1}^n \frac{(1 - 2e^{-\left(\frac{\sigma}{x_i}\right)^\mu})^2}{\left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu}\right]^2}.$$

For interval estimation of (μ, σ, λ) , the observed information matrix is obtained since its expectation requires numerical integration. The 3×3 observed information matrix $J(\theta)$ is

$$J(\theta) = - \begin{pmatrix} I_{\mu\mu} & I_{\mu\sigma} & I_{\mu\lambda} \\ I_{\sigma\mu} & I_{\sigma\sigma} & I_{\sigma\lambda} \\ I_{\lambda\mu} & I_{\lambda\sigma} & I_{\lambda\lambda} \end{pmatrix}$$

whose elements are given above.

Under regularity conditions, the asymptotic distribution of

$$\sqrt{n}(\hat{\theta} - \theta) \text{ is } N_3(0, I(\theta)^{-1}),$$

where $I(\theta)$ is the expected information matrix. This asymptotic behavior is valid if $I(\theta)$ is replaced by $J(\hat{\theta})$, i.e., the observed information matrix evaluated at $\hat{\theta}$. The multivariate normal $N_4(0, J(\hat{\theta})^{-1})$ distribution can be used to construct approximate confidence intervals for the individual parameters and for the hazard rate and survival functions. An $100(1-\gamma)$ asymptotic confidence interval for each parameter θ_i is given by

$$(\hat{\theta}_i - Z_{\gamma/2} \sqrt{\hat{J}_{\theta_i\theta_i}}, \hat{\theta}_i + Z_{\gamma/2} \sqrt{\hat{J}_{\theta_i\theta_i}}),$$

where $\hat{J}_{\theta_i\theta_i}$ is the (i, i) diagonal element of $J(\hat{\theta})^{-1}$ (variance-covariance matrix) for $i = 1, 2, 3$ and $Z_{\gamma/2}$ is the quantile $1 - \gamma/2$ of the standard normal distribution.

A Numerical Example:

Here we fit the transmuted Fréchet distribution to two examples of uncensored data.

Data Set 1: The following data is an uncensored data set consisting of 100 observations on breaking stress of carbon fibers (in Gba): 0.92, 0.928, 0.997, 0.9971, 1.061, 1.117, 1.162, 1.183, 1.187, 1.192, 1.196, 1.213, 1.215, 1.2199, 1.22, 1.224, 1.225, 1.228, 1.237, 1.24, 1.244, 1.259, 1.261, 1.263, 1.276, 1.31, 1.321, 1.329, 1.331, 1.337, 1.351, 1.359, 1.388, 1.408, 1.449, 1.4497, 1.45, 1.459, 1.471, 1.475, 1.477, 1.48, 1.489, 1.501, 1.507, 1.515, 1.53, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.562, 1.566, 1.585, 1.586, 1.599, 1.602, 1.614, 1.616, 1.617, 1.628, 1.684, 1.711, 1.718, 1.733, 1.738, 1.743, 1.759, 1.777, 1.794, 1.799, 1.806, 1.814, 1.816, 1.828, 1.83, 1.884, 1.892, 1.944, 1.972, 1.984, 1.987, 2.02, 2.0304, 2.029, 2.035, 2.037, 2.043, 2.046, 2.059, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.306.

We treated with the data as complete data and obtained the maximum likelihood estimates for the TF distribution as follow:

$$\hat{\mu} = 4.25, \quad \hat{\sigma} = 1.421 \quad \text{and} \quad \hat{\lambda} = 0.12.$$

Data Set 2: This data set is generated data to simulate the strengths of glass fibers. The data set is: 1.014, 1.081, 1.082, 1.185, 1.223, 1.248, 1.267, 1.271, 1.272, 1.275, 1.276, 1.278, 1.286, 1.288, 1.292, 1.304, 1.306, 1.355, 1.361, 1.364, 1.379, 1.409, 1.426, 1.459, 1.46, 1.476, 1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541, 1.568, 1.579, 1.581, 1.591, 1.593, 1.602, 1.666, 1.67, 1.684, 1.691, 1.704, 1.731, 1.735, 1.747, 1.748, 1.757, 1.800, 1.806, 1.867, 1.876, 1.878, 1.91, 1.916, 1.972, 2.012, 2.456, 2.592, 3.197, 4.121.

We treated with the data as complete data and obtained the maximum likelihood estimates for the TF distribution as follow:

$$\hat{\mu} = 5.395, \quad \hat{\sigma} = 1.416 \quad \text{and} \quad \hat{\lambda} = 0.036.$$

The popular Kolmogorov-Smirnov goodness of fit test was carried out at 5% level of significance.

Given the cumulative distribution function $F_0(x)$ of the hypothesized distribution (here TF distribution) and the empirical distribution function $F_{\text{data}}(x)$ of the observed data, the test statistic is given by: $D = \sup_x |F_0(x) - F_{\text{data}}(x)|$.

For data set 1, Kolmogorov-Smirnov test statistic $D=0.0819$ with p-value $0.513 > 0.05$.

For data set 2, Kolmogorov-Smirnov test statistic $D=0.0887$ with p-value $0.410 > 0.05$.

In any of the cases we had no reason to reject the null hypothesis that generated data follows TF distribution.

5. Bayesian Estimation:

In this section, approximate Bayes estimates are computed using the Gibbs sampling procedure to generate samples from the posterior distributions. The approximate Bayes estimators are obtained under the assumptions of non-informative priors.

We consider the TF model with density function (2) and a non-informative joint prior distribution for μ , σ and λ given by:

$$\pi_0(\mu, \sigma, \lambda) \propto \frac{1}{\mu\sigma\lambda}, \quad (10)$$

where μ, σ and $\lambda > 0$. The joint posterior distribution for these parameters can be written as

$$\pi(\mu, \sigma, \lambda | x) \propto \pi_0(\mu, \sigma, \lambda) \exp\{l(x; \mu, \sigma, \lambda)\} \quad (11)$$

where $l(x; \mu, \sigma, \lambda)$ is the logarithm of the log likelihood function given by (6), which is

$$l = n \ln \mu - n \ln \sigma - \sum_{i=1}^n \left(\frac{\sigma}{x_i}\right)^\mu + (\mu + 1) \sum_{i=1}^n \ln \left(\frac{\sigma}{x_i}\right) + \sum_{i=1}^n \ln \left[1 + \lambda - 2\lambda e^{-\left(\frac{\sigma}{x_i}\right)^\mu}\right].$$

Consider the reparametrization $\rho_1 = \log(\mu)$ and $\rho_2 = \log(\sigma)$ and $\rho_3 = \log(\lambda)$. We obtain from (10) a non-informative prior for ρ_1, ρ_2 and ρ_3 , namely

$$\pi(\rho_1, \rho_2, \rho_3) = \text{constant}, \quad \text{where} \quad -\infty < \rho_1, \rho_2 \text{ and } \rho_3 < \infty.$$

The choice of the values of hyper-parameters of the uniform priors is required to obtain convergence of the Gibbs sampling algorithm. In practical terms, one can consider a uniform prior distribution $U(-a_i, a_i)$ for $i=1, 2, 3$ with larger values for a_i to produce approximate non-informative priors for ρ_1, ρ_2 and ρ_3 and proper joint posterior distribution.

Using the above reparametrization, the joint posterior distributions for ρ_1, ρ_2 and ρ_3 reduces to

$$\pi(\rho_1, \rho_2, \rho_3 | x) \propto \pi(\rho_1, \rho_2, \rho_3) \exp \left\{ n\rho_1 - n\rho_2 - \sum_{i=1}^n \left(\frac{\exp(\rho_2)}{x_i}\right)^{\exp(\rho_1)} + (\exp(\rho_1) + 1) \sum_{i=1}^n \ln \left(\frac{\exp(\rho_2)}{x_i}\right) + \sum_{i=1}^n \ln \left[1 + \exp(\rho_3) - 2 \exp(\rho_3) \cdot \exp \left(-\left(\frac{\exp(\rho_2)}{x_i}\right)^{\exp(\rho_1)}\right)\right] \right\}. \quad (12)$$

If we assume the prior $\pi(\rho_1, \rho_2, \rho_3) = \text{constant}$, the conditional posterior distributions used in the Gibbs sampling algorithm are given by:

$$\pi(\rho_1 | \rho_2, \rho_3, x) \propto \exp \left\{ n\rho_1 - \sum_{i=1}^n \left(\frac{\exp(\rho_2)}{x_i} \right)^{\exp(\rho_1)} + (\exp(\rho_1)) \right. \\ \left. + 1 \right\} \sum_{i=1}^n \ln \left(\frac{\exp(\rho_2)}{x_i} \right) + \sum_{i=1}^n \ln \left[1 + \exp(\rho_3) - 2 \exp(\rho_3) \cdot \exp \left(- \left(\frac{\exp(\rho_2)}{x_i} \right)^{\exp(\rho_1)} \right) \right],$$

$$\pi(\rho_2 | \rho_1, \rho_3, x) \propto \exp \left\{ -n\rho_2 - \sum_{i=1}^n \left(\frac{\exp(\rho_2)}{x_i} \right)^{\exp(\rho_1)} + (\exp(\rho_1)) \right. \\ \left. + 1 \right\} \sum_{i=1}^n \ln \left(\frac{\exp(\rho_2)}{x_i} \right) + \sum_{i=1}^n \ln \left[1 + \exp(\rho_3) - 2 \exp(\rho_3) \cdot \exp \left(- \left(\frac{\exp(\rho_2)}{x_i} \right)^{\exp(\rho_1)} \right) \right],$$

and

$$\pi(\rho_3 | \rho_1, \rho_2, x) \propto \exp \left\{ \sum_{i=1}^n \ln \left[1 + \exp(\rho_3) - 2 \exp(\rho_3) \cdot \exp \left(- \left(\frac{\exp(\rho_2)}{x_i} \right)^{\exp(\rho_1)} \right) \right] \right\}.$$

Posterior summaries of interest can be derived from the generated samples for the joint posterior distribution for the new parameters using the Gibbs sampling procedure. However, this involves very lengthy and complicated computations. A considerable simplification in the computation can be achieved using the WinBUGS software which requires only the specification of the joint distribution for the data and the prior distributions for the model parameters.

A Numerical Example:

Consider the data sets mentioned in section 4. We consider the TF distribution with density (2) under the reparametrization $\rho_1 = \log(\mu)$, $\rho_2 = \log(\sigma)$ and $\rho_3 = \log(\lambda)$. We assume approximate non-informative prior uniform $U(-1, 2)$, $U(-1, 3)$ and $U(-4, -3)$ distributions for ρ_1 , ρ_2 and ρ_3 respectively.

A set of 10000 Gibbs samples was generated after a "burn-in-sample" of size 1000 to eliminate the initial values considered for the Gibbs sampling algorithm. All the calculations are performed using the WinBUGS software. Convergence of the Gibbs sampling algorithm is verified from a formal convergence diagnostic (the BGR) which was suggested by Brooks and Gelman (1998). They generalized the method proposed by Gelman and Rubin (1992) for monitoring the convergence of iterative simulations by comparing between and within variances of multiple chains, in order to obtain a family of tests of convergence. The BGR can be implemented using the Winbugs package (the option bgrdiag), but it requires multiple chain simulation. The bgr diagnostic in WinBUGS examines the widths of the 80% equal-tailed intervals of the pooled sample, and of each individual sample. The average width of the individual samples (blue curve) should approach to the width from the pooled sample (green curve), thus the ratio (red curve) should approach to 1. In our example one can notes that the ratio (R) approach to 1.

Once the convergence achieved, one need to run the simulation for a further number of iterations to obtain samples that can be used for posterior inference. One way to assess the accuracy of the posterior estimates is by calculating the Monte Carlo error (MC error) for each parameter. This is an estimate of the difference between the mean of sampled values and the true posterior mean. The simulation should be run until the MC error for each parameter of interest is less than about 5% of the sample standard deviation. One can notes in our example that MC error less than 5% of the sample standard deviation.

Tables list the posterior descriptive summaries of interest for the TF model. The posterior kernel densities for the parameters are given in Figures 1-2.

Table 1: Summary results for the posterior parameters in the case of transmuted Fréchet model based on 100 breaking stress data.

Parameter	Estimate	Standard Deviation	MC	95% Credible Interval
μ	4.3430	0.3238	0.0037	(3.7260, 4.9870)
σ	1.3970	0.0341	3.254E-4	(1.3330, 1.4650)
λ	0.0252	0.0064	9.825E-5	(0.0185, 0.0427)

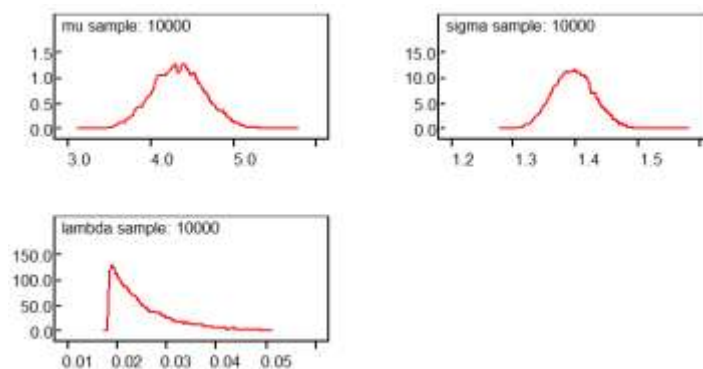


Fig.1: Kernel density for the parameters in the case of transmuted Fréchet model based on 100 breaking stress data.

Table 2: Summary results for the posterior parameters in the case of transmuted Fréchet model based on 63 strengths of glass fibers data.

Parameter	Estimate	Standard Deviation	MC	95% Credible Interval
μ	4.044	0.4249	0.0054	(3.2450, 4.9030)
σ	1.442	0.0475	4.481E-4	(1.3300, 1.5390)
λ	0.0269	0.0074	9.585E-5	(0.0186, 0.04562)

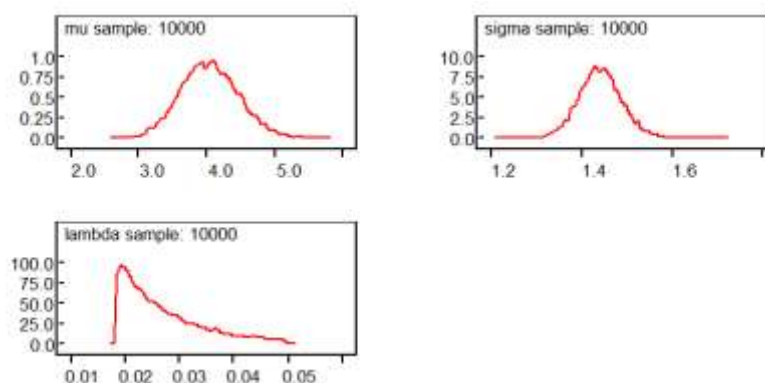


Fig. 2: Kernel density for the parameters in the case of transmuted Fréchet model based on 63 strengths of glass fibers data.

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