Solved of Sheet (1)

Chapter (6): Electrostatic Boundary –Value Problems

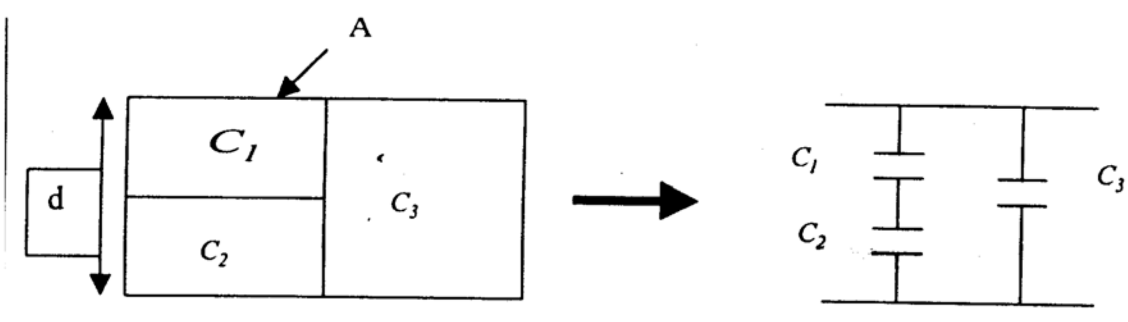
6.27 In an integrated circuit, a capacitor is formed by growing a silicon dioxide layer ($\varepsilon_r = 4$) of thickness 1 μm over the conducting silicon substrate and covering it with a metal electrode of area S. Determine S if a capacitance of 2 nF is desired.

Solution

$$C = \frac{\varepsilon S}{d} \longrightarrow S = \frac{Cd}{\varepsilon_o \varepsilon_r} = \frac{2x10^{-9}\, x10^{-6}}{4x10^{-9}/36\pi}\ \text{m}^2 = \underline{\underline{0.5655\ \text{cm}^2}}$$

6.28 The parallel-plate capacitor of Figure 6.34 is quarter-filled with mica ($\varepsilon_r = 6$). Find the capacitance of the capacitor.

Solution



From the figure above,

$$C = \frac{C_1 C_2}{C_1 + C_2} + C_3$$

where

$$C_1 = \frac{\varepsilon_o A/2}{d/2} = \frac{\varepsilon_o A}{d}, \qquad C_2 = \frac{\varepsilon_o \varepsilon_r A}{d}, \qquad C_3 = \frac{\varepsilon_o A}{2d}$$

$$C = \frac{\varepsilon_o^2 \varepsilon_r A^2/d^2}{\varepsilon_o(\varepsilon_r + 1)A/d} + \frac{\varepsilon_o A}{2d} = \frac{\varepsilon_o A}{d}\left(\frac{1}{2} + \frac{\varepsilon_r}{\varepsilon_r + 1}\right) = \frac{10^{-9}}{36\pi}\frac{10x10^{-4}}{2x10^{-3}}\left(\frac{1}{2} + \frac{6}{7}\right) \cong \underline{\underline{6\ \text{pF}}}$$

***6.29** An air-filled parallel plate capacitor of length L, width a, and plate separation d has its plates maintained at constant potential difference V_0. If a dielectric slab of dielectric constant ε_r is slid between the plates and is withdrawn until only a length x remains between the plates as in Figure 6.35, show that the force tending to restore the slab to its original position is

$$F = \frac{\varepsilon_0(\varepsilon_r - 1)\, a\, V_0^2}{2d}$$

Solution

$$F dx = dW_E \longrightarrow F = \frac{dW_E}{dx}$$

$$W_E = \int \frac{1}{2}\varepsilon |E|^2 dv = \frac{1}{2}\varepsilon_o \varepsilon_r E^2 xad + \frac{1}{2}\varepsilon_o E^2 da(l - x)$$

where $E = V_o / d$.

$$\frac{dW_E}{dx} = \frac{1}{2}\varepsilon_o \frac{V_o^2}{d^2}(\varepsilon_r - 1)da \longrightarrow F = \frac{\varepsilon_o(\varepsilon_r - 1)V_o^2 a}{2d}$$

Alternatively, $W_E = \frac{1}{2}CV_o^2$, where

$$C = C_1 + C_2 = \frac{\varepsilon_o \varepsilon_r ax}{d} + \frac{\varepsilon_o \varepsilon_r (l - x)}{d}$$

$$\frac{dW_E}{dx} = \frac{1}{2}\varepsilon_o \frac{V_o^2 a}{d}(\varepsilon_r - 1)$$

$$F = \frac{\varepsilon_o(\varepsilon_r - 1)V_o^2 a}{2d}$$

6.30 A parallel-plate capacitor has plate area 200 cm^2 and plate separation 3 mm. The charge density is 1 $\mu C/m^2$ with air as dielectric. Find

(a) The capacitance of the capacitor

(b) The voltage between the plates

(c) The force with which the plates attract each other

Solution

$$C = \frac{\varepsilon_o S}{d} = \frac{10^{-9}}{36\pi}\frac{200x10^{-4}}{3x10^{-3}} = 59 \text{ pF}$$

(b) $\rho_s = D_n = 10^{-6}$ nC / m². But

$$D_n = \varepsilon E_n = \frac{\varepsilon_o V_o}{d} = \rho_s$$

or

$$V_o = \frac{\rho_s d}{\varepsilon_o} = 10^{-6} x3x10^{-3} x36\pi x10^{9} = 339.3 \text{ V}$$

(c)

$$F = \frac{Q^2}{2S\varepsilon_o} = \frac{\rho_s{}^2 S}{2\varepsilon_o} = \frac{10^{-12} x200x10^{-4} x36\pi x10^{9}}{2} = 1.131 \text{ mN}$$

6.36 Determine the capacitance of a conducting sphere of radius 5 cm deeply immersed in sea water ($\varepsilon_r = 80$).

Solution

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

Since b $\longrightarrow \infty$,

$$C = 4\pi a\varepsilon_o\varepsilon_r = 4\pi x5x10^{-2} x80x\frac{10^{-9}}{36\pi} = \underline{\underline{444 \text{ pF}}}$$

6.37 A conducting sphere of radius 2 cm is surrounded by a concentric conducting sphere of radius 5 cm. If the space between the spheres is filled with sodium chloride ($\varepsilon_r = 5.9$), calculate the capacitance of the system.

Solution

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi x 5.9 x 10^{-9} / 36\pi}{\left(\frac{1}{2} - \frac{1}{5}\right) x 10^{-2}} = \underline{\underline{21.85 \text{ pF}}}$$

Solved of Sheet (2)

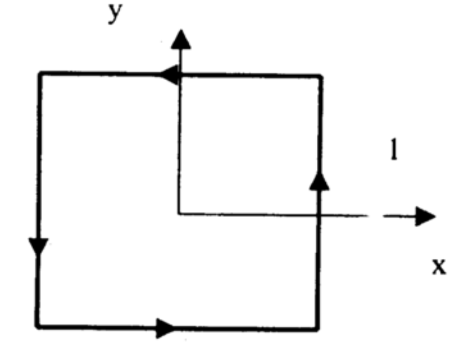
Chapter (7): Magneto static Fields:

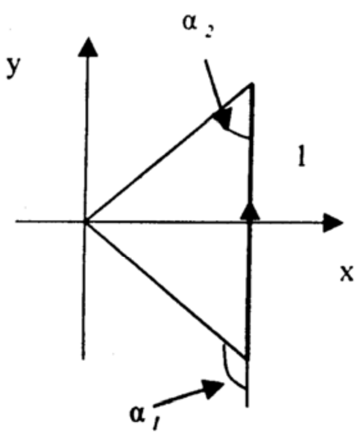
7.10 A square conducting loop of side $2a$ lies in the $z = 0$ plane and carries a current I in the counterclockwise direction. Show that at the center of the loop

$$\mathbf{H} = \frac{\sqrt{2}I}{\pi a}\mathbf{a}_z$$

Solution

Prob. 7.10





H = 4H₁, where H₁ is due to side 1.

$$H_1 = \frac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)a_\phi$$

$$\rho = a, \quad \alpha_2 = 45^o, \quad \alpha_1 = 135^o, \quad a_\phi = a_y x - a_x = a_z$$

$$H_1 = \frac{I}{4\pi\rho}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)a_z = \frac{2I}{4\pi a\sqrt{2}}a_z$$

7.18 (a) An infinitely long solid conductor of radius a is placed along the z-axis. If the conductor carries current I in the $+z$ direction, show that

$$\mathbf{H} = \frac{I\rho}{2\pi a^2}\mathbf{a}_\phi$$

within the conductor. Find the corresponding current density.

(b) If $I = 3$ A and $a = 2$ cm in part (a), find $\mathbf{H}$ at (0, 1 cm, 0) and (0, 4 cm, 0).

Solution

(a) Applying Ampere's law,

$$H_\phi \cdot 2\pi\rho = I \cdot \frac{\pi\rho^2}{\pi a^2} \rightarrow H_\phi = I \cdot \frac{I\rho^2}{2\pi a^2}$$

i.e $$\overline{H} = \frac{I\rho}{2\pi a^2}\bar{a}_\phi$$

$$\bar{J} = \overline{\nabla} \cdot \overline{H} = -\frac{\partial H_\phi}{\partial z}\bar{a}_\rho + \frac{I}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\bar{a}_z$$

$$= \frac{I}{\rho}\frac{1}{2\pi a^2} \cdot 2\rho\bar{a}_z = \frac{I}{\pi a^2}\bar{a}_z$$

(b) From Prob. 7.15,

$$H_\phi = \begin{cases} \frac{I\rho}{2\pi a^2}, & \rho < a \\ \frac{I}{2\pi\rho}, & \rho > a \end{cases}$$

At $(0, 1\,\text{cm}, 0)$,

$$H_\phi = \frac{3 \times 1 \times 10^{-2}}{2\pi \times 4 \times 10^{-4}} = \frac{300}{8\pi}$$

$$\overline{H} = \underline{\underline{11.94\,\bar{a}_\phi \text{ A/m}}}$$

At $(0, 4\,\text{cm}, 0)$,

$$H_\phi = \frac{3}{2\pi \times 4 \times 10^{-2}} = \frac{300}{8\pi}$$

$$\overline{H} = \underline{\underline{11.94\,\bar{a}_\phi \text{ A/m}}}$$

7.20 In a certain conducting region,

$$\mathbf{H} = yz(x^2 + y^2)\mathbf{a}_x - y^2xz\mathbf{a}_y + 4x^2y^2\mathbf{a}_z \text{ A/m}$$

(a) Determine $\mathbf{J}$ at $(5, 2, -3)$

(b) Find the current passing through $x = -1, 0 < y, z < 2$

(c) Show that $\nabla \cdot \mathbf{B} = 0$

Solution

(a) $$\bar{J} = \nabla \times \bar{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz(x^2+y^2) & -y^2 \times z & -4x^2y^2 \end{vmatrix}$$

$$= (8x^2y + xy^2)\bar{a}_x + [y(x^2+y^2) - 4xy^2]\bar{a}_y$$
$$+ [-y^2z - z(x^2+y^2)]\bar{a}_z$$

At $(5, 2, -3)$, $x = 5$, $y = 2$, $z = -3$

$$\bar{J} = 420\,\bar{a}_z - 22\,\bar{a}_y + 99\,\bar{a}_z \text{ A/m}^2$$

(b) $$I = \int J \cdot dS = \iint (8x^2y + xy^2)\,dydz\Big|_{x=-1}$$

$$= \int_0^2 dz \int_0^2 (8y - y^2)dz = 2\left(4y^2 - \frac{y^3}{3}\right)\Big|_0^2$$

$$= 4\left(16 - \frac{8}{3}\right) = 53.33 \text{ A}$$

(c) $\bar{B} = \mu\bar{H}$, $\nabla \cdot \bar{B} = 0 \rightarrow \nabla \cdot \bar{H} = 0$

$$\nabla \cdot \bar{H} = \frac{\partial}{\partial x}H_x + \frac{\partial}{\partial y}H_y + \frac{\partial}{\partial z}H_z = 2xy - 2yxz = 0$$

Hence $\nabla \cdot \bar{B} = 0$

7.26 Consider the following arbitrary fields. Find out which of them can possibly represent electrostatic or magnetostatic field in free space.

(a) $\mathbf{A} = y \cos ax\mathbf{a}_x + (y + e^{-x})\mathbf{a}_z$

(b) $\mathbf{B} = \dfrac{20}{\rho}\mathbf{a}_\rho$

(c) $\mathbf{C} = r^2 \sin\theta\, \mathbf{a}_\phi$

Solution

(a)

$$\nabla \cdot \overline{A} = -ya \sin ax \neq 0$$

$$\nabla \times \overline{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y \cos ax & 0 & y + e^{x} \end{vmatrix}$$

$$= \overline{a}_x + e^{-x}\overline{a}_y - \cos a_x \overline{a}_z \neq 0$$

$\overline{A}$ is neither electrostatic nor magnetostatic field

(b)

$$\nabla \cdot \overline{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho B_\rho\right) = \frac{1}{\rho}\frac{\partial}{\partial \rho}(20) = 0$$

$$\nabla \times \overline{B} = 0$$

$\overline{B}$ can be $\overline{E}$ - field in a charge - free region.

(c)

$$\nabla \cdot \overline{C} = \frac{1}{r^2} 4r^3 \sin\theta \neq 0$$

$$\nabla \times \overline{C} = \frac{1}{r \sin\theta}\frac{\partial}{\partial r}\left(r^2 \sin^2\theta\right) \neq 0$$

$\overline{C}$ is neither or $\overline{E}$ nor $\overline{H}$ field.

7.29 The magnetic vector potential of a current distribution in free space is given by

$$\mathbf{A} = 15e^{-\rho} \sin\phi \, \mathbf{a}_z \text{ Wb/m}$$

Find **H** at $(3, \pi/4, -10)$. Calculate the flux through $\rho = 5, 0 \le \phi \le \pi/2, 0 \le z \le 10$.

Solution

$$\overline{B} = \overline{\nabla} \times \overline{A} = \frac{1}{\rho}\frac{\partial A_z}{\partial \phi}\bar{a}_\rho - \frac{\partial A_z}{\partial \rho}\bar{a}_\phi$$

$$= \frac{15}{\rho} e^{-\rho} \cos\phi \, \bar{a}_\rho + 15 e^{-\rho} \sin\phi \, \bar{a}_\phi$$

$$\overline{B}\left(3, \frac{\pi}{4}, -10\right) = 5 e^{-3} \frac{1}{\sqrt{2}} \bar{a}_\rho + 15 e^{-3} \frac{1}{\sqrt{2}} \bar{a}_\phi$$

$$\overline{H} = \frac{\overline{B}}{\mu_o} = \frac{10^7}{4\pi} \frac{15}{\sqrt{2}} e^{-3} \left(\frac{1}{3}\bar{a}_\rho + \bar{a}_\phi\right)$$

$$\overline{H} = \left(14\,\bar{a}_\rho + 42\,\bar{a}_\phi\right) \cdot 10^4 \text{ A/m}$$

$$\psi = \int \overline{B} \cdot d\bar{s} = \iint \frac{15}{\rho} e^{-\rho} \cos\phi \, \rho \, d\phi \, dz$$

$$= 15\,z\Big|_0^{10} \left(-\sin\phi\right)\Big|_0^{\pi/2} e^{-5} = -150\,e^{-5} \quad \Rightarrow \quad \psi = -1.011 \text{ Wb}$$

7.33 Find the current density **J** to

$$\mathbf{A} = \frac{10}{\rho^2}\,\mathbf{a}_z \text{ Wb/m}$$

in free space.

Solution

$$\bar{J} = \bar{\nabla} \times \bar{H} = \bar{\nabla} \times \frac{\bar{\nabla} \times \bar{A}}{\mu_o} = \frac{1}{\mu_o}\bar{\nabla} \times \bar{\nabla} \times \bar{A}$$

$$\bar{\nabla} \times \bar{A} = \frac{\partial A_z}{\partial \rho}\bar{a}_\psi = \frac{+20}{\rho^3}\bar{a}_\psi$$

$$\bar{\nabla} \times \bar{\nabla} \times \bar{A} = \frac{1}{\rho}\frac{\partial}{\partial d}\left(\rho A_\phi\right)\bar{a}_z = -\frac{40}{\rho^4}\bar{a}_z$$

$$\bar{J} = -\frac{40}{\mu_o \rho^4}\bar{a}_z \text{ A/m}^2$$

or $\quad \bar{\nabla}^2 \bar{A} = -\mu_o \bar{J}$

or $\quad \bar{J} = -\frac{1}{\mu_o}\bar{\nabla}^2 \bar{A} = -\frac{1}{\mu_o}\bar{\nabla}^2 A_z \bar{a}_z$

$$= -\frac{1}{\mu_o}\bar{a}_z\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial A_z}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2}\right]$$

$$= \frac{1}{\mu_o}\bar{a}_z\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{20}{\rho^2}\right) = -\frac{40}{\mu_o \rho^4}\bar{a}_z \text{ A/m}^2$$

Solved of Sheet (3)

8.1 An electron with velocity $\mathbf{u} = (3\mathbf{a}_x + 12\mathbf{a}_y - 4\mathbf{a}_z) \times 10^5$ m/s experiences no net force at a point in a magnetic field $\mathbf{B} = 10\mathbf{a}_x + 20\mathbf{a}_y + 30\mathbf{a}_z$ mWb/m^2. Find $\mathbf{E}$ at that point.

Solution

Prob. 8.1

$$\bar{F} = q\left(\bar{E} + u \times \bar{B}\right)$$

If $\bar{F} = 0, \quad \bar{E} = -u \times \bar{B} = \bar{B} \times u$

$$= \begin{vmatrix} 10 & 20 & 30 \\ 3 & 12 & -4 \end{vmatrix} \times 10^5 \times 10^{-3}$$

$$\bar{E} = -4.4\bar{a}_x + 1.3\bar{a}_x + 11.4\bar{a}_x \text{ kV/}$$

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8.2 A charged particle of mass 1 kg and charge 2 C starts at the origin with velocity $10\mathbf{a}_z$ m/s in a magnetic field $\mathbf{B} = 1\mathbf{a}_x$ Wb/m^2. Find the location and the kinetic energy of the particle at $t = 2$ s.

Solution

Prob.8.2

$$\bar{F} = ma = q\,u \times B$$

$$\bar{a} = \frac{q}{m} u \times \bar{B}$$

$$\frac{d}{dt}(u_x, u_y, u_z) = 0 \quad \rightarrow \quad \frac{2}{1}\begin{vmatrix} u_x & u_y & u_z \\ 1 & 0 & 0 \end{vmatrix} = 2(0, u_z, -u_y)$$

$$\frac{du_x}{dt} = 0 \quad \rightarrow \quad u_x = C_o \qquad \ldots \quad (1)$$

$$\frac{du_y}{dt} = 2u_z, \quad \frac{du_z}{dt} = -2u_y$$

$$\frac{d^2u_y}{dt^2} = 2, \quad \frac{du_z}{dt} = -4u_y$$

$$\ddot{u}_y + 4u_y = 0$$

$$u_y = C_1 \cos 2t + C_2 \sin 2t \qquad \ldots \quad (2)$$

$$u_z = \frac{1}{2}\frac{du_y}{dt} = -C_1 \sin 2t + C_2 \cos 2t \qquad \ldots \quad (3)$$

At $t = 0$, $u_x = 0 \rightarrow c_0 = 0$

$u_y = 0 \rightarrow c_1 = 0$

$u_z = 10 \rightarrow c_2 = 10$

Hence,

$$\bar{u} = (0, 10 \sin 2t, 10 \cos 2t)$$

$$u_x = \frac{dx}{dt} = 0 \quad \rightarrow \quad x = c_4$$

$$u_y = \frac{dy}{dt} = 10 \sin 2t \quad \rightarrow \quad y = -5 \cos 2t + c_5$$

$$u_z = 10 \cos 2t \quad \rightarrow \quad z = 5 \sin 2t + c_6$$

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***8.3** A particle with mass 1 kg and charge 2 C starts from rest at point (2, 3, −4) in a region where $\mathbf{E} = -4\mathbf{a}_y$ V/m and $\mathbf{B} = 5\mathbf{a}_x$ Wb/m^2. Calculate

(a) The location of the particle at $t = 1$ s

(b) Its velocity and K.E. at that location

Solution

(a) $F = m\vec{a} = Q(\vec{E} + \vec{u} \times \vec{B})$

$$\frac{d}{dt}(u_x, u_y, u_z) = 2\left[-4\vec{a}_y + \begin{vmatrix} u_x & u_y & u_z \\ 5 & 0 & 0 \end{vmatrix}\right] = -8\vec{a}_y + 10u_z\vec{a}_y - 10u_y\vec{a}_z$$

i.e. $\frac{du_x}{dt} = 0 \rightarrow u_x = A_1$ (1)

$\frac{du_y}{dt} = -8 + 10u_z$ (2)

$\frac{du_z}{dt} = -10u_y$ (3)

$$\frac{d^2u_y}{dt^2} = 0 + 10\frac{du_z}{dt} = -100u_y$$

$$\ddot{u}_y + 100u_y = 0 \rightarrow u_y = B_1\cos 10t + B_2\sin 10t$$

From (2),

$$10u_z = 8 + \dot{u}_y = 8 - 10B_1\sin 10t + 10B_2\cos 10t$$

$$u_z = 0.8 - B_1\sin 10t + B_2\cos 10t$$

At t=0, $\bar{u} = 0 \rightarrow A_1 = 0, B_1 = 0, B_2 = -0.8$

$$\bar{u} = (0,\ 0.8\sin 10t,\ 0.8 - 0.8\cos 10t)$$

$$u_x = \frac{dx}{dt} = 0 \rightarrow x = c_1$$

$$u_y = \frac{dy}{dt} = -0.8\sin 10t \rightarrow y = 0.08\cos 10t + c_2$$

$$u_z = \frac{dz}{dt} = 0.8 - 0.8\cos 10t \rightarrow z = 0.8t + c_3 - 0.08\sin 10t$$

At t=0, (x, y, z) = (2, 3, -4) $\Rightarrow$ c_1=2, c_2=2.92, c_3=-4

Hence (x, y, z) = (2, 2 + 0.08cos10t, 0.8t – 0.08sin10t – 4)

At t=1,

(x, y, z) = <u>(2, 1.933, -3.156)</u>

(b) From (4), at t=1, $\vec{u} = (0, 0.435, 1.471)$ m/s

$$\text{K.E.} = \frac{1}{2}m|\vec{u}|^2 = \frac{1}{2}(1)(0.435^2 + 1.471^2) = \underline{1.177\text{J}}$$

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8.4 A -2-mC charge starts at point (0, 1, 2) with a velocity of $5\mathbf{a}_x$ m/s in a magnetic field $\mathbf{B} = 6\mathbf{a}_y$ Wb/m^2. Determine the position and velocity of the particle after 10 s assuming that the mass of the charge is 1 gram. Describe the motion of the charge.

<u>Solution</u>

$$m\vec{a} = Q\vec{u} \times \vec{B}$$

$$10^{-3}\vec{a} = -2\times10^{-3}\begin{vmatrix} u_x & u_y & u_z \\ 0 & 6 & 0 \end{vmatrix}$$

$$\frac{d}{dt}(u_x, u_y, u_z) = (12u_z, 0, -12u_x)$$

i.e. $\frac{du_x}{dt} = -12u_z$ (1)

$\frac{du_y}{dt} = 0 \rightarrow u_y = A_1$ (2)

$\frac{du_z}{dt} = -12u_x$ (3)

From (1) and (2),

$$\ddot{u}_x = -12\dot{u}_z = -144u_x$$

or

$$\ddot{u}_x + 144u_x = 0 \rightarrow u_x = c_1\cos 12t + c_2\sin 12t$$

From (1), $u_z = -c_1\sin 12t + c_2\cos 12t$

At t=0,

$u_x=2$, $u_y=0$, $u_z=0 \rightarrow A_1=0=c_2$, $c_1=5$

Hence,

$$\vec{u} = (5\cos 12t, 0, -5\sin 12t)$$

$$\vec{u}(t = 10s) = (5\cos 120, 0, -5\sin 120) = \underline{\underline{4.071\vec{a}_x - 2.903\vec{a}_z}} \text{ m/s}$$

$$u_x = \frac{dx}{dt} = 5\cos 12t \rightarrow x = \frac{5}{12}\sin 12t + B_1$$

$$u_y = \frac{dy}{dt} = 0 \rightarrow y = B_2$$

$$u_z = \frac{dz}{dt} = -5\sin 12t \rightarrow z = \frac{5}{12}\cos 12t + B_3$$

At t=0, (x, y, z) = (0, 1, 2) $\rightarrow B_1=0$, $B_2=1$, $B_3=\frac{19}{12}$

$$(x, y, z) = \left(\frac{5}{12}\sin 12t, 1, \frac{5}{12}\cos 12t + \frac{19}{12}\right) \qquad (4)$$

At t=10s,

$$(x, y, z) = \left(\frac{5}{12}\sin 120, 1, \frac{5}{12}\cos 120 + \frac{19}{12}\right) = \underline{(0.2419, 1, 1.923)}$$

By eliminating t from (4),

$x^2 + (z = 19/12) = (5/12)^2$, $y = 1$ which is a helix with axis on line y=1, z=$\frac{19}{12}$

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