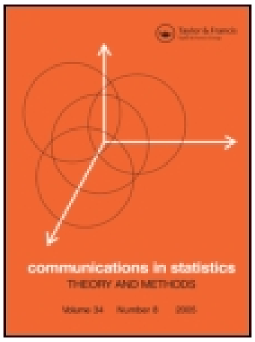

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Fisher Information Matrix for the Generalized Feller-Pareto Distribution

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In this article, the exact form of Fisher information matrix for the generalized Feller-Pareto (GFP) distribution is determined. The GFP family is a general distribution which includes a variety of distributions as special cases. For example:

- *generalized Singh-Maddala distribution which in turn includes Burr, Fisk, and Lomax distribution (see Kleiber and Kotz, 2003);*
- *a Pareto IV distribution which includes a hierarchy of Pareto models, omitted an additional location parameter (see Arnold, 1983, 2008); and*
- *beta Lomax distribution which includes, for example, beta II and Lomax distributions.*

Application of these distributions covers a wide spectrum of areas ranging from actuarial science, economics, finance to bioscience, telecommunications, and medicine.

Keywords Fisher information; Beta Lomax; Generalized Singh-Maddala; Generalized Feller-Pareto; Beta Burr XII.

Mathematics Subject Classification 62E15.

1. Introduction

In this article, the exact Fisher information matrix for the generalized Feller-Pareto (GFP) distribution is determined. It is well known that Fisher information matrix serves as a valuable tool for derivation of covariance matrix in the asymptotic distribution of maximum likelihood estimators (MLE).

The GFP distribution is a five-parameter family of distributions, four important four-parameter daughter distributions can be obtained from the five-parameter parent GFP distribution either as special cases or as limiting cases. These distributions are commonly referred to as the generalized beta distribution of the second kind, beta Weibull, beta

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Lomax and generalized Singh-Maddala distributions. They can, in turn, give rise to special or limiting case three- parameter families of distributions: for example, generalized gamma and Burr XII distributions. In summary, the GFP family is indeed an attractive, flexible, elegant, and ingenious family but it involves five parameters. For further discussion and diverse applications, see Arnold (2008), Brazauskas (2002), Brazauskas (2003), Kleiber and Kotz (2003), and Paranaíba et al. (2011).

This article is organized as follows. In Sec. 2, two different representations of the GFP family are described and some general distributions which the GFP family includes as special cases are specified. In Sec. 3, we provide elements of the Fisher information matrix for GFP and some special cases.

2. Generalized Feller-Pareto and Related Distribution

The Generalized Feller-Pareto (GFP) family was defined by Zandonatti in 2001 as mentioned in Kleiber and Kotz (2003). Let random variable V is distributed as $Beta(q, p)$. For some$a, \theta, b > 0$, use $x = b(v^{-\frac{1}{\theta}} - 1)^{1/a}$ to make a transformation to obtain

$$f(x) = \frac{a\theta x^{a-1}}{b^a B(p,q)} \left(1 + \left(\frac{x}{b}\right)^a\right)^{-\theta q - 1} \left(1 - \left(1 + \left(\frac{x}{b}\right)^a\right)^{-\theta}\right)^{p-1}, \qquad x > 0, \tag{1}$$

where the four parameters $a, \theta, p$and$q > 0$ are shape parameters, $b > 0$ is a scale parameter and $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. Kleiber and Kotz (2003) omited an additional location parameter. In this case, X is said to have a generalized Feller-Pareto distribution. We denoted this distribution by $GFP(a, \theta, b, p, q)$. Alternatively, the GFP distribution is defined and investigated by Paranaíba et al. (2011) under different name which is Beta Burr XII (BBXII) distribution using a generalized class of distributions was defined by Eugene et al. (2002). More specifically, if a generalized class of distributions for $p > 0$ and $q > 0$ is defined as

$$F(x) = \int_0^{G(x)} \frac{1}{B(p,q)} v^{q-1} (1-v)^{p-1}\, dv, \tag{2}$$

where p and q are shape parameters. When $G(x) = 1 - (1 + (\frac{x}{b})^a)^{-\theta}$ is the cumulative function of Burr XII distribution, then (2) is the corresponding cumulative function to GFP distribution (1).

Some known distributions can be special case from GFP distribution in (1). For example, beta Weibull distribution is a limiting distribution as $\theta \to \infty$ (Paranaíba et al., 2011) and generalized beta distribution of the second kind (Feller-Pareto) distribution is obtained for $\theta = 1$ (Kleiber and Kotz, 2003). And the beta Lomax distribution is given by

$$f(x) = \frac{\theta}{bB(p,q)} \left(1 + \left(\frac{x}{b}\right)\right)^{-\theta q - 1} \left(1 - \left(1 + \left(\frac{x}{b}\right)\right)^{-\theta}\right)^{p-1}, \qquad x > 0$$

which is special case from generalized Feller-Pareto distribution if $a = 1$ in (1).

A new distribution defined in Zandonatti 2001 as mentioned in Kleiber and Kotz (2003) is obtained by putting $q = 1$ in (1), as

$$f(x) = a\theta p \left(\frac{x}{b}\right)^{a-1} \left(1 + \left(\frac{x}{b}\right)^a\right)^{-\theta - 1} \left(1 - \left(1 + \left(\frac{x}{b}\right)^a\right)^{-\theta}\right)^{p-1} \qquad x > 0. \tag{3}$$

Distribution (3) will be known as the generalized Singh-Maddala distribution. It includes Singh-Maddala, Lomax, and Fisk distributions by putting in (3) $p = 1, a = 1, \theta = 1$, respectively. Furthermore; the distribution obtained when $p = 1$ in (1) can be a Pareto IV distribution with density function

$$f(x) = \frac{a\theta q}{b^a} x^{a-1} \left(1 + \left(\frac{x}{b}\right)^a\right)^{-\theta q - 1}, \qquad x > 0 \tag{4}$$

Distribution (4) includes a hierarchy of Pareto models, with an additional location parameter which discussed in Arnold (1983) by putting $q = \theta = 1$ in (4). For further discussion and diverse applications, see Arnold (2008), Brazauskas (2002, 2003), and Kleiber and Kotz (2003).

3. Information Matrix for Generalized Feller-Pareto

Suppose X is a random variable with the probability density function $f_\Lambda(x)$ where $\Lambda = (\lambda_1, \ldots, \lambda_k)$and $f_\Lambda(x)$ has second derivatives $f_\Lambda(x)/\partial\lambda_i\partial\lambda_j$ for all i and j. Then the information matrix $I(\Lambda)$ is the $k \times k$ symmetric matrix with elements

$$I_{ij}(\Lambda) = -\mathrm{E}_\Lambda\left[\frac{\partial^2 \log(f_\Lambda(x))}{\partial\lambda_i\partial\lambda_j}\right]$$

For the *GFP*(a, θ, b, p, q) distribution all second derivatives exist. Thus, we have $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (a, \theta, b, p, q)$, the log-likelihood

$$\log(L(x)) = n\log\left(\frac{a\theta}{bB(p,q)}\right) + (a-1)\sum_{i=1}^n \log\left(\frac{x_i}{b}\right) - (\theta q + 1)\sum_{i=1}^n \log\left(1 + \left(\frac{x_i}{b}\right)^a\right)$$
$$+ (p-1)\sum_{i=1}^n \log\left(1 - \left(1 + \left(\frac{x_i}{b}\right)^a\right)^{-\theta}\right).$$

Elements of the Fisher information matrix of the GFP distribution were obtained using second partial derivatives and some of them can be written in a simpler form by using the digamma function $\Psi(\gamma) = \frac{\Gamma'(\gamma)}{\Gamma(\gamma)}$ and the trigamma function $\Psi'(\gamma) = \frac{d}{d\gamma}(\frac{\Gamma'(\gamma)}{\Gamma(\gamma)})$

$$I_{55}(\Lambda) = n[\Psi'(q) - \Psi'(p+q)]$$

$$I_{54}(\Lambda) = -n\Psi'(p+q)$$

$$I_{53}(\Lambda) = \frac{-na\theta}{bB(p,q)}\left[B(p,q) - B\left(p, q + \frac{1}{\theta}\right)\right]$$

$$I_{52}(\Lambda) = \frac{n}{\theta}(\Psi(p+q) - \Psi(q))$$

$$I_{51}(\Lambda) = \frac{n}{a}(\Psi(p+q) - \Psi(q))$$
$$+ \frac{n}{aB(p,q)}\left[B\left(q + \frac{1}{\theta}, p\right)\left(\Psi\left(q + \frac{1}{\theta}\right) - \Psi\left(p + q + \frac{1}{\theta}\right)\right)\right.$$

$$+\theta\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\,\frac{\Gamma(p)}{\alpha!\Gamma(p-\alpha)}}{(q+\alpha)(\theta q+\theta\alpha+1)}\left(\Psi(1)-\Psi(\alpha\theta+\theta q+1)\right)$$

$$+\theta^{2}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\,\frac{\Gamma(p)}{\alpha!\Gamma(p-\alpha)}}{(\theta q+\theta\alpha+1)^{2}}$$

$$I_{44}(\Lambda)=n[\Psi'(p)-\Psi'(p+q)]$$

$$I_{43}(\Lambda)=\frac{na\theta}{bB(p,q)}\left[B(p-1,q+1)-B\left(p-1,q+\frac{1}{\theta}+1\right)\right]$$

$$I_{42}(\Lambda)=\frac{nq}{\theta(p-1)}(\Psi(q+1)-\Psi(p+q))$$

$$I_{41}(\Lambda)=\frac{nq}{a(p-1)}(\Psi(q+1)-\Psi(p+q))$$

$$-\frac{nB\left(q+\frac{1}{\theta}+1,p-1\right)}{aB(p,q)}\left(\Psi\left(q+\frac{1}{\theta}+1\right)-\Psi\left(p+q+\frac{1}{\theta}\right)\right)$$

$$-\frac{n\theta}{aB(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\,\frac{\Gamma(p-1)}{\alpha!\Gamma(p-\alpha-1)}}{(q+\alpha+1)(\theta q+\theta\alpha+\theta+1)}$$

$$\times\left(\Psi(1)-\Psi(\alpha\theta+\theta q+\theta+1)\right)-\frac{n\theta^{2}}{aB(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\,\frac{\Gamma(p-1)}{\alpha!\Gamma(p-\alpha-1)}}{(\theta q+\theta\alpha+\theta+1)^{2}}$$

$$I_{33}(\Lambda)=\frac{-na}{b^{2}}+\frac{na(\theta q+1)}{b^{2}B(p,q)}\left[B(p,q)-B\left(p,q+\frac{1}{\theta}\right)\right]$$

$$+\frac{na^{2}(\theta q+1)}{b^{2}B(p,q)}\left[B\left(p,q+\frac{1}{\theta}\right)-B\left(p,q+\frac{2}{\theta}\right)\right]$$

$$-\frac{na\theta(p-1)}{b^{2}B(p,q)}\left[B(p-1,q+1)-B\left(p-1,q+\frac{1}{\theta}+1\right)\right]$$

$$-\frac{n\theta a^{2}(p-1)}{b^{2}B(p,q)}\left[B\left(p-1,q+\frac{1}{\theta}+1\right)-B\left(p-1,q+\frac{2}{\theta}+1\right)\right]$$

$$+\frac{n\theta^{2}a^{2}(p-1)}{b^{2}B(p,q)}\left[B\left(p-2,q+\frac{2}{\theta}+1\right)-2B\left(p-2,q+\frac{1}{\theta}+1\right)\right]$$

$$+\frac{n\theta^{2}a^{2}(p-1)}{b^{2}B(p,q)}B(p-2,q+1)$$

$$I_{32}(\Lambda)=\frac{-naq}{bB(p,q)}\left[B(p,q)-B\left(p,q+\frac{1}{\theta}\right)\right]$$

$$-\frac{na(p-1)}{bB(p,q)}B(p-2,q+\frac{1}{\theta}+1)\left[\Psi\left(p+q+\frac{1}{\theta}-1\right)-\Psi\left(q+\frac{1}{\theta}+1\right)\right]$$

$$+\frac{na(p-1)}{bB(p,q)}\left[B(p-1,q+1)-B\left(p-1,q+\frac{1}{\theta}+1\right)\right]$$

$$+\frac{na(p-1)}{bB(p,q)}B(p-2,q+1)\left[\Psi(q+1)-\Psi(p+q-1)\right]$$

$$I_{31}(\Lambda)=\frac{n}{b}-\frac{n(\theta q+1)}{bB(p,q)}\left[B(p,q)-B\left(p,q+\frac{1}{\theta}\right)\right]$$

$$+\frac{n\theta(p-1)}{bB(p,q)}\left[B(p-1,q+1)-B\left(p-1,q+\frac{1}{\theta}+1\right)\right]$$

$$+\frac{n(\theta q+1)}{b\theta B(p,q)}B\left(p,q+\frac{1}{\theta}\right)\left[\Psi\left(q+\frac{1}{\theta}\right)-\Psi\left(p+q+\frac{1}{\theta}\right)\right]$$

$$-\frac{n(\theta q+1)}{b\theta B(p,q)}B\left(p,q+\frac{2}{\theta}\right)\left[\Psi\left(q+\frac{2}{\theta}\right)-\Psi\left(p+q+\frac{2}{\theta}\right)\right]$$

$$-\frac{n(p-1)}{bB(p,q)}B\left(p-1,q+1+\frac{1}{\theta}\right)\left[\Psi\left(q+\frac{1}{\theta}+1\right)-\Psi\left(p+q+\frac{1}{\theta}\right)\right]$$

$$+\frac{n(p-1)}{bB(p,q)}B\left(p-1,q+1+\frac{2}{\theta}\right)\left[\Psi\left(q+\frac{2}{\theta}+1\right)-\Psi\left(p+q+\frac{2}{\theta}\right)\right]$$

$$+\frac{n\theta(p-1)}{bB(p,q)}B(p-2,q+1)\left[\Psi(q+1)-\Psi(p+q-1)\right]$$

$$-2\frac{n\theta(p-1)}{bB(p,q)}B\left(p-2,q+1+\frac{1}{\theta}\right)$$

$$\times\left[\Psi\left(q+\frac{1}{\theta}+1\right)-\Psi\left(p+q+\frac{1}{\theta}-1\right)\right]$$

$$+\frac{n\theta(p-1)}{bB(p,q)}B\left(p-2,q+1+\frac{2}{\theta}\right)$$

$$\times\left[\Psi\left(q+\frac{2}{\theta}+1\right)-\Psi\left(p+q+\frac{2}{\theta}-1\right)\right]$$

$$-\frac{n\theta(\theta q+1)}{bB(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p)}{\alpha!\Gamma(p-\alpha)}\left(\Psi(2)-\Psi(\alpha\theta+\theta q+3)\right)}{(\theta q+\theta\alpha+2)(\theta q+\theta\alpha+1)}$$

$$+\frac{n\theta^{2}(p-1)}{bB(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-1)}{\alpha!\Gamma(p-\alpha-1)}\left(\Psi(2)-\Psi(\alpha\theta+\theta q+\theta+3)\right)}{(\theta q+\theta\alpha+\theta+1)(\theta q+\theta\alpha+\theta+2)}$$

$$-2\frac{n\theta^{2}(p-1)}{bB(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}\left(\Psi(3)-\Psi(\theta q+\theta\alpha+\theta+3)\right)}{(q+\alpha+1)(\theta q+\theta\alpha+\theta+1)(\theta q+\theta\alpha+\theta+2)}$$

$$I_{22}(\Lambda)=\frac{n}{\theta^{2}}+\frac{nq(p+q-1)}{\theta^{2}(p-2)}\left\{\Psi(q+1)-\Psi(p+q-1)\right\}^{2}$$

$$+\frac{nq(p+q-1)}{\theta^{2}(p-2)}\left(\Psi'(q+1)-\Psi'(p+q-1)\right)$$

$$I_{21}(\Lambda) = \frac{nq}{aB(p,q)} \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha} \frac{\Gamma(p)}{\alpha!\Gamma(p-\alpha)} (\Psi(1) - \Psi(\alpha\theta + \theta q + 1))}{(\alpha + q)(\alpha\theta + \theta q + 1)}$$

$$+ \frac{nq\theta}{aB(p,q)} \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha} \frac{\Gamma(p)}{\alpha!\Gamma(p-\alpha)}}{(\alpha\theta + \theta q + 1)^2}$$

$$- \frac{nq}{a\theta} \left[(\Psi(q) - \Psi(p+q)) + \frac{B\left(q + \frac{1}{\theta}, p\right)}{B(p,q)} \right.$$

$$\left. \times \left(\Psi\left(q + \frac{1}{\theta}\right) - \Psi\left(p + q + \frac{1}{\theta}\right) \right) \right]$$

$$+ \frac{nq(p+q-1)}{a\theta(p-2)} \left[\{\Psi(q+1) - \Psi(p+q-1)\}^2 + \Psi(q+1) - \Psi(p+q-1) \right]$$

$$- \frac{n(p-1)B(q + \frac{1}{\theta}, p-2)}{a\theta B(p,q)} \left\{ \Psi\left(q + \frac{1}{\theta} + 1\right) - \Psi\left(p + q + \frac{1}{\theta} - 1\right) \right\}^2$$

$$- \frac{n(p-1)B(q + \frac{1}{\theta}, p-2)}{a\theta B(p,q)} \left[\Psi(q + \frac{1}{\theta} + 1) - \Psi(p + q + \frac{1}{\theta} - 1) \right]$$

$$+ \frac{n(p-1)}{a\theta^2 B(p,q)} \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha} \frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)} (1 + 2\theta(q + \alpha + 1))}{[(\alpha + q)(\alpha\theta + \theta q + \theta + 1)]^2}$$

$$\times (\Psi(1) - \Psi(\alpha\theta + \theta q + \theta + 1)) + \frac{n(p-1)}{aB(p,q)} \sum_{\alpha=0}^{\infty} (-1)^{\alpha} \frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}$$

$$\times \left[\frac{1}{(\theta\alpha + \theta q + \theta + 1)^3} - \frac{\Psi(\alpha\theta + \theta q + \theta + 1)}{\theta(\theta\alpha + \theta q + \theta + 1)(\alpha + q + 1)} \right]$$

$$I_{11}(\Lambda) = \frac{n}{a^2} + \frac{\theta n(\theta q + 1)}{a^2 B(p,q)} \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha} \frac{\Gamma(p)}{\alpha!\Gamma(p-\alpha)} (\Psi(2) - \Psi(\alpha\theta + \theta q + 1))^2}{(\alpha\theta + \theta q + 1)(\alpha\theta + \theta q + 2)}$$

$$+ \frac{\theta n(\theta q + 1)}{a^2 B(p,q)} \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha} \frac{\Gamma(p)}{\alpha!\Gamma(p-\alpha)} (\Psi'(2) + \Psi'(\alpha\theta + \theta q + 1))}{(\alpha\theta + \theta q + 1)(\alpha\theta + \theta q + 2)}$$

$$- \frac{\theta^2 n(p-1)}{a^2 B(p,q)} \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha} \frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)} (\Psi'(1) + \Psi'(\alpha\theta + \theta q + 2\theta + 2))}{(\alpha\theta + \theta q + 2\theta + 2)}$$

$$- \frac{\theta^2 n(p-1)}{a^2 B(p,q)} \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha} \frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)} (\Psi(1) - \Psi(\alpha\theta + \theta q + 2\theta + 2))^2}{(\alpha\theta + \theta q + 2\theta + 2)}$$

$$+ \frac{\theta^2 n(p-1)}{a^2 B(p,q)} \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha} \frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)} (\theta + 1)(\Psi'(1) + \Psi'(\alpha\theta + \theta q + \theta + 2))}{(\alpha\theta + \theta q + \theta + 2)}$$

$$+\frac{\theta^2 n(p-1)}{a^2 B(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}(\theta+1)(\Psi(1)-\Psi(\alpha\theta+\theta q+\theta+2))^2}{(\alpha\theta+\theta q+\theta+2)}$$

$$+\frac{\theta^2 n(p-1)}{a^2 B(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}(\Psi'(1)+\Psi'(\alpha\theta+\theta q+2\theta+1))}{(\alpha\theta+\theta q+2\theta+1)}$$

$$+\frac{\theta^2 n(p-1)}{a^2 B(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}(\Psi(1)-\Psi(\alpha\theta+\theta q+2\theta+1))^2}{(\alpha\theta+\theta q+2\theta+1)}$$

$$-\frac{\theta^2 n(p-1)}{a^2 B(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}(2\theta+1)(\Psi'(1)+\Psi'(\alpha\theta+\theta q+\theta+1))}{(\alpha\theta+\theta q+\theta+1)}$$

$$-\frac{\theta^2 n(p-1)}{a^2 B(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}(\Psi(1)-\Psi(\alpha\theta+\theta q+\theta+1))^2}{(\alpha\theta+\theta q+\theta+1)}$$

$$+\frac{\theta^2 n(p-1)}{a^2 B(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}(\Psi'(1)+\Psi'(\alpha\theta+\theta q+\theta))}{(\alpha+q+1)}$$

$$+\frac{\theta^2 n(p-1)}{a^2 B(p,q)}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}(\Psi(1)-\Psi(\alpha\theta+\theta q+\theta))^2}{(\alpha+q+1)}.$$

Noted that, Paranaiba et al. (2011) provided the observed information. Indeed, the GFP family is an attractive, flexible, elegant, and ingenious family but it involves five parameters.

4. Special Cases

In this section, we provide Fisher information matrix for special distributions from GFP family $GFP(a, \theta, b, p, q)$. For Feller-Pareto family, Fisher information matrix elements, given in Brazauskas (2002), is obtained when $\theta = 1$. Thus, elements for Fisher information matrix of generalized gamma and transformed beta distributions can be obtained.

4.1. Generalized Singh-Maddala Distribution

The elements that represent information for the parameters of the generalized Singh-Maddala distribution$GFP(a, \theta, b, p, 1)$ are given as

$$I_{44}(\Lambda) = \frac{n}{p^2}$$

$$I_{43}(\Lambda) = \frac{na\theta p}{b}\left[\frac{1}{p(p-1)} - B\left(p-1, \frac{1}{\theta}+2\right)\right]$$

$$I_{42}(\Lambda) = \frac{n}{\theta(p-1)}(\Psi(2) - \Psi(p+1))$$

$$I_{41}(\Lambda) = \frac{n}{a(p-1)}(\Psi(2) - \Psi(p+1)) - \frac{npB\left(\frac{1}{\theta}+2, p-1\right)}{a}$$

$$\times\left(\Psi\left(\frac{1}{\theta}+2\right)-\Psi\left(p+1+\frac{1}{\theta}\right)\right)-\frac{np\theta}{a}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\ \frac{\Gamma(p-1)}{\alpha!\Gamma(p-\alpha-1)}}{(\alpha+2)(\theta\alpha+2\theta+1)}$$

$$\times\left(\Psi(1)-\Psi(\alpha\theta+2\theta+1)\right)-\frac{np\theta^{2}}{a}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\ \frac{\Gamma(p-1)}{\Gamma(p-\alpha-1)}}{\alpha!(\theta\alpha+2\theta+1)^{2}}$$

$$I_{33}(\Lambda)=\frac{n\theta^{2}a^{2}p}{b^{2}(p-2)}+\frac{nap\,(a\theta q+a-\theta-1)}{b^{2}}B\left(p,1+\frac{1}{\theta}\right)$$

$$-\frac{na^{2}p\,(\theta q+1)}{b^{2}}B\left(p,1+\frac{2}{\theta}\right)+\frac{n\theta^{2}a^{2}p\,(p-1)}{b^{2}}$$

$$\times\left[B\left(p-2,\frac{2}{\theta}+2\right)-2B\left(p-2,\frac{1}{\theta}+2\right)\right]$$

$$+\frac{n\theta ap\,(p-1)(1-a)}{b^{2}}B\left(p-1,\frac{1}{\theta}+2\right)+\frac{n\theta a^{2}p\,(p-1)}{b^{2}}B\left(p-1,\frac{2}{\theta}+2\right)$$

$$I_{32}(\Lambda)=\frac{nap}{b}B\left(p,1+\frac{1}{\theta}\right)+\frac{nap}{b\,(p-2)}\left[\Psi(2)-\Psi(p)\right]$$

$$-\frac{nap\,(p-1)}{b}B\left(p-1,2+\frac{1}{\theta}\right)-\frac{nap\,(p-1)}{b}B\left(p-2,2+\frac{1}{\theta}\right)$$

$$\times\left[\Psi\left(p+\frac{1}{\theta}\right)-\Psi\left(2+\frac{1}{\theta}\right)\right]$$

$$I_{31}(\Lambda)=\frac{np\,(\theta+1)}{b}B\left(p,1+\frac{1}{\theta}\right)-\frac{n\theta p\,(p-1)}{b}B\left(p-1,\frac{1}{\theta}+2\right)$$

$$+\frac{n\theta p}{b\,(p-2)}\left[\Psi(2)-\Psi(p)\right]+\frac{np\,(\theta+1)}{b\theta}B\left(p,1+\frac{1}{\theta}\right)$$

$$\times\left[\Psi\left(1+\frac{1}{\theta}\right)-\Psi\left(p+1+\frac{1}{\theta}\right)\right]-\frac{np\,(\theta+1)}{b\theta}B\left(p,1+\frac{2}{\theta}\right)$$

$$\times\left[\Psi\left(1+\frac{2}{\theta}\right)-\Psi\left(p+1+\frac{2}{\theta}\right)\right]$$

$$-\frac{np\,(p-1)}{b}B\left(p-1,2+\frac{1}{\theta}\right)\left[\Psi\left(\frac{1}{\theta}+2\right)-\Psi\left(p+1+\frac{1}{\theta}\right)\right]$$

$$+\frac{np\,(p-1)}{b}B\left(p-1,2+\frac{2}{\theta}\right)\left[\Psi\left(\frac{2}{\theta}+2\right)-\Psi\left(p+1+\frac{2}{\theta}\right)\right]$$

$$-2\frac{n\theta p\,(p-1)}{b}B\left(p-2,2+\frac{1}{\theta}\right)\left[\Psi\left(\frac{1}{\theta}+2\right)-\Psi\left(p+\frac{1}{\theta}\right)\right]$$

$$+\frac{n\theta p\,(p-1)}{b}B\left(p-2,2+\frac{2}{\theta}\right)\left[\Psi\left(\frac{2}{\theta}+2\right)-\Psi\left(p+\frac{2}{\theta}\right)\right]$$

$$-\frac{n\theta p\,(\theta+1)}{b}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\ \frac{\Gamma(p)}{\alpha!\Gamma(p-\alpha)}\left(\Psi(2)-\Psi(\alpha\theta+\theta+3)\right)}{(\theta+\theta\alpha+2)(\theta+\theta\alpha+1)}$$

$$+\frac{n\theta^2 p(p-1)}{b}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-1)}{\alpha!\Gamma(p-\alpha-1)}(\Psi(2)-\Psi(\alpha\theta+2\theta+3))}{(\theta\alpha+2\theta+1)(\theta\alpha+2\theta+2)}$$

$$-2\frac{n\theta^2 p(p-1)}{b}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}(\Psi(3)-\Psi(\theta\alpha+2\theta+3))}{(\alpha+2)(\theta\alpha+2\theta+1)(\theta\alpha+2\theta+2)}$$

$$I_{22}(\Lambda)=\frac{n}{\theta^2}+\frac{np}{\theta^2(p-2)}\left(\Psi'(2)-\Psi'(p)\right)+\frac{np}{\theta^2(p-2)}\{\Psi(2)-\Psi(p)\}^2$$

$$I_{21}(\Lambda)=\frac{np}{a}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p)}{\alpha!\Gamma(p-\alpha)}(\Psi(1)-\Psi(\alpha\theta+\theta+1))}{(\alpha+1)(\alpha\theta+\theta+1)}+\frac{np\theta}{a}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p)}{\alpha!\Gamma(p-\alpha)}}{(\alpha\theta+\theta+1)^2}$$

$$-\frac{n}{a\theta}\left[(\Psi(1)-\Psi(p+1))+pB\left(1+\frac{1}{\theta},p\right)\left(\Psi(1+\frac{1}{\theta})-\Psi(p+1+\frac{1}{\theta})\right)\right]$$

$$+\frac{np}{a\theta(p-2)}\left[\{\Psi(2)-\Psi(p)\}^2+\Psi(2)-\Psi(p)\right]-\frac{n(p-1)pB(1+\frac{1}{\theta},p-2)}{a\theta}$$

$$\times\left\{\Psi\left(2+\frac{1}{\theta}\right)-\Psi\left(p+\frac{1}{\theta}\right)\right\}^2-\frac{n(p-1)pB(1+\frac{1}{\theta},p-2)}{a\theta}\Bigg]\Psi\left(\frac{1}{\theta}+2\right)$$

$$-\Psi(p+\frac{1}{\theta})]+\frac{np(p-1)}{a\theta^2}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}(\Psi(1)-\Psi(\alpha\theta+2\theta+1))}{[(\alpha+1)(\alpha\theta+2\theta+1)]^2}$$

$$(1+2\theta(\alpha+2))+\frac{np(p-1)}{a}\sum_{\alpha=0}^{\infty}(-1)^{\alpha}\frac{\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}$$

$$\times\left[\frac{1}{(\theta\alpha+2\theta+1)^3}-\frac{\Psi(\alpha\theta+2\theta+1)}{\theta(\theta\alpha+2\theta+1)(\alpha+2)}\right]$$

$$I_{11}(\Lambda)=\frac{n}{a^2}+\frac{\theta np(\theta+1)}{a^2}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\Gamma(p)}{\alpha!\Gamma(p-\alpha)}$$

$$\times\frac{\left(\Psi'(2)+\Psi'(\alpha\theta+\theta+1)+(\Psi(2)-\Psi(\alpha\theta+\theta+1))^2\right)}{(\alpha\theta+\theta+1)(\alpha\theta+\theta+2)}$$

$$-\frac{\theta^2 np(p-1)}{a^2}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}$$

$$\times\frac{\left(\Psi'(1)+\Psi'(\alpha\theta+3\theta+2)+(\Psi(1)-\Psi(\alpha\theta+3\theta+2))^2\right)}{(\alpha\theta+3\theta+2)}$$

$$+\frac{\theta^2 np(p-1)}{a^2}(\theta+1)\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\Gamma(p-2)}{\alpha!\Gamma(p-\alpha-2)}$$

$$\times\frac{\left(\Psi'(1)+\Psi'(\alpha\theta+2\theta+2)+(\Psi(1)-\Psi(\alpha\theta+2\theta+2))^2\right)}{(\alpha\theta+2\theta+2)}$$

$$+\frac{\theta^2 np\,(p-1)}{a^2}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\,\Gamma\,(p-2)}{\alpha!\Gamma\,(p-\alpha-2)}$$
$$\times\frac{\left(\Psi'\,(1)+\Psi'\,(\alpha\theta+3\theta+1)+(\Psi\,(1)-\Psi\,(\alpha\theta+3\theta+1))^2\right)}{(\alpha\theta+3\theta+1)}$$
$$-\frac{\theta^2 np\,(p-1)}{a^2}(2\theta+1)\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\,\Gamma\,(p-2)}{\alpha!\Gamma\,(p-\alpha-2)}$$
$$\times\frac{\left(\Psi'\,(1)+\Psi'\,(\alpha\theta+2\theta+1)+(\Psi\,(1)-\Psi\,(\alpha\theta+2\theta+1))^2\right)}{(\alpha\theta+2\theta+1)}$$
$$+\frac{\theta^2 np\,(p-1)}{a^2}\sum_{\alpha=0}^{\infty}\frac{(-1)^{\alpha}\,\Gamma\,(p-2)}{\alpha!\Gamma\,(p-\alpha-2)}$$
$$\times\frac{\left(\Psi'\,(1)+\Psi'\,(\alpha\theta+2\theta)+(\Psi\,(1)-\Psi\,(\alpha\theta+2\theta))^2\right)}{(\alpha+2)}.$$

4.2. A Pareto IV Distribution

The elements of the Fisher information matrix of the generalized Pareto IV $GFP\,(a,\theta,b,1,q)$ are given (where $\Lambda=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=(a,\theta,b,q)$)

$$I_{44}(\Lambda)=\frac{n}{q^2}$$
$$I_{43}(\Lambda)=\frac{na\theta}{b(q\theta+1)}$$
$$I_{42}(\Lambda)=\frac{n}{\theta q}$$
$$I_{41}\,(\Lambda)=\frac{nq}{a}\left[\frac{\Psi\,(1)-\Psi\,(\theta q+1)}{q}-\frac{\theta^2\,(\Psi\,(1)-\Psi\,(\theta q+2))}{\theta q+1}+\frac{2\theta q+1}{(q(q\theta+1))^2}\right]$$
$$I_{33}(\Lambda)=\frac{na^2\theta q}{b^2(q\theta+2)}$$
$$I_{32}(\Lambda)=\frac{-naq}{b(q\theta+1)}$$
$$I_{31}(\Lambda)=\frac{-nq\theta}{b(q\theta+2)}\,(\Psi\,(2)-\Psi\,(\theta q+3))-\frac{nq\theta(3+2\theta q)}{b(q\theta+2)^2(q\theta+1)}$$
$$I_{22}(\Lambda)=\frac{n}{\theta^2}$$
$$I_{21}(\Lambda)=\frac{nq\theta}{a(q\theta+1)}\,(\Psi\,(2)-\Psi\,(\theta q))$$
$$I_{11}(\Lambda)=\frac{n}{a^2}+\frac{\theta nq\left\{\Psi'\,(2)+\Psi'\,(\theta q+1)+(\Psi\,(2)-\Psi\,(\theta q+1))^2\right\}}{a^2(\theta q+2)}.$$

4.3. The Beta Lomax Distribution

The beta Lomax distribution, $GFP(1, \theta, b, p, q)$, can have the following elements for its Fisher information matrix where $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\theta, b, p, q)$

$$I_{44}(\Lambda) = n[\Psi'(q) - \Psi'(p+q)]$$

$$I_{43}(\Lambda) = -n\Psi'(p+q)$$

$$I_{42}(\Lambda) = \frac{-n\theta}{bB(p,q)}\left[B(p,q) - B\left(p, q+\frac{1}{\theta}\right)\right]$$

$$I_{41}(\Lambda) = \frac{n}{\theta}(\Psi(p+q) - \Psi(q))$$

$$I_{33}(\Lambda) = n\left[\Psi'(p) - \Psi'(p+q)\right]$$

$$I_{32}(\Lambda) = \frac{n\theta}{bB(p,q)}\left[B(p-1, q+1) - B\left(p-1, q+\frac{1}{\theta}+1\right)\right]$$

$$I_{31}(\Lambda) = \frac{nq}{\theta(p-1)}(\Psi(q+1) - \Psi(p+q))$$

$$\begin{aligned} I_{22}(\Lambda) = {} & \frac{n\theta q}{b^2} - \frac{n(\theta q+1)}{b^2 B(p,q)} B\left(p, q+\frac{2}{\theta}\right) + \frac{n\theta^2 q(p+q-1)}{b^2(p-2)} \\ & + \frac{n\theta(p-1)}{b^2 B(p,q)}\left[B\left(p-1, q+\frac{2}{\theta}+1\right) - B(p-1, q+1)\right] \\ & + \frac{n\theta^2(p-1)}{b^2 B(p,q)}\left[B\left(p-2, q+\frac{2}{\theta}+1\right) - 2B\left(p-2, q+\frac{1}{\theta}+1\right)\right] \end{aligned}$$

$$\begin{aligned} I_{21}(\Lambda) = {} & \frac{nq}{bB(p,q)} B\left(p, q+\frac{1}{\theta}\right) - \frac{n(p-1)}{bB(p,q)} B\left(p-1, q+\frac{1}{\theta}+1\right) \\ & + \frac{nq(p+q-1)}{b(p-2)}[\Psi(q+1) - \Psi(p+q-1)] \\ & - \frac{n(p-1)B(p-2, q+\frac{1}{\theta}+1)}{bB(p,q)}\left[\Psi\left(p+q+\frac{1}{\theta}-1\right) - \Psi\left(q+\frac{1}{\theta}+1\right)\right] \end{aligned}$$

$$\begin{aligned} I_{11}(\Lambda) = {} & \frac{n}{\theta^2} + \frac{nq(p+q-1)}{\theta^2(p-2)}\left(\Psi'(q+1) - \Psi'(p+q-1)\right) \\ & + \frac{nq(p+q-1)}{\theta^2(p-2)}\{\Psi(q+1) - \Psi(p+q-1)\}^2 \end{aligned}$$

In summary, we introduced exact Fisher matrix for the generalized Feller-Pareto (GFP) distribution and for Beta Lomax and generalized Singh-Maddala and a Pareto IV distributions. For diverse applications, see Arnold (2008), Brazauskas (2002, 2003), Kleiber and Kotz (2003), and Paranaíba et al. (2011). There are some useful integrals are used to compute the elements of Fisher matrix for GFP distribution will be in the following section.

5. Useful Integrals

The first-order and second-order partial derivatives of beta function enable us to get the exact Fisher matrix for GFP distribution. Useful integrals are in the following:

$$\int_0^1 v^{q-1}(1-v)^{p-1}\,ln(v)dv = B(p,q)[\Psi(q)-\Psi(p+q)]$$

$$\int_0^1 v^{q-1}(1-v)^{p-1}\,ln(1-v)dv = B(p,q)[\Psi(p)-\Psi(p+q)]$$

$$\int_0^1 v^{q-1}(1-v)^{p-1}\,ln(v)ln(1-v)dv$$

$$= B(p,q)\{[\Psi(q)-\Psi(p+q)][\Psi(p)-\Psi(p+q)]-\Psi'(p+q)\}$$

$$\int_0^1 v^{q-1}(1-v)^{p-1}\,ln^2(v)dv = B(p,q)\{[\Psi(q)-\Psi(p+q)]^2+\Psi'(q)-\Psi'(p+q)\}$$

$$\int_0^1 v^{q-1}(1-v)^{p-1}\,ln^2(1-v)dv = B(p,q)\{[\Psi(p)-\Psi(p+q)]^2+\Psi'(p)-\Psi'(p+q)\}.$$

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