Solved problem for Electromagnetism subject (231)

Chapter (1): Vector algebra

Problems:

(1.1, 1.3, 1.5, 1.7, 1.9 .1.11, 1.13, 1.15, 1.17 1.19, 1.21)

1.1 Find the unit vector along the line joining point (2, 4, 4) to point (−3, 2, 2).

$$\boldsymbol{r} = (-3,2,2) - (2,4,4) = (-5,-2,-2)$$

$$\boldsymbol{a}_r = \frac{\boldsymbol{r}}{|\boldsymbol{r}|} = \frac{(-5,-2,-2)}{\sqrt{25+4+4}} = -0.8703\boldsymbol{a}_x - 0.3482\boldsymbol{a}_y - 0.3482\boldsymbol{a}_z$$

1.3 If

$$\mathbf{A} = 2\mathbf{a}_x + \mathbf{a}_y - 3\mathbf{a}_z$$

$$\mathbf{B} = \mathbf{a}_y - \mathbf{a}_z$$

$$\mathbf{C} = 3\mathbf{a}_x + 5\mathbf{a}_y + 7\mathbf{a}_z$$

determine:

(a) $\mathbf{A} - 2\mathbf{B} + \mathbf{C}$

(b) $\mathbf{C} - 4(\mathbf{A} + \mathbf{B})$

(c) $\dfrac{2\mathbf{A} - 3\mathbf{B}}{|\mathbf{C}|}$

(d) $\mathbf{A} \cdot \mathbf{C} - |\mathbf{B}|^2$

(e) $\frac{1}{2}\mathbf{B} \times \left(\frac{1}{3}\mathbf{A} + \frac{1}{4}\mathbf{C}\right)$

(a) $\boldsymbol{A}-2\boldsymbol{B}=(2,1,-3)-(0,2,-2)=(2,-1,-1)$

$$\boldsymbol{A}-2\boldsymbol{B}+\boldsymbol{C}=\underline{\underline{5\boldsymbol{a}_x+4\boldsymbol{a}_y+6\boldsymbol{a}_z}}$$

(b) $\boldsymbol{A}+\boldsymbol{B}=(2,2,-4)$

$$\boldsymbol{C}-4(\boldsymbol{A}+\boldsymbol{B})=(3,5,7)-(8,8,-16)=\underline{\underline{-5\boldsymbol{a}_x-3\boldsymbol{a}_y+23\boldsymbol{a}_z}}$$

(c) $2\boldsymbol{A}-3\boldsymbol{B}=(4,2,-6)-(0,3,-3)=(4,-1,-3)$

$$|\boldsymbol{C}|=\sqrt{9+25+49}=9.11$$

$$\frac{2\boldsymbol{A}-3\boldsymbol{B}}{|\boldsymbol{C}|}=\underline{\underline{0.439\boldsymbol{a}_x-0.11\boldsymbol{a}_y-0.3293\boldsymbol{a}_z}}$$

(d) $A\cdot C=6+5-21=-10,$

$$|B|=\sqrt{2}$$

$$A\cdot C-|B|^2=-10+2=\underline{\underline{-8}}$$

(e) $\frac{1}{3}A+\frac{1}{4}C=\left(\frac{2}{3},\frac{1}{3},-1\right)+\left(\frac{3}{4},\frac{5}{4},\frac{7}{4}\right)=(1.4167,1.5833,0.75)$

$$\frac{1}{2}B\times\left(\frac{1}{3}A+\frac{1}{4}C\right)=\frac{1}{2}\begin{vmatrix}0 & 1 & -1\\ 1.4167 & 1.5833 & 0.75\end{vmatrix}=\underline{\underline{1.1667\boldsymbol{a}_x-0.7084\boldsymbol{a}_y-0.7084\boldsymbol{a}_z}}$$

1.5 If

$$\mathbf{A} = 5\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z$$

$$\mathbf{B} = -\mathbf{a}_x + 4\mathbf{a}_y + 6\mathbf{a}_z$$

$$\mathbf{C} = 8\mathbf{a}_x + 2\mathbf{a}_y$$

find the values of α and β such that $\alpha\mathbf{A} + \beta B + C$ is parallel to the y-axis.

Let $\boldsymbol{D} = \alpha\boldsymbol{A} + \beta\boldsymbol{B} + \boldsymbol{C}$

$$= (5\alpha - \beta + 8)\boldsymbol{a}_x + (3\alpha + 4\beta + 2)\boldsymbol{a}_y + (-2\alpha + 6\beta)\boldsymbol{a}_z$$

$$\boldsymbol{D}_x = 0 \rightarrow 5\alpha - \beta + 8 = 0 \qquad (1)$$

$$\boldsymbol{D}_z = 0 \rightarrow -2\alpha + 6\beta = 0 \rightarrow \alpha = 3\beta \qquad (2)$$

Substituting (2) into (1),

$$15\beta - \beta + 8 = 0 \rightarrow \beta = -\frac{8}{14} = -\frac{4}{7}$$

Thus

$$\underline{\underline{\alpha = -\frac{12}{7}, \beta = -\frac{4}{7}}}$$

1.7 (a) Show that

$$(\mathbf{A} \cdot \mathbf{B})^2 + (\mathbf{A} \times \mathbf{B})^2 = (AB)^2$$

(b) Show that

$$\mathbf{a}_x = \frac{\mathbf{a}_y \times \mathbf{a}_z}{\mathbf{a}_x \cdot \mathbf{a}_y \times \mathbf{a}_z}, \qquad \mathbf{a}_y = \frac{\mathbf{a}_z \times \mathbf{a}_x}{\mathbf{a}_x \cdot \mathbf{a}_y \times \mathbf{a}_z}, \qquad \mathbf{a}_z = \frac{\mathbf{a}_x \times \mathbf{a}_y}{\mathbf{a}_x \cdot \mathbf{a}_y \times \mathbf{a}_z}$$

(a) $\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{AB} \cos\theta_{AB}$

$\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{AB} \sin\theta_{AB} \boldsymbol{a}_n$

$(\boldsymbol{A} \cdot \boldsymbol{B})^2 + |\boldsymbol{A} \times \boldsymbol{B}|^2 = (\boldsymbol{AB})^2 \left(\cos^2\theta_{AB} + \sin^2\theta_{AB}\right) = (\boldsymbol{AB})^2$

(b) $\boldsymbol{a}_x \cdot \left(\boldsymbol{a}_y \times \boldsymbol{a}_z\right) = \boldsymbol{a}_x \cdot \boldsymbol{a}_x = 1$. Hence,

$$\frac{\boldsymbol{a}_y \times \boldsymbol{a}_z}{\boldsymbol{a}_x \cdot \boldsymbol{a}_y \times \boldsymbol{a}_z} = \frac{\boldsymbol{a}_x}{1} = \boldsymbol{a}_x$$

$$\frac{\boldsymbol{a}_z \times \boldsymbol{a}_x}{\boldsymbol{a}_x \cdot \boldsymbol{a}_y \times \boldsymbol{a}_z} = \frac{\boldsymbol{a}_y}{1} = \boldsymbol{a}_y$$

$$\frac{\boldsymbol{a}_x \times \boldsymbol{a}_y}{\boldsymbol{a}_x \cdot \boldsymbol{a}_y \times \boldsymbol{a}_z} = \frac{\boldsymbol{a}_z}{1} = \boldsymbol{a}_z$$

1.9 Given vectors $\mathbf{T} = 2\mathbf{a}_x - 6\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{S} = \mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z$, find: (a) the scalar projection of **T** on **S**, (b) the vector projection of **S** on **T**, (c) the smaller angle between **T** and **S**.

(a) $\boldsymbol{T}_s = \boldsymbol{T} \cdot \boldsymbol{a}_s = \dfrac{\boldsymbol{T} \cdot \boldsymbol{S}}{|\boldsymbol{S}|} = \dfrac{(2,-6,-3)\cdot(1,2,1)}{\sqrt{6}} = \dfrac{-7}{\sqrt{6}} = \underline{\underline{-2.8577}}$

(b) $\boldsymbol{S}_T = (\boldsymbol{S} \cdot \boldsymbol{a}_T)\boldsymbol{a}_T = \dfrac{(\boldsymbol{S}\cdot\boldsymbol{T})\boldsymbol{T}}{\boldsymbol{T}^2} = \dfrac{-7(2,-6,3)}{7^2}$

$= \underline{\underline{-0.2857\boldsymbol{a}_x + 0.8571\boldsymbol{a}_y - 0.4286\boldsymbol{a}_z}}$

(c) $\sin\theta_{TS} = \dfrac{|\boldsymbol{T}\times\boldsymbol{S}|}{|\boldsymbol{T}||\boldsymbol{S}|} = \begin{vmatrix} 2 & -6 & 3 \\ 1 & 2 & 1 \end{vmatrix} = \dfrac{|(-12,1,10)|}{7\sqrt{6}} = \dfrac{\sqrt{245}}{7\sqrt{6}} = 0.9129$

$\Rightarrow \theta_{TS} = \underline{\underline{65.91^\circ}}$

1.11 Calculate the angles that vector $\mathbf{H} = 3\mathbf{a}_x + 5\mathbf{a}_y - 8\mathbf{a}_z$ makes with the x-, y-, and z-axes.

$$\cos\theta = \frac{\boldsymbol{H}\cdot\boldsymbol{a}_x}{|\boldsymbol{H}|} = \frac{3}{\sqrt{9+25+64}} = \frac{3}{98}$$

$$\theta_x = \underline{\underline{72.36^\circ}}$$

$$\cos\theta = \frac{\boldsymbol{H}\cdot\boldsymbol{a}_y}{|\boldsymbol{H}|} = \frac{5}{\sqrt{9+25+64}} = \frac{5}{98}$$

$$\theta_y = \underline{\underline{59.66^\circ}}$$

$$\cos\theta = \frac{\boldsymbol{H}\cdot\boldsymbol{a}_z}{|\boldsymbol{H}|} = \frac{-8}{\sqrt{9+25+64}} = \frac{-8}{98}$$

$$\theta_z = \underline{\underline{143.91^\circ}}$$

1.13 Simplify the following expressions:

(a) $\mathbf{A} \times (\mathbf{A} \times \mathbf{B})$

(b) $\mathbf{A} \times [\mathbf{A} \times (\mathbf{A} \times \mathbf{B})]$

(a) Using the fact that

$$(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{A},$$

we get

$$\boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{B}) = -(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{A} = \underline{\underline{(\boldsymbol{B} \cdot \boldsymbol{A})\boldsymbol{A} - (\boldsymbol{A} \cdot \boldsymbol{A})\boldsymbol{B}}}$$

(b)
$$\boldsymbol{A} \times (\boldsymbol{A} \times (\boldsymbol{A} \times \boldsymbol{B})) = \boldsymbol{A} \times [(\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{A} - (\boldsymbol{A} \cdot \boldsymbol{A})\boldsymbol{B}]$$
$$= \underline{\underline{(\boldsymbol{A} \cdot \boldsymbol{B})(\boldsymbol{A} \times \boldsymbol{A}) - (\boldsymbol{A} \cdot \boldsymbol{A})(\boldsymbol{A} \times \boldsymbol{B})}}$$

1.15 Points $P_1(1, 2, 3)$, $P_2(-5, 2, 0)$, and $P_3(2, 7, -3)$ form a triangle in space. Calculate the area of the triangle.

$$\boldsymbol{P_1P_2} = \boldsymbol{r}_{P_2} - \boldsymbol{r}_{P_1} = (-6, 0, -3)$$

$$\boldsymbol{P_1P_3} = \boldsymbol{r}_{P_3} - \boldsymbol{r}_{P_1} = (1, 5, -6)$$

$$\boldsymbol{P_1P_2} \times \boldsymbol{P_1P_3} = \begin{vmatrix} -6 & 0 & -3 \\ 1 & 5 & -6 \end{vmatrix} = (15, 39, -30)$$

$$\text{Area of the triangle} = \frac{1}{2}\left|\boldsymbol{P_1P_2} \times \boldsymbol{P_1P_3}\right| = \frac{1}{2}\sqrt{15^2 + 39^2 + 30^2} = \underline{\underline{25.72}}$$

1.17 Points P, Q, and R are located at $(-1, 4, 8)$, $(2, -1, 3)$, and $(-1, 2, 3)$, respectively. Determine: (a) the distance between P and Q, (b) the distance vector from P to R, (c) the angle between QP and QR, (d) the area of triangle PQR, (e) the perimeter of triangle PQR.

(a) $\boldsymbol{r}_{PQ} = \boldsymbol{r}_Q - \boldsymbol{r}_P = (2,-1,3) - (-1,4,8) = (3,-5,-5)$

$$r_{PQ} = |\boldsymbol{r}_{PQ}| = \sqrt{9+25+25} = \underline{\underline{7.681}}$$

(b) $\boldsymbol{r}_{PR} = \boldsymbol{r}_R - \boldsymbol{r}_P = (-1,2,3) - (-1,4,8) = (0,-2,-5) = \underline{\underline{-2\boldsymbol{a}_y - 5}}$

(c) $\boldsymbol{r}_{QP} = -\boldsymbol{r}_{PQ} = -3\boldsymbol{a}_x + 5\boldsymbol{a}_y + 5\boldsymbol{a}_z$

$$\boldsymbol{r}_{QR} = \boldsymbol{r}_Q - \boldsymbol{r}_R = (2,-1,3) - (-1,2,3) = 3\boldsymbol{a}_x - 3\boldsymbol{a}_y$$

$$\cos\theta = \frac{\boldsymbol{r}_{QP} \cdot \boldsymbol{r}_{QR}}{|\boldsymbol{r}_{QP}||\boldsymbol{r}_{QR}|} = \frac{-9-15}{\sqrt{9+25+25}\sqrt{9+9}} = \frac{-24}{\sqrt{18}\sqrt{59}}$$

$$\underline{\underline{\theta = 137.43^\circ}}$$

(d) $\text{Area} = \frac{1}{2}|\boldsymbol{r}_{QP} \times \boldsymbol{r}_{QR}|$

$$\boldsymbol{r}_{QP} \times \boldsymbol{r}_{QR} = \begin{vmatrix} -3 & 5 & 5 \\ 3 & -3 & 0 \end{vmatrix} = 15\boldsymbol{a}_x + 15\boldsymbol{a}_y - 6\boldsymbol{a}_z$$

$$\text{Area} = \frac{1}{2}\sqrt{15^2 + 15^2 + 6^2} = \underline{\underline{11.02}}$$

(e) Perimeter $= QP + PR + RQ = \boldsymbol{r}_{QP} + \boldsymbol{r}_{PR} + \boldsymbol{r}_{QR}$

$$= \sqrt{59} + \sqrt{4+25} + \sqrt{18}$$

$$= 7.681 + 5.385 + 4.243$$

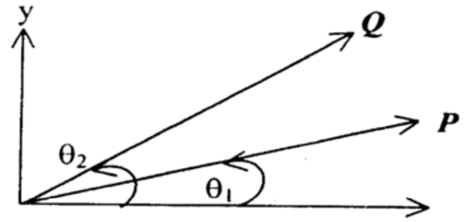
$$= \underline{17.31}$$

***1.19** (a) Prove that $\mathbf{P} = \cos\theta_1 \mathbf{a}_x + \sin\theta_1 \mathbf{a}_y$ and $\mathbf{Q} = \cos\theta_2 \mathbf{a}_x + \sin\theta_2 \mathbf{a}_y$ are unit vectors in the xy-plane respectively making angles θ_1 and θ_2 with the x-axis.

(b) By means of dot product, obtain the formula for $\cos(\theta_2 - \theta_1)$. By similarly formulating $\mathbf{P}$ and $\mathbf{Q}$, obtain the formula for $\cos(\theta_2 + \theta_1)$.

(c) If θ is the angle between $\mathbf{P}$ and $\mathbf{Q}$, find $\frac{1}{2}|\mathbf{P} - \mathbf{Q}|$ in terms of θ.

(a) Let P and Q be as shown below:



$$|\boldsymbol{P}| = \cos^2\theta_1 + \sin^2\theta_1 = 1, |\boldsymbol{Q}| = \cos^2\theta_2 + \sin^2\theta_2 = 1,$$

Hence **P** and **Q** are unit vectors.

(b) $\boldsymbol{P}\cdot\boldsymbol{Q} = (1)(1)\cos(\theta_2 - \theta_1)$

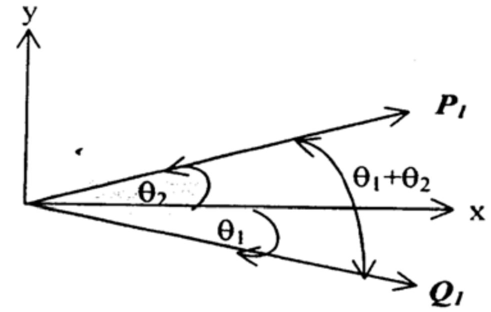
But $\boldsymbol{P}\cdot\boldsymbol{Q} = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$. Thus,

$$\underline{\underline{\cos(\theta_2 - \theta_1) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}}$$

Let $\boldsymbol{P}_1 = \boldsymbol{P} = \cos\theta_1\boldsymbol{a}_x + \sin\theta_1\boldsymbol{a}_y$ and

$\boldsymbol{Q}_1 = \cos\theta_2\boldsymbol{a}_x - \sin\theta_2\boldsymbol{a}_y$.

$\boldsymbol{P}_1$ and $\boldsymbol{Q}_1$ are unit vectors as shown below:



$\boldsymbol{P}_1\cdot\boldsymbol{Q}_1 = (1)(1)\cos(\theta_1 + \theta_2)$

But $\boldsymbol{P}_1\cdot\boldsymbol{Q}_1 = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$,

$$\underline{\underline{\cos(\theta_2 + \theta_1) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}}$$

Alternatively, we can obtain this formula from the previous one by replacing θ_2 by $-\theta_2$ in $\boldsymbol{Q}$.

(c)

$$\frac{1}{2}|P-Q| = \frac{1}{2}|(\cos\theta_1 - \cos\theta_2)a_x + (\sin\theta_1 - \sin\theta_2)a_y$$

$$= \frac{1}{2}\sqrt{\cos^2\theta_1 + \sin^2\theta_1 + \cos^2\theta_2 + \sin^2\theta_2 - 2\cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2}$$

$$= \frac{1}{2}\sqrt{2 - 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)} = \frac{1}{2}\sqrt{2 - 2\cos(\theta_2 - \theta_1)}$$

Let $\theta_2 - \theta_1 = \theta$, the angle between **P** and **Q.**

$$\frac{1}{2}|P-Q| = \frac{1}{2}\sqrt{2 - 2\cos\theta}$$

But $\cos 2A = 1 - 2\sin^2 A$.

$$\frac{1}{2}|P-Q| = \frac{1}{2}\sqrt{2 - 2 + 4\sin^2\theta/2} = \sin\theta/2$$

Thus,

$$\frac{1}{2}|P-Q| = |\sin\frac{\theta_2 - \theta_1}{2}|$$

1.21 Given $\mathbf{A} = x^2y\mathbf{a}_x - yz\mathbf{a}_y + yz^2\mathbf{a}_z$, determine:

(a) The magnitude of $\mathbf{A}$ at point $T(2, -1, 3)$

(b) The distance vector from T to S if S is 5.6 units away from T and in the same direction as $\mathbf{A}$ at T

(c) The position vector of S

(a) At T, $\boldsymbol{A} = (-4,3,-9)$

$$|\boldsymbol{A}| = \sqrt{16+9+81} = \sqrt{106} = \underline{\underline{10.3}}$$

(b) Let $\boldsymbol{r}_{TS} = \boldsymbol{B} = B\boldsymbol{a}_B$

$$B = 5.6, \boldsymbol{a}_B = \boldsymbol{a}_A = \frac{(-4,3,-9)}{10.3}$$

$$\boldsymbol{r}_{TS} = \boldsymbol{B} = \frac{5.6(-4,3,9)}{10.3}$$

$$= \underline{\underline{-2.175\boldsymbol{a}_x + 1.631\boldsymbol{a}_y - 4.893\boldsymbol{a}_z}}$$

(c) $\boldsymbol{r}_{TS} = \boldsymbol{r}_S - \boldsymbol{r}_T \rightarrow \boldsymbol{r}_S = \boldsymbol{r}_T + \boldsymbol{r}_{TS}$

$$\therefore \underline{\underline{\boldsymbol{r}_S = -0.175\boldsymbol{a}_x + 0.631\boldsymbol{a}_y - 1.893\boldsymbol{a}_z}}$$

2.7 Convert the following vectors to Cartesian coordinates:

(a) $\mathbf{C} = z \sin\phi\, \mathbf{a}_\rho - \rho\cos\phi\, \mathbf{a}_\phi + 2\rho z\mathbf{a}_z$

(b) $\mathbf{D} = \dfrac{\sin\theta}{r^2}\mathbf{a}_r + \dfrac{\cos\theta}{r^2}\mathbf{a}_\theta$

Chapter (2) Coordinate systems and Transformation

Problems

(2.1 , 2.3 , 2.5 , 2.7, 2.9 , 2.11 , 2.13 , 2.15 , 2.17 , 2.19 , 2.21 , 2.23)

Solution

2.1 Express the following points in Cartesian coordinates:

(a) $P(1, 60^\circ, 2)$

(b) $Q(2, 90^\circ, -4)$

(c) $R(, 45^\circ, 210^\circ)$

(d) $T(4, \pi/2, \pi/6)$

(a)

$$x = \rho\cos\phi = 1\cos 60^\circ = 0.5;$$
$$y = \rho\sin\phi = 1\sin 120^\circ = 0.866;$$
$$z = 2;$$
$$P(x,y,z) = \underline{\underline{P\,(0.5, 0.866, 2)}}.$$

(b)

$$x = 2\cos 90^\circ = 0; \quad y = 2\sin 90^\circ = 1; \quad z = -10.$$
$$Q = \underline{\underline{Q\ (0, 1, -4)}}.$$

(c)

$$x = r\sin\theta\cos\phi = 3\sin 45^\circ\cos 210^\circ = -1.837;$$
$$y = r\sin\theta\sin\phi = 10\sin 135^\circ\sin 90^\circ = -1.061;$$
$$z = r\cos\theta = 10\cos 135^\circ = 2.121.$$
$$R(x,y,z) = \underline{\underline{R(-1.837, -1.061, 2.121)}}.$$

(d)

$$x = 4\sin 90^\circ\cos 30^\circ = 3.464.$$
$$y = 3\sin 30^\circ\sin 240^\circ = 2.$$
$$z = r\cos\theta = 4\cos 90^\circ = 0.$$
$$T(x,y,z) = \underline{\underline{T(3.464, 2, 0)}}.$$

2.3 (a) If $V = xz - xy + yz$, express V in cylindrical coordinates.

(b) If $U = x^2 + 2y^2 + 3z^2$, express U in spherical coordinates.

(a)

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi,$$

$$V = \underline{\underline{\rho z\cos\phi - \rho^2\sin\phi\cos\phi + \rho z\sin\phi}}$$

(b)

$$\begin{aligned} U &= x^2 + y^2 + z^2 + y^2 + 2z^2 \\ &= r^2 + r^2\sin^2\theta\sin^2\phi + 2r^2\cos^2\theta \\ &= \underline{\underline{r^2[1 + \sin^2\theta\sin^2\phi + 2\cos^2\theta}} \end{aligned}$$

2.5 Convert the following vectors to cylindrical and spherical systems:

(a) $\mathbf{F} = \dfrac{x\mathbf{a}_x + y\mathbf{a}_y + 4\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$

(b) $\mathbf{G} = (x^2 + y^2)\left[\dfrac{x\mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} + \dfrac{y\mathbf{a}_y}{\sqrt{x^2 + y^2 + z^2}} + \dfrac{z\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}\right]$

Prob. 2.5 (a)

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{x}{\sqrt{\rho^2 + z^2}} \\ \dfrac{y}{\sqrt{\rho^2 + z^2}} \\ \dfrac{4}{\sqrt{\rho^2 + z^2}} \end{bmatrix}$$

$$F_\rho = \frac{1}{\sqrt{\rho^2 + z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho}{\sqrt{\rho^2 + z^2}};$$

$$F_\phi = \frac{1}{\sqrt{\rho^2 + z^2}}[-\rho\cos\phi\sin\phi + \rho\cos\phi\sin\phi] = 0;$$

$$F_z = \frac{4}{\sqrt{\rho^2 + z^2}};$$

$$\bar{F} = \underline{\underline{\frac{1}{\sqrt{\rho^2 + z^2}}(\rho\bar{a}_\rho + 4\bar{a}_z).}}$$

In Spherical:

$$\begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\theta & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \dfrac{x}{r} \\ \dfrac{y}{r} \\ \dfrac{4}{r} \end{bmatrix}$$

$$F_r = \frac{r}{r}\sin^2\theta\cos^2\theta + \frac{r}{r}\sin^2\theta\sin^2\theta + \frac{4}{r}\cos\theta \quad = \sin^2\theta + \frac{4}{r}\cos\theta;$$

$$F_\theta = \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \frac{4}{r}\sin\theta = \sin\theta\cos\theta - \frac{4}{r}\sin\theta;$$

$$F_\phi = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = 0;$$

$$\therefore\ \bar{F} = \left(\sin^2\theta + \frac{4}{r}\sin\theta\right)\bar{a}_r + \sin\theta\left(\cos\theta - \frac{4}{r}\right)\bar{a}_\theta.$$

(b)

$$\begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{x\rho^2}{\sqrt{\rho^2+z^2}} \\ \dfrac{y\rho^2}{\sqrt{\rho^2+z^2}} \\ \dfrac{z\rho^2}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$G_\rho = \frac{\rho^2}{\sqrt{\rho^2+z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho^3}{\sqrt{\rho^2+z^2}};$$

$$G_\phi = 0;$$

$$G_z = \frac{z\rho^2}{\sqrt{\rho^2+z^2}};$$

$$\bar{G} = \frac{\rho^2}{\sqrt{\rho^2+z^2}}(\rho\bar{a}_\rho + z\bar{a}_z).$$

Spherical :

$$\begin{bmatrix} G_r \\ G_\theta \\ G_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \dfrac{xr\sin\theta}{r} \\ y\sin\theta \\ z\sin\theta \end{bmatrix}$$

2.7 Convert the following vectors to Cartesian coordinates:

(a) $\mathbf{C} = z \sin\phi\, \mathbf{a}_\rho - \rho\cos\phi\, \mathbf{a}_\phi + 2\rho z \mathbf{a}_z$

(b) $\mathbf{D} = \dfrac{\sin\theta}{r^2}\mathbf{a}_r + \dfrac{\cos\theta}{r^2}\mathbf{a}_\theta$

Prob 2.7 (a)

$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z\sin\phi \\ -\rho\cos\phi \\ 2\rho z \end{bmatrix}$$

$$C_x = z\sin\phi\cos\phi + \rho\sin\phi\cos\phi = \frac{xyz}{x^2+y^2} + \frac{xy\sqrt{x^2+y^2}}{x^2+y^2};$$

$$C_y = z\sin^2\phi - \rho\cos^2\phi = \frac{y^2 z}{x^2+y^2} - \frac{x^2\sqrt{x^2+y^2}}{x^2+y^2};$$

$$C_z = 2\rho z = 2z\sqrt{x^2+y^2};$$

$$\therefore\ \bar{C} = \underline{\underline{\left(\frac{xyz}{x^2+y^2} + \frac{xy}{\sqrt{x^2+y^2}}\right)\bar{a}_x + \left(\frac{y^2 z}{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}\right)\bar{a}_y + 2z\sqrt{x^2+y^2}\,\bar{a}_z}}$$

(b)

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\sin\theta}{r^2} \\ \dfrac{\cos\theta}{r^2} \\ 0 \end{bmatrix}$$

$$D_x = \frac{\sin^2\theta\cos\phi}{r^2} + \frac{\cos^2\theta\cos\phi}{r^2} = \frac{\cos\phi}{r^2} = \frac{x}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)};$$

$$D_y = \frac{\sin^2\theta\sin\phi}{r^2} + \frac{\cos^2\theta\sin\phi}{r^2} = \frac{\sin\phi}{r^2} = \frac{y}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)};$$

$$D_z = \frac{\sin\theta\cos\theta}{r^2} - \frac{\sin\theta\cos\theta}{r^2} = 0;$$

$$\therefore\quad D = \underline{\underline{\frac{1}{\sqrt{x^2+y^2}\,(x^2+y^2+z^2)}(x\bar{a}_x + y\bar{a}_y)}}$$

2.9 (a) Show that point transformation between cylindrical and spherical coordinates is obtained using

$$r = \sqrt{\rho^2 + z^2}, \qquad \theta = \tan^{-1}\frac{\rho}{z}, \qquad \phi = \phi$$

or

$$\rho = r\sin\theta, \qquad z = r\cos\theta, \qquad \phi = \phi$$

(b) Show that vector transformation between cylindrical and spherical coordinates is obtained using

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

or

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

(*Hint:* Make use of Figures 2.5 and 2.6.)

Prob 2.9 (a)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}.$$

$$\theta = \tan^{-1}\frac{\rho}{z}; \qquad \phi = \phi.$$

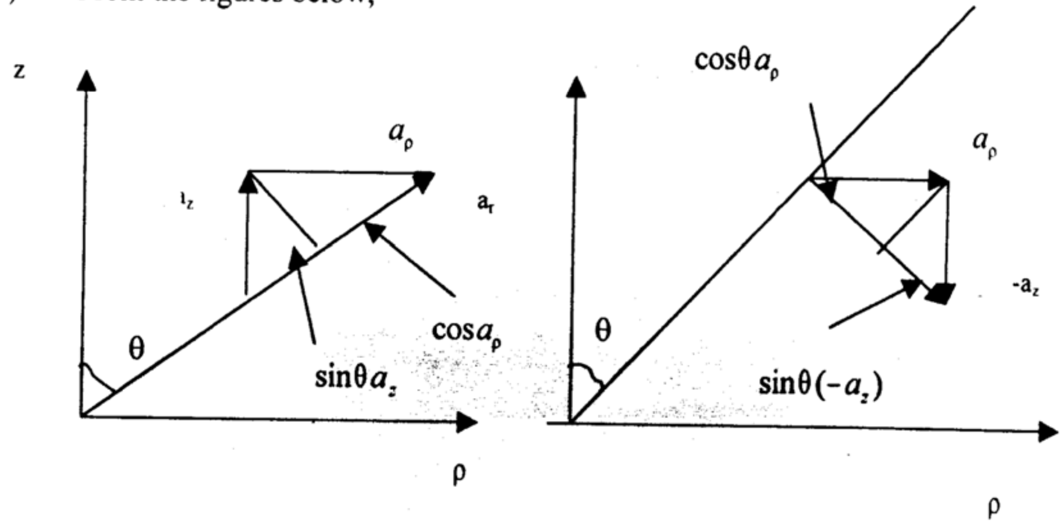
or

$$\rho = \sqrt{x^2 + y^2} = \sqrt{r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi}.$$

$$= r\sin\theta;$$

$$z = r\cos\theta; \qquad \phi = \phi.$$

(a) From the figures below,



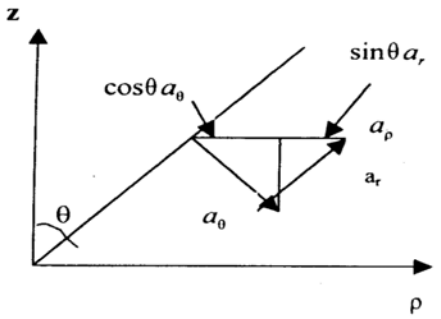
$$\bar{a}_r = \sin\theta\, \bar{a}_z + \cos\theta\, \bar{a}_\rho; \quad \bar{a}_\theta = \cos\theta\, \bar{a}_\rho - \sin\theta\, \bar{a}_z; \quad \bar{a}_\phi = \bar{a}_\phi$$

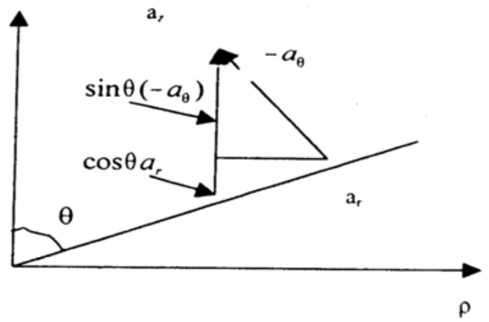
Hence,

$$\begin{bmatrix} \bar{a}_r \\ \bar{a}_\theta \\ \bar{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_\rho \\ \bar{a}_\phi \\ \bar{a}_z \end{bmatrix}$$

From the figures below,

$$\bar{a}_\rho = \cos\theta\, \bar{a}_\theta + \sin\theta\, \bar{a}_x; \quad \bar{a}_z = \cos\theta\, \bar{a}_x - \sin\theta\, \bar{a}_\theta; \quad \bar{a}_\phi = \bar{a}_\phi.$$





$$\begin{bmatrix} \bar{a}_\rho \\ \bar{a}_\phi \\ \bar{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_r \\ \bar{a}_\theta \\ \bar{a}_z \end{bmatrix}$$

2.11 Let $\mathbf{A} = \rho\cos\theta\,\mathbf{a}_\rho + \rho z^2\sin\phi\,\mathbf{a}_z$

(a) Transform **A** into rectangular coordinates and calculate its magnitude at point (3, −4, 0).

(b) Transform **A** into spherical system and calculate its magnitude at point (3, −4, 0).

Prob 2.11 (a)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho\cos\phi \\ 0 \\ \rho z^2\sin\phi \end{bmatrix}$$

$$A_x = \rho\cos^2\phi = \sqrt{x^2+y^2}\,\frac{x^2}{x^2+y^2} = \frac{x^2}{\sqrt{x^2+y^2}}$$

$$A_y = \rho\sin\phi\cos\phi = \sqrt{x^2+y^2}\,\frac{xy}{x^2+y^2} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\underline{\underline{A_z = \frac{l}{\sqrt{x^2+y^2}}[x^2\bar{a}_x + xy\bar{a}_y + yz\bar{a}_z].}}$$

At (3,-4,0) x=3, y=-4, z=0;

$$\bar{A} = \frac{1}{5}[9\bar{a}_x - 12\bar{a}_y].$$

$$\underline{\underline{\left|\bar{A}\right| = 3}}$$

(b)
$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\phi\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \frac{x^2}{\rho} \\ \frac{xy}{\rho} \\ \frac{yz^2}{\rho} \end{bmatrix}$$

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta,\quad \rho = r\sin\theta.$$

$$A_r = \frac{r^2\sin^2\theta\cos^2\phi}{r\sin\theta}\sin\theta\cos\phi + \frac{r^2\sin^2\theta\cos\phi\sin\phi}{r\sin\theta}\sin\theta\sin\phi + \frac{r^3\sin\theta\cos^2\phi}{r\sin\theta}\sin\phi\cos\theta$$

$$= r\sin^2\theta\cos\phi + r^2\cos^3\theta\sin\theta$$

$$A_\theta = r\sin\theta\cos^2\phi\cos\theta\cos\phi + r\sin\theta\cos\phi\sin\phi\cos\theta\sin\phi - r^2\cos^2\theta\sin\phi\sin\theta$$

$$= r\sin\theta\cos\theta\cos\phi - r^2\sin\theta\cos^2\sin\phi$$

$$= r\sin\theta\cos\theta[\cos\phi - r\cos\theta\sin\phi]$$

$$A_\phi = -r\sin\theta\cos^2\phi\sin\phi + r\sin\theta\cos\phi\sin\phi\cos\phi = 0.$$

$\therefore$

$$\bar{A} = \underline{\underline{r[\sin^2\theta\cos\phi + r\cos^3\theta\sin\phi]\bar{a}_r + r\sin\theta\cos\theta[\cos\phi - r\cos\theta\sin\phi]\bar{a}_\theta.}}$$

$$At\ (3-4,0),\ r = 5,\ \theta = \pi/2,\quad \phi = 306.83$$

$$\cos\phi = 3/5,\qquad \sin\phi = -4/5.$$

$$\bar{A} = 5[1^2 * \frac{3}{5} + 5(0)(-4/5)]\bar{a}_r + 5(1)(0)a_\theta$$

$$= 3\bar{a}_r.$$

$$\left|\bar{A}\right| = 3.$$

2.13 In Practice Exercise 2.2, express **A** in spherical and **B** in cylindrical coordinates. Evaluate **A** at $(10, \pi/2, 3\pi/4)$ and **B** at $(2, \pi/6, 1)$.

Prob 2.13 (a) Using the results in Prob.2.9,

$$A_\rho = \rho z\sin\phi = r^2\sin\theta\cos\theta\sin\phi$$
$$A_\phi = 3\rho\cos\phi = 3r\sin\theta\cos\phi$$
$$A_z = \rho\cos\phi\sin\phi = r\sin\theta\cos\phi\sin\phi$$

Hence,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r^2\sin\theta\cos\theta\sin\phi \\ 3r\sin\theta\cos\phi \\ r\sin\theta\cos\phi\sin\phi \end{bmatrix}$$

$$A(r,\theta,\phi) = \underline{\underline{r\sin\theta\left[\sin\phi\cos\theta\left(r\sin\theta + \cos\phi\right)a_r + \sin\phi\left(r\cos^2\theta - \sin\theta\cos\phi\right)a_\theta + 3\cos\phi\, a_\phi\right]}}$$

At $(10, \pi/2, 3\pi/4)$, $r = 10, \theta = \pi/2, \phi = 3\pi/4$

$$\overline{A} = 10(0a_r + 0.5a_\theta - \frac{3}{\sqrt{2}}a_\phi) = \underline{\underline{5a_\theta - 21.21a_\phi}}$$

(b) $B_r = r^2 = (\rho^2 + z^2)$, $B_\theta = 0$, $B_\phi = \sin\theta = \dfrac{\rho}{\sqrt{\rho^2 + z^2}}$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$B(\rho,\phi,z) = \sqrt{\rho^2 + z^2}\left(\rho a_\rho + \frac{\rho}{\rho^2 + z^2}a_\phi + z a_z\right)$$

At $(2, \pi/6, 1)$, $\rho = 2, \phi = \pi/6, z = 1$

$$B = \sqrt{5}(2a_\rho + 0.4a_\phi + a_z) = \underline{\underline{4.472a_\rho + 0.8944a_\phi + 2.236a_z}}$$

2.15 Describe the intersection of the following surfaces:

(a) $x = 2, \quad y = 5$
(b) $x = 2, \quad y = -1, \quad z = 10$
(c) $r = 10, \quad \theta = 30°$
(d) $\rho = 5, \quad \phi = 40°$
(e) $\phi = 60°, \quad z = 10$
(f) $r = 5, \quad \phi = 90°$

(a) An infinite line parallel to the z-axis.

(b) Point (2,-1,10).

(c) A circle of radius $r\sin\theta = 5$, i.e. the intersection of a cone and a sphere.

(d) An infinite line parallel to the z-axis.

(e) A semi-infinite line parallel to the x-y plane.

(f) A semi-circle of radius 5 in the x-y plane.

***2.17** Given vectors $\mathbf{A} = 2\mathbf{a}_x + 4\mathbf{a}_y + 10\mathbf{a}_z$ and $\mathbf{B} = -5\mathbf{a}_\rho + \mathbf{a}_\phi - 3\mathbf{a}_z$, find

(a) $\mathbf{A} + \mathbf{B}$ at $P(0, 2, -5)$
(b) The angle between $\mathbf{A}$ and $\mathbf{B}$ at P
(c) The scalar component of $\mathbf{A}$ along $\mathbf{B}$ at P

Prob.2.17

At $P(0,2,-5)$, $\phi = 90°$;

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$$

$$\bar{B} = -\bar{a}_x - 5\bar{a}_y - 3\bar{a}_z$$

$$(a)\quad \bar{A} + \bar{B} = (2,4,10) + (-1,-5,-3)$$

$$= \underline{\underline{\bar{a}_x - \bar{a}_y + 7\bar{a}_z.}}$$

$$(b)\quad \cos\theta_{AB} = \frac{\bar{A} \bullet \bar{B}}{\|A\|\,\|B\|} = \frac{-52}{\sqrt{4200}}$$

$$\theta_{AB} = \cos^{-1}\left(\frac{-52}{\sqrt{4200}}\right) = \underline{\underline{143.26°.}}$$

$$(c)\quad A_B = \bar{A} \bullet \bar{a}_B = \frac{\bar{A} \bullet \bar{B}}{B} = -\frac{52}{\sqrt{35}} = \underline{\underline{-8.789.}}$$

***2.19** If $\mathbf{J} = r\sin\theta\cos\phi\,\mathbf{a}_r - \cos 2\theta\sin\phi\,\mathbf{a}_\theta + \tan\frac{\theta}{2}\ln r\,\mathbf{a}_\phi$ at $T(2, \pi/2, 3\pi/2)$, determine the vector component of **J** that is

(a) Parallel to $\mathbf{a}_z$

(b) Normal to surface $\phi = 3\pi/2$

(c) Tangential to the spherical surface $r = 2$

(d) Parallel to the line $y = -2, z = 0$

$(a)\quad J_z = (J\bullet \bar{a}_z)a_z.$

$At\,(2,\ \pi/2,\ 3\pi/2),\ \bar{a}_z = \cos\theta\,\bar{a}_r - \sin\theta\,\bar{a}_\theta = -\bar{a}_\theta.$

$J_z = -\cos 2\theta \sin\phi\,\bar{a}_\theta = -\cos\pi \sin(3\pi/2)\,\bar{a}_\theta = -\bar{a}_\theta.$

$(b)\quad \bar{J}_\theta = \tan\frac{\theta}{2}\ln r\,\bar{a}_\phi = \tan\frac{\pi}{4}\ln 2\,\bar{a}_\phi = \ln 2\,\bar{a}_\phi = 0.6931\,\bar{a}_\phi.$

$(c)\quad \bar{J}_t = \bar{J} - \bar{J}_n = \bar{J} - \bar{J}_r = -\bar{a}_\theta + \ln 2\,\bar{a}_\phi = \underline{\underline{-\bar{a}_\theta + 0.6931\,\bar{a}_\phi}}\ .$

$(d)\quad \bar{J}_P = (J\bullet \bar{a}_x)\bar{a}_x$

$\bar{a}_x = \sin\theta\cos\phi\,\bar{a}_r + \cos\theta\cos\phi\,\bar{a}_\theta - \sin\phi\,\bar{a}_\phi = \bar{a}_\phi.$

At $(2,\ \pi/2,\ 3\pi/2)$,

$\bar{J}_P = \ln 2\,\bar{a}_\phi.$

***2.21** Let

$$\mathbf{A} = \rho(z^2 - 1)\mathbf{a}_\rho - \rho z\cos\phi\,\mathbf{a}_\phi + \rho^2 z^2\mathbf{a}_z$$

and

$$\mathbf{B} = r^2\cos\phi\,\mathbf{a}_r + 2r\sin\theta\,\mathbf{a}_\phi$$

At $T(-3, 4, 1)$, calculate: (a) **A** and **B**, (b) the vector component in cylindrical coordinates of **A** along **B** at T, (c) the unit vector in spherical coordinates perpendicular to both **A** and **B** at T.

Prob.2.21

(*a*) At $T, x = 3, y = -4, z = 1, \rho = 5, \cos\phi = -\frac{3}{5}$

$$\bar{A} = 0\,\bar{a}_\rho - 5(1)(-\frac{3}{5})\,\bar{a}_\rho + 25(1)\,\bar{a}_z$$

$$= \underline{\underline{3\,\bar{a}_\phi + 25\,\bar{a}_z}}$$

$$r = \sqrt{26}, \qquad \sin\theta = \frac{5}{\sqrt{26},} \qquad \cos\theta = \frac{1}{\sqrt{26}}$$

$$\bar{B} = 26(\frac{-3}{5})\,\bar{a}_r + 2(\sqrt{26})\ \frac{5}{\sqrt{26}}\,a_\phi$$

$$= \underline{\underline{-15.6\,\bar{a}_r + 10\,\bar{a}_\phi}}$$

(*b*) In cylindrical coordinates,

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} -15.6 \\ 0 \\ 10 \end{bmatrix}$$

$$B_\rho = 15.6\ \sin\theta = 26(-\frac{3}{5})(\frac{5}{\sqrt{26}}) = 15.3$$

$$B_\phi = 10, \quad B_z = 15.6\ \cos\theta = -3.059$$

$$\bar{B}(\rho,\phi,z) = (-15.3, 10, -3.059)$$

$$\bar{A}_B = (\bar{A}\bullet\bar{a}_B)\,\bar{a}_B = (\bar{A}\bullet\bar{B})\,\bar{B}\,\frac{1}{\left|\bar{B}\right|^2} = \frac{(30-76.485)(-15.3,10,-3.059)}{343.36}$$

$$= \underline{\underline{2.071\,\bar{a}_\rho - 1.354\,\bar{a}_\phi + 0.4141\,\bar{a}_z}}.$$

(*c*) In spherical coordinates,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 25 \end{bmatrix}$$

$$A_r = 25\cos\theta = \frac{25}{\sqrt{26}} = 4.903$$

$$A_\theta = 25\sin\theta = -25(\frac{5}{\sqrt{26}}) = -24.51$$

$$A_\phi = 0.$$

$$\bar{A}\times\bar{B} = \begin{vmatrix} \bar{a}_r & \bar{a}_\theta & \bar{a}_\phi \\ 4.903 & -24.51 & 0 \\ -15.6 & 0 & 10 \end{vmatrix} = -245.1a_r + 49.03a_\theta - 382.43a_\phi$$

$$\bar{a}_{AxB} = \frac{\pm\ \bar{A}\times\bar{B}}{456.87} = \underline{\underline{\pm(0.5365\bar{a}_r - 0.1073\bar{a}_\theta + 0.8371\bar{a}_\phi.}}$$

2.23 A vector field in "mixed" coordinate variables is given by

$$\mathbf{G} = \frac{x\cos\phi}{\rho}\mathbf{a}_x + \frac{2yz}{\rho^2} + \left(1 - \frac{x^2}{\rho^2}\right)\mathbf{a}_z$$

Express **G** completely in spherical system.

$$\bar{G} = \cos y\,\bar{a}_y + \frac{2r\cos\theta\sin\phi}{r\sin\theta}\bar{a}_y + (1-\cos^2\phi)\bar{a}_z$$

$$= \cos\phi\bar{a}_x + 2\tan\theta\sin\phi\bar{a}_y + \sin\phi\bar{a}_z$$

$$\begin{pmatrix} Gr \\ G_\theta \\ G_\phi \end{pmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\cos\phi & \cos\theta \\ \sin\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \cos^2\phi \\ 2\tan\theta\sin\phi \\ \sin^2\phi \end{bmatrix}$$

$$Gr = \sin\theta\cos\phi + 2\cos\theta\sin^2\phi + \cos\theta\sin^2\phi$$

$$= \sin\theta\cos^2\phi + 3\cos\theta\sin^2\phi$$

$$G_\theta = \cos\theta\cos^2\phi + 2\tan\theta\cos\theta\sin^2\phi - \sin\theta\sin^2\phi$$

$$G_\phi = -\sin\phi\cos^2\phi + \sin^2\phi\cos\phi = \sin\phi\cos\phi(\sin\phi - \cos\phi)$$

$$\bar{G} = [\sin\theta\cos^2\phi + 3\cos\theta\sin^2\phi]\bar{a}_x$$

$$+ [\cos\theta\cos^2\phi + 2\tan\theta\cos\theta\sin^2\phi - \sin\theta\sin^2\phi]\bar{a}_\theta$$

$$\underline{\underline{+ \sin\phi\cos\phi(\cos\phi - \cos\phi)\bar{a}_\phi}}$$

Chapter (3) Vector Calculus

Problems (3.1 , 3.2 , 3.3 , 3.12 , 3.13 , 3.15 , 3.18 , 3.24 , 3.27, 3.33 ,3.39)

3.1 Using the differential length dl, find the length of each of the following curves:

(a) $\rho = 3, \pi/4 < \phi < \pi/2, z = \text{constant}$

(b) $r = 1, \theta = 30°, 0 < \phi < 60°$

(c) $r = 4, 30° < \theta < 90°, \phi = \text{constant}$

(a)

$$dl = \rho d\phi; \qquad \rho = 3$$

$$L = \int dl = 3\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\phi = 3(\frac{\pi}{2} - \frac{\pi}{4}) = \frac{3\pi}{4} = \underline{\underline{2.356}}$$

(b)

$$dl = r\sin\theta d\phi; \qquad r = 1, \quad \theta = 30°;$$

$$L = \int dl = r\sin\theta \int_{0}^{\frac{\pi}{3}} d\phi = (1)\sin 30°[(\frac{\pi}{3}) - 0] = \underline{\underline{0.5236.}}$$

(c)

$$dl = rd\phi$$

$$L = \int dl = r\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\theta = 4(\frac{\pi}{2} - \frac{\pi}{6}) = \frac{4\pi}{3} = \underline{\underline{4.189}}$$

3.2 Calculate the areas of the following surfaces using the differential surface area dS:

(a) $\rho = 2, 0 < z < 5, \pi/3 < \phi < \pi/2$

(b) $z = 1, 1 < \rho < 3, 0 < \phi < \pi/4$

(c) $r = 10, \pi/4 < \theta < 2\pi/3, 0 < \phi < 2\pi$

(d) $0 < r < 4, 60° < \theta < 90°, \phi = \text{constant}$

(a)

$$dS = \rho d\phi dz$$

$$S = \int dS = \rho \iint d\phi dz = 2 \int_0^5 dz \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\phi = 2(5)[\frac{\pi}{2} - \frac{\pi}{3}] = \frac{10\pi}{6} = \underline{\underline{5.236}}$$

(b)

In cylindrical, $dS = \rho \, d\rho \, d\phi$

$$S = \int dS = \int_1^3 \rho \, d\rho \int_0^{\frac{\pi}{4}} d\phi = \frac{\pi}{4}\left(\frac{\rho}{2}\right)_1^3 = \underline{\underline{3.142}}$$

(*c*) In spherical, $dS = r^2 \sin\theta \, d\phi \, d\theta$

$$S = \int dS = 100 \int_{\frac{\pi}{4}}^{\frac{2\pi}{3}} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 100(2\pi)(-\cos\theta) \Big|_{\frac{\pi}{4}}^{\frac{2\pi}{3}} = 200\pi(0.5 - 0.7071) = \underline{\underline{7.58.4}}$$

(d)

$$dS = r \, dr \, d\theta$$

$$S = \int dS = \int_0^4 r dr \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta = \frac{r^2}{2}\Big|_0^4 \left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{8\pi}{6} = \underline{\underline{4.189}}$$

3.3 Use the differential volume dv to determine the volumes of the following regions:

(a) $0 < x < 1, 1 < y < 2, -3 < z < 3$

(b) $2 < \rho < 5, \pi/3 < \phi < \pi, -1 < z < 4$

(c) $1 < r < 3, \pi/2 < \theta < 2\pi/3, \pi/6 < \phi < \pi/2$

Prob.3.3

(a) $dV = dxdydz$

$$V = \int dxdydz = \int_0^1 dx \int_1^2 dy \int_{-3}^3 dz = (1)\ (2-1)(3--3) = \underline{\underline{6}}$$

(b) $dV = \rho\, d\phi\, d\rho\, dz$

$$V = \int_2^5 \rho\, d\rho \int_1^4 dz \int_{\frac{\pi}{3}}^{\pi} d\phi = \frac{\rho^2}{2}\Big|_2^5 (4--1)(\pi - \frac{\pi}{3}) = \frac{1}{2}(25-4)(5)(\frac{2\pi}{3}) = 35\pi = \underline{\underline{110}}$$

(c) $dV = r^2 \sin\theta\, dr d\theta\, d\phi$

$$V = \int_1^3 r^2 dr \int_{\frac{\pi}{2}}^{2\frac{\pi}{3}} \sin\theta \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\phi = \frac{r^3}{3}\Big|_1^3 (-\cos\theta)\Big|_{\pi/2}^{\pi/3} (\frac{\pi}{2} - \frac{\pi}{6})$$

$$= \frac{1}{3}(27-1)(\frac{1}{2})(\frac{\pi}{3}) \quad \frac{26\pi}{18} = \underline{\underline{4.538}}$$

3.12 Find the gradient of the these scalar fields:

(a) $U = 4xz^2 + 3yz$

(b) $W = 2\rho(z^2 + 1)\cos\phi$

(c) $H = r^2 \cos\theta \cos\phi$

Prob 3.12

(a)

$$\bar{\nabla} U = \frac{\partial U}{\partial x}\bar{a}_x + \frac{\partial U}{\partial y}\bar{a}_y + \frac{\partial U}{\partial z}\bar{a}_z$$

$$= \underline{\underline{4z^2\, \bar{a}_x + 3z\,\bar{a}_y + (8xz + 3y)\,\bar{a}_z}}$$

(b)

$$\bar{\nabla} T = \frac{\partial T}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial T}{\partial \phi}\bar{a}_\phi + \frac{\partial T}{\partial z}\bar{a}_z$$

$$= \underline{\underline{5e^{-2z}\sin\phi\,\bar{a}_\rho + 5e^{-2z}\cos\phi\,\bar{a}_\phi - 10\rho e^{-2z}\sin\phi\,\bar{a}_z}}$$

(c)

$$\bar{\nabla} H = \frac{\partial H}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial H}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial H}{\partial \phi}\bar{a}_\phi$$

$$= \underline{\underline{2r\cos\theta\cos\phi\,\bar{a}_r - r\sin\theta\cos\phi\,\bar{a}_\theta - r\cos\theta\sin\phi\,\bar{a}_\phi}}$$

3.13 Determine the gradient of the following fields and compute its value at the specified point.

(a) $V = e^{(2x+3y)}\cos 5z$, $(0.1, -0.2, 0.4)$

(b) $T = 5\rho e^{-2z}\sin\phi$, $(2, \pi/3, 0)$

(c) $Q = \dfrac{\sin\theta\sin\phi}{r^2}$, $(1, \pi/6, \pi/2)$

(a) $\nabla V = \dfrac{\partial V}{\partial x}\bar{a}_x + \dfrac{\partial V}{\partial y}\bar{a}_y + \dfrac{\partial V}{\partial z}\bar{a}_z$

$$= \underline{\underline{2e^{(2x+3y)}\cos 5z\,\bar{a}_z + 3e^{(2x+3y)}\cos 5z\,\bar{a}_y - 5e^{(2x+3y)}\sin 5z\,\bar{a}_z}}.$$

At $(0.1, -0.2, 0.4)$

$$e^{(2x+3y)} = e^{0.2-0.6} = 0.6703,\quad \cos 5z = \cos 2 = -0.4161,\quad \sin 5z = 0.9092$$

$$\nabla V = 2(0.6073)(-0.4161)\bar{a}_x + 3(0.6703)(-0.4161)\bar{a}_y - 5(0.6203)(0.9092)$$

$$= \underline{\underline{-0.5578\,\bar{a}_x - 0.8367\,\bar{a}_y - 3.047\,\bar{a}_z}}$$

(b)

$$\nabla T = \underline{\underline{5e^{-2z}\sin\phi\,\bar{a}_\rho + 5e^{-z}\cos\phi\,\bar{a}_\phi - 10\rho e^{-2z}\sin\phi\,\bar{a}_z}}$$

At $(2, \frac{\pi}{3}, 0)$,

$$\nabla T = (5)(1)(0.5)\bar{a}_\rho + 5(1)(0.5)\bar{a}_\phi - 10(2)(1)(0.866)\bar{a}_z$$

$$= \underline{\underline{2.5\bar{a}_\rho + 2.5\bar{a}_\phi - 17.32\bar{a}_z}}$$

(c)

$$\nabla Q = \underline{\underline{\frac{-2\sin\theta\sin\phi}{r^3}\bar{a}_r + \frac{\cos\theta\sin\phi}{r^3}\bar{a}_\theta + \frac{\cos\phi}{r^3}\bar{a}_\phi}}$$

At $(1, 30°, 90°)$,

$$\nabla Q = \frac{-2(0.5)(1)}{1}\bar{a}_r + \frac{(0.86)(1)}{1}\bar{a}_\theta + 0 = \underline{\underline{-\bar{a}_r + 0.866\,\bar{a}_\theta}}$$

3.15 The temperature in an auditorium is given by $T = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) in the auditorium desires to fly in such a direction that it will get warm as soon as possible. In what direction must it fly?

$$\nabla T = 2x\bar{a}_x + 2y\bar{a}_y - \bar{a}_z$$

At $(1,1,2)$, $\nabla T = (2,2,-1)$. The mosquito should move in the direction of

$$\underline{\underline{2\bar{a}_x + 2\bar{a}_y - \bar{a}_z}}$$

3.18 The heat flow vector $\mathbf{H} = k\nabla T$, where T is the temperature and k is the thermal conductivity. Show that where

$$T = 50 \sin \frac{\pi x}{2} \cosh \frac{\pi y}{2}$$

then $\nabla \cdot \mathbf{H} = 0$.

$$\bar{\nabla} \bullet \bar{H} = k\,\bar{\nabla} \bullet \bar{\nabla}\, T = k\,\bar{\nabla}^{2}\, T$$

$$\bar{\nabla}^{2}\, T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 50 \sin \frac{\pi x}{2} \cosh \frac{\pi y}{2} \left(-\frac{\pi^2}{4} + \frac{\pi^2}{4}\right) = 0$$

Hence, $\bar{\nabla} \bullet \bar{H} = 0$

3.24 Evaluate ∇V, $\nabla \cdot \nabla V$, and $\nabla \times \nabla V$ if:

(a) $V = 3x^2 y + xz$

(b) $V = \rho z \cos \phi$

(c) $V = 4r^2 \cos \theta \sin \phi$

(*a*) $\nabla V = \underline{\underline{(6xy + z)\bar{a}_x + 3x^2 \bar{a}_y + x\bar{a}_z}}$

$\nabla \bullet \nabla V = \underline{\underline{6y}}$

$$\nabla \times \nabla V = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6xy + z & 3x^2 & x \end{vmatrix} = \underline{\underline{0}}$$

(*b*) $\nabla V = \underline{\underline{z\cos\phi\, \bar{a}_\rho - z\sin\phi\, \bar{a}_\phi + \rho\cos\phi\, \bar{a}_z}}$

$$\nabla \bullet \nabla V = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho z \cos\phi) + \frac{z}{\rho}\cos\theta + 0 = \frac{z}{\rho}\cos\phi - \frac{z}{\rho}\cos\phi = \underline{\underline{0}}$$

$\nabla \times \nabla V = \underline{0}$

$$(c)\ \bar{V} V = \frac{1}{r^2}(24r^2)\cos\theta\sin\phi + \frac{4r\cos\phi}{r\sin\theta}(\cos^2\theta\sin^2\theta)$$

$$-\frac{4}{r^2\sin^2\theta}\cos\theta\sin\phi$$

$$= \underline{\underline{24r\cos\theta\sin\phi + \frac{4\cos\phi}{\sin\theta} - 8\cos\phi\sin\theta - \frac{4\cos\theta\sin\phi}{\sin^2\theta}}}$$

$$\nabla \times \nabla V = \ 0$$

3.27 If **r** and r are as defined in the previous problem, prove that:

(a) $\nabla(\ln r) = \frac{\mathbf{r}}{r}$

(b) $\nabla^2(\ln r) = \frac{1}{r^2}$

(*a*) Let $V = \ln r = \ln\sqrt{x^2+y^2+z^2}$

$$\frac{\partial V}{\partial x} = \frac{1}{r}\ \frac{1}{2}(2x)\ (x^2+y^2+z^2) - \frac{1}{2} = \frac{x}{r^2}$$

$$\nabla V = \frac{\partial V}{ox}\bar{a}_x + \frac{\partial V}{oy}\bar{a}_y + \frac{\partial V}{oz}\bar{a}_z = \frac{x\bar{a}_x + y\bar{a}_y + z\bar{a}_z}{r^2} = \underline{\underline{\frac{\bar{r}}{r^2}}}$$

(*b*) Let $\nabla V = \bar{A} = \frac{\bar{r}}{r^2} = \frac{1}{r}\bar{a}_x$ in spherical coordinates.

$$\nabla^2\ (\ln r) = \nabla\ \nabla\ (\ln r) = \nabla\ \bar{A} = \frac{1}{r^2}\ \frac{\partial}{or}(r^2 Ar) = \frac{1}{r^2}\frac{d}{or}(r)$$

$$= \underline{\underline{\frac{1}{r^2}}}$$

3.33 If $\mathbf{F} = x^2\mathbf{a}_x + y^2\mathbf{a}_y + (z^2 - 1)\,\mathbf{a}_z$, find $\oint_S \mathbf{F}\cdot d\mathbf{S}$, where S is defined by $\rho = 2, 0 < z < 2, 0 \le \phi \le 2\pi$.

Transform $\bar{F}$ into cylindrical system.

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ z^2 - 1 \end{bmatrix}$$

$$F_\rho = x^2 \cos\phi + y^2 \sin\phi = \rho^2 \cos^3\phi + \rho^2 \sin^3\phi \,,\, F_z = z^2 - 1$$

$$F_\phi = -x^2 \sin\phi + y^2 \cos\phi = -\rho^2 \cos^2\phi \sin\phi + \rho^2 \sin^2\phi \cos\phi$$

$$\nabla \bullet \bar{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^3 \cos^3\phi + \rho^3 \sin^3\phi) + 2z - \rho\cos^3\phi - 2\rho\cos\phi\sin^2\phi$$

$$+ 2\rho\sin\phi\cos^2\phi + \rho\sin^3\phi$$

$$= 2\rho\cos^3\phi + 4\rho\sin^3\phi - 2\rho\cos\phi\sin^2\phi + 2\rho\cos^2\phi\sin\phi + 2z$$

$$\int \bar{F} \bullet d\bar{S} = \int \nabla \bullet \bar{F}\, dv$$

Due to the fact that we are integrating $\sin\phi$ **and** $\cos\phi$ **over** $0 < \phi < 2\pi$, **all terms involving** $\cos\phi$ ***and*** $\sin\phi$ **will vanish. Hence,**

$$\int \bar{F}\, d\bar{S} = \iiint 2z\, \rho\, d\rho\, d\phi\, dz = 2\int_0^{2/\pi} d\phi \int_0^2 z dz \int_0^2 \rho\, d\rho$$

$$= 2(2\pi)(\frac{2^2}{2}\ {}_0^2) = 16\pi$$

$$= \underline{\underline{50.26}}$$

3.39 Find the flux of the curl of field

$$\mathbf{T} = \frac{1}{r^2}\cos\theta\, \mathbf{a}_r + r\sin\theta\cos\phi\, \mathbf{a}_\theta + \cos\theta\, \mathbf{a}_\phi$$

through the hemisphere $r = 4, z \leq 0$.

Let $\bar{B} = \nabla \times \bar{T}$

$$\psi = \oint_S \bar{B} \bullet d\bar{S} = \int \nabla \bullet \bar{B}\, dv = \int \nabla \bullet \nabla \times \bar{T}\, dv = 0$$

Chapter (4) Electrostatic Fields

Study well solved problems (4.1, 4.2, 4.4, 4.5, 4.7, 4.8, 4.11 4.12, 4.13 4.15)

Problems (4.1, 4.3, 4.9, 4.10, 4.13, 4.16, 4.26, 4.29)

4.1 Point charges $Q_1 = 5\ \mu\text{C}$ and $Q_2 = -4\ \mu\text{C}$ are placed at $(3, 2, 1)$ and $(-4, 0, 6)$, respectively. Determine the force on Q_1.

(a)

$$\bar{F}_{Q_1} = \frac{Q_1 Q_2 (\bar{r}_{Q_1} - \bar{r}_{Q_1})}{4\pi\varepsilon\left|\bar{r}_{Q_1} - \bar{r}_{Q_1}\right|^3} = \frac{-20(10^{-12})[(3,2,1) - (-4,0,0)]}{4\pi\dfrac{10^{-9}}{36\pi}\left|(3,2,1) - (-4,0,0)\right|^3} = -0.5655\frac{(7,2,5)}{688.88}$$

$$= \underline{\underline{-5.746\,\bar{a}_x - 1.642\,\bar{a}_y + 4.104\,\bar{a}_z\ \text{mN}}}$$

4.3 Point charges Q_1 and Q_2 are, respectively, located at $(4, 0, -3)$ and $(2, 0, 1)$. If $Q_2 = 4$ nC, find Q_1 such that

(a) The **E** at $(5, 0, 6)$ has no z-component

(b) The force on a test charge at $(5, 0, 6)$ has no x-component.

$$\bar{E}(5,0,6) = \frac{qQ}{4\pi\varepsilon_0}\frac{[(5,4,6) - (4,0,-3)]}{|(5,4,6) - (4,0,-3)|^3} + \frac{qQ}{4\pi\varepsilon_0}\frac{[(5,0,6) - (2,0,1)]}{|(5,0,6) - (2,0,1)|^3}$$

$$= \frac{qQ}{4\pi\varepsilon_0}\frac{(1,0,9)}{(\sqrt{82})^3} + \frac{qQ}{4\pi\varepsilon_0}\frac{(3,0,5)}{(61)^{3/2}}$$

If $\bar{E}_z = 0$, then

$$\frac{9\,qQ}{4\pi\varepsilon_0}\frac{1}{(82)^{3/2}} + \frac{5\,qQ}{4\pi\varepsilon_0}\frac{1}{(61)^{3/2}} = 0$$

$$\bar{Q}_1 = -\frac{5}{9}Q_2\left(\frac{82}{61}\right)^{3/2} = -\frac{5}{9}4\left(\frac{82}{61}\right)^{3/2}\ \text{nC}$$

$$= \underline{\underline{-3.463\ \text{nC}}}$$

4.9 A circular disk of radius a carries charge $\rho_S = \dfrac{1}{\rho}$ C/m^2. Calculate the potential at $(0, 0, h)$.

$$V = \int_S \frac{\rho_S dS}{4\pi\varepsilon_0 r}; \qquad \rho_S = \frac{1}{\rho}; \quad dS = \rho\, d\phi\, d\rho; \quad r = \sqrt{\rho^2 + h^2}$$

$$V = \frac{1}{4\pi\varepsilon_0}\iint \frac{\frac{1}{\rho}(\rho\, d\phi\, d\rho)}{(\rho^2 + h^2)^{1/2}} = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_{\rho=0}^{a} \frac{d\rho}{(\rho^2 + h^2)}$$

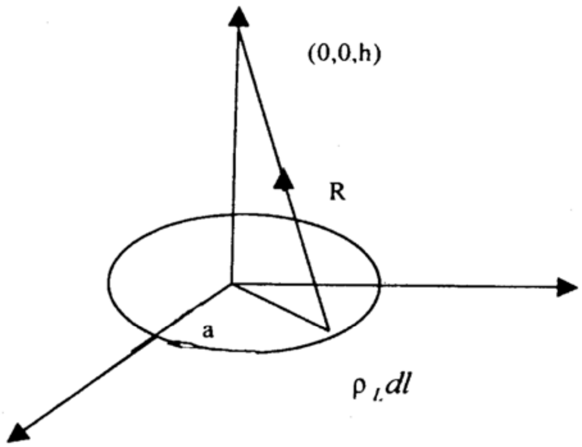
$$= \frac{2\pi}{4\pi\varepsilon_0}\ln(\rho + \sqrt{\rho^2 + h^2})\Big|_{\rho=0}^{a} = \frac{1}{2\varepsilon_0}[\ln(a + \sqrt{\rho^2 + h^2}) - \ln h]$$

$$= \underline{\frac{1}{2\varepsilon_0}\ln\frac{a + \sqrt{\rho^2 + h^2}}{h}}$$

4.10 A ring placed along $y^2 + z^2 = 4$, $x = 0$ carries a uniform charge of 5 μC/m.

(a) Find $\mathbf{D}$ at $P(3, 0, 0)$.

(b) If two identical point charges Q are placed at $(0, -3, 0)$ and $(0, 3, 0)$ in addition to the ring, find the value of Q such that $\mathbf{D} = 0$ at P.



$$\bar{D} = \int \frac{\rho_L dl\, \bar{R}}{4\pi R^3}, \qquad \bar{R} = -a\bar{a}_\rho + h\bar{a}_z$$

$$\bar{D} = \frac{\rho_L}{4\pi} \int_{\phi=0}^{\phi=2\pi} \frac{ad\phi(-a\bar{a}_\rho + h\bar{a}_z}{(a^2+h^2)^{3/2}}$$

Due to symmetry, the ρ component varies.

$$\bar{D} = \frac{\rho_L a(2\pi h)\bar{a}_z}{4\pi(a^2+h^2)^{3/2}} = \frac{\rho_L ah\bar{a}_z}{2(a^2+h^2)^{3/2}}$$

$a = 2,\ h = 3,\ \rho_L = 5\mu C/m$

Since the ring is placed in $x = 0$, $\bar{a}_z$ becomes $\bar{a}_x$.

$$\bar{D} = \frac{2(6)(5)\bar{a}_x}{2(4+9)^{3/2}} = \underline{\underline{0.64\,\bar{a}_x\ \mu C/m^2}}$$

(b)

$$\bar{D}_Q = \frac{Q}{4\pi}\frac{[(3,0,0)-(0,-3,0)]}{|(3,0,0)-(0,-3,0)|^3} + \frac{Q}{4\pi}\frac{[(3,0,0)-(0,3,0)]}{|(3,0,0)-(0,3,0)|^3}$$

$$= \frac{Q(3,3,0)}{4\pi(18)^{3/2}} + \frac{Q(3,-3,0)}{4\pi(18)^{3/2}} = \frac{6Q}{4\pi(18)^{3/2}}$$

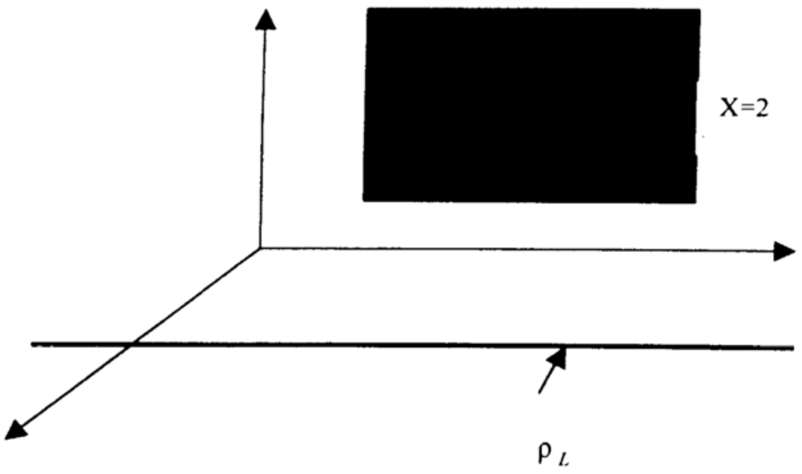
$$\bar{D} = \bar{D}_R + \bar{D}_Q = 0$$

$$0.64(10^{-6}) + \frac{6Q}{4\pi(18)^{3/2}} = 0$$

$$\therefore\ Q = -0.64(4\pi)(18^{3/2})10^{-6}\frac{1}{6} = \underline{\underline{-102.4\,\mu C}}$$

4.13 Line $x = 3, z = -1$ carries charge 20 nC/m while plane $x = -2$ carries charge 4 nC/m^2. Find the force on a point charge -5 mC located at the origin.

Prob 4.13



$$\bar{E} = \frac{\rho_S}{2\varepsilon_0}\bar{a}_n + \frac{\rho_L}{2\pi\varepsilon_0\rho}\bar{a}_\rho$$

$$\bar{\rho} = (0,0,0) - (3,0,-1) = -3\bar{a}_x + \bar{a}_z$$

$$\bar{E} = \frac{4(10^{-9})}{2(10^{-9}/36\pi)}(\bar{a}_x) + \frac{20(10^{-9})}{2\pi(10^{-9}/36\pi)}\frac{(-3\bar{a}_x + \bar{a}_z)}{(9+1)}$$

$$= 72\pi\,\bar{a}_x + 36(-3\bar{a}_x + \bar{a}_z)$$

$$\bar{F} = q\,\bar{E} = -5(36)\,[(2\pi - 3)\bar{a}_x + \bar{a}_z]\,mN$$

$$= -0.591\bar{a}_x - 0.18\,\bar{a}_z \text{ N}$$

4.16 Determine the charge density due to each of the following electric flux densities:

(a) $\mathbf{D} = 8xy\mathbf{a}_x + 4x^2\mathbf{a}_y$ C/m^2

(b) $\mathbf{D} = \rho\sin\phi\,\mathbf{a}_\rho + 2\rho\cos\phi\,\mathbf{a}_\phi + 2z^2\mathbf{a}_z$ C/m^2

(c) $\mathbf{D} = \dfrac{2\cos\theta}{r^3}\mathbf{a}_r + \dfrac{\sin\theta}{r^3}\mathbf{a}_\theta$ C/m^2

(a)

$$\rho_V = \nabla \bullet \bar{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 8y + 0 = \underline{\underline{8y \ \ \mathrm{C/m^3}}}$$

(b)

$$\rho_V = \nabla \bullet \bar{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(2\rho\cos\phi) + \frac{\partial}{\partial z}(2z^2)$$

$$= 2\sin\phi - 2\sin\phi + 4z = 4z \ \ \mathrm{C/m^3}$$

(c)

$$\rho_V = \nabla \bullet \bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(\frac{2}{r}\cos\theta) + \frac{1}{r^4 \sin\theta}\frac{\partial}{\partial \theta}(\sin^2\theta)$$

$$= \frac{-2}{r^3}\cos\theta + \frac{1}{r^4 \sin\theta}(2\sin\theta\cos\theta) = \underline{\underline{0}}$$

4.26 Given that the electric field in a certain region is

$$\mathbf{E} = (z+1)\sin\phi\,\mathbf{a}_\rho + (z+1)\cos\,\mathbf{a}_\phi + \rho\sin\phi\,\mathbf{a}_z \text{ V/m}$$

determine the work done in moving a 4-nC charge from

(a) $A(1, 0, 0)$ to $B(4, 0, 0)$

(b) $B(4, 0, 0)$ to $C(4, 30°, 0)$

(c) $C(4, 30°, 0)$ to $D(4, 30°, -2)$

(d) A to D

(a)

$$W_{AB} = q \int \bar{E} \bullet d\bar{l}, \qquad d\bar{l} = d\rho\, \bar{a}_\rho$$

$$\frac{-W_{AB}}{q} = \int (z+1) \sin\phi \, d\rho \Big|_{\phi=0, z=0} = 0$$

$$W_{AB} = 0$$

(b)

$$\frac{-W_{BC}}{q} = \int_{\phi=0}^{30} (z+1) \cos\phi \, \rho \, d\phi \Big|_{\rho=4, z=0} = 4 \sin\phi \Big|_{0}^{30^\circ} = 2$$

$$W_{BC} = -2q = \underline{\underline{-8 \text{ nJ}}}$$

(c)

$$\frac{-W_{CD}}{q} = \int_{z=0}^{-2} \rho \sin\phi \, dz \Big|_{\substack{\phi=30^\circ \\ \rho=4}} = 4 \sin 30^\circ \left(z \Big|_{0}^{-2}\right) = -4$$

$$W_{CD} = 4q = \underline{\underline{16 \text{ nJ}}}$$

(d)

$$W_{AD} = W_{AB} + W_{BC} + W_{CD} = 0 - 8 + 16 = \underline{\underline{8 \text{ nJ}}}$$

4.29 Determine the electric field due to the following potentials:

(a) $V = x^2 + 2y^2 + 4z^2$

(b) $V = \sin(x^2 + y^2 + z^2)^{1/2}$

(c) $V = \rho^2(z + 1)\sin \phi$

(d) $V = e^{-r} \sin \theta \cos 2\phi$

(a)

$$\bar{E} = -\nabla V = -(2x\bar{a}_x + 4y\bar{a}_y + 8z\bar{a}_z)$$

$$= \underline{\underline{-2x\bar{a}_x + 4y\bar{a}_y + 8z\bar{a}_z \text{ V/m}}}$$

(b)

$$-\bar{E} = \frac{\partial V}{dx}\bar{a}_x + \frac{\partial V}{dy}\bar{a}_y + \frac{\partial V}{dz}\bar{a}_z$$

$$= \cos(x^2 + y^2 + z^2)^{1/2}[2x\bar{a}_x + 2y\bar{a}_y + 2z\bar{a}_z](\frac{1}{2})$$

$$= \underline{\underline{-(x\bar{a}_x + y\bar{a}_y + z\bar{a}_z)\cos(x^2 + y^2 + z^2)^{1/2} \quad \text{V/m}}}$$

(c)

$$-\bar{E} = \frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}a_z$$

$$= 2\rho(z+1)\sin\phi\bar{a}_\rho + \rho(z+1)\cos\phi\bar{a}_\rho + \rho^2\sin\phi\bar{a}_z$$

$$= \underline{\underline{-2\rho(z+1)\sin\phi\bar{a}_\rho - \rho(z+1)\cos\phi\bar{a}_\rho - \rho^2\sin\phi\bar{a}_z}}$$

(d)

$$\bar{E} = \frac{\partial V}{\partial}\bar{a}_z + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi$$

$$-\bar{E} = -e^x\sin\theta\cos 2\phi\,\bar{a}_r + \frac{1}{r}e^{-r}\cos\theta\cos 2\phi\,\bar{a}_\theta + \frac{e^{-r}}{r}(-2\sin 2\phi)\bar{a}_\phi$$

$$\underline{\underline{\bar{E} = e^x\sin\theta\cos 2\phi\,\bar{a}_r - \frac{1}{r}e^{-r}\cos\theta\cos 2\phi\,\bar{a}_\theta + \frac{2e^{-r}}{r}(\sin 2\phi)\bar{a}_\phi}} \quad \text{V/m}$$

Chapter (5)

5.1 In a certain region, $\mathbf{J} = 3r^2 \cos\theta\, \mathbf{a}_r - r^2 \sin\theta\, \mathbf{a}_\theta$ A/m, find the current crossing the surface defined by $\theta = 30°, 0 < \phi < 2\pi, 0 < r < 2$ m.

The Solution:

$$I = \int J \bullet dS, \qquad dS = r\sin\theta d\phi dr a_\theta$$

$$I = -\int_{r=0}^{2}\int_{\phi=0}^{2\pi} r^3 \sin^2\theta\, d\phi\, dr \Big|_{\theta=30^o} = -(\sin 30^o)^2 \left.\frac{r^4}{4}\right|_0^2 (2\pi) = -2\pi = \underline{\underline{-6.283}}\text{ A}$$

……

5.3 The current density in a cylindrical conductor of radius a is

$$\mathbf{J} = 10e^{-(1-\rho/a)}\mathbf{a}_z \text{ A/m}^2$$

Find the current through the cross section of the conductor.

The Solution:

$$I = \int J \bullet dS \quad = 10\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} e^{-(1-\rho/a)}\rho\, d\phi\, d\rho = 20\pi \int_{\rho=0}^{a} \rho e^{-(1-\rho/a)} d\rho$$

But $\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax - 1),$

$$I = 20\pi e^{-1} a^2 \left(\frac{\rho}{a} - 1\right) e^{\rho/a}\Big|_0^a = \frac{20\pi a^2}{e}(1+0) = \underline{\underline{23.11a^2}}\ \text{A}$$

……

5.12 A dielectric material contains 2×10^{19} polar molecules/m^3, each of dipole moment 1.8×10^{-27} C/m. Assuming that all the dipoles are aligned in the direction of the electric field $\mathbf{E} = 10^5\,\mathbf{a}_x$ V/m, find $\mathbf{P}$ and ε_r.

The Solution:

$$P = \frac{\sum_{i=1}^{N} q_i d_i}{v} = \frac{\sum_{i=1}^{N} p_i}{v}$$

$$|P| = \frac{N}{v}|p| = 2x10^{19}\,x1.8x10^{-27} = 3.6x10^{-8}$$

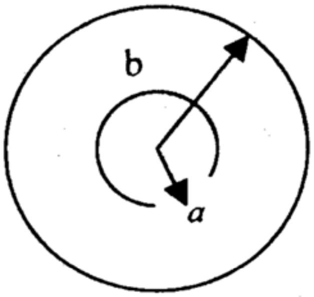
$$P = |P|a_x = \underline{\underline{3.6x10^{-18}\,a_x\ \mathrm{C/m^2}}}$$

But $\quad P = \chi_e \varepsilon_o E \quad$ or $\quad \chi_e = \dfrac{P}{\varepsilon_o E} = \dfrac{3.6x36\pi x10^9\,x10^{-18}}{10^5} = 0.0407$

$$\varepsilon_r = 1 + \chi_e = \underline{\underline{1.0407}}$$

5.18 At the center of a hollow dielectric sphere ($\varepsilon = \varepsilon_0\varepsilon_r$) is placed a point charge Q. If the sphere has inner radius a and outer radius b, calculate $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$.

The Solution:



For $0 < r < a$.

$$D = \frac{Q}{4\pi r^2} a_r \quad \longrightarrow \quad E = \frac{Q}{4\pi \varepsilon_o r^2} a_r, \quad P = 0$$

For $a < r < b$.

$$D = \frac{Q}{4\pi r^2} a_r \quad \longrightarrow \quad E = \frac{Q}{4\pi \varepsilon \varepsilon_r r^2} a_r, \quad P = \chi_e \varepsilon_o E = \frac{\varepsilon_r - 1}{\varepsilon_r} \frac{Q}{4\pi\ r^2} a_r$$

For $r > b$.

$$D = \frac{Q}{4\pi r^2} a_r \quad \longrightarrow \quad E = \frac{Q}{4\pi \varepsilon_o r^2} a_r, \quad P = 0$$

Thus.

$$D = \frac{Q}{4\pi \varepsilon r^2} a_r, \quad r > 0$$

$$E = \begin{cases} \dfrac{Q}{4\pi \varepsilon \varepsilon_r r^2} a_r, & a < r < b \\ \dfrac{Q}{4\pi \varepsilon_o r^2} a_r, & \text{otherwise} \end{cases}$$

$$P = \begin{cases} \dfrac{\varepsilon_r - 1}{\varepsilon_r} \dfrac{Q}{4\pi\ r^2} a_r, & a < r < b \\ 0, & \text{otherwise} \end{cases}$$

5.20 For static (time-independent) fields, which of the following current densities are possible?

(a) $\mathbf{J} = 2x^3y\mathbf{a}_x + 4x^2z^2\mathbf{a}_y - 6x^2yz\mathbf{a}_z$

(b) $\mathbf{J} = xy\mathbf{a}_x + y(z+1)\mathbf{a}_y + 2y\mathbf{a}_z$

(c) $\mathbf{J} = \dfrac{z^2}{\rho}\mathbf{a}_\rho + z\cos\phi\,\mathbf{a}_z$

(d) $\mathbf{J} = \dfrac{\sin\theta}{r^2}\mathbf{a}_r$

The Solution:

Prob. 5.20 Since $\dfrac{\partial \rho_v}{\partial t} = 0$, $\nabla \bullet J = 0$ must hold.

(a) $\nabla \bullet J = 6x^2y + 0 - 6x^2y = 0 \longrightarrow$ This is possible.

(b) $\nabla \bullet J = y + (z+1) \neq 0 \longrightarrow$ This is not possible.

(c) $\nabla \bullet J = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(z^2) + \cos\phi \neq 0 \longrightarrow$ This is not possible.

(d) $\nabla \bullet J = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(\sin\theta) = 0 \longrightarrow$ This is possible.

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5.27 Region 1 ($z < 0$) contains a dielectric for which $\varepsilon_r = 2.5$, while region 2 ($z > 0$) is characterized by $\varepsilon_r = 4$. Let $\mathbf{E}_1 = -30\mathbf{a}_x + 50\mathbf{a}_y + 70\mathbf{a}_z$ V/m and find: (a) $\mathbf{D}_2$, (b) $\mathbf{P}_2$, (c) the angle between $\mathbf{E}_1$ and the normal to the surface.

The Solution:

Prob. 5.27 (a) $E_{2t} = E_{1t} = -300a_x + 50a_y, \quad E_{1n} = 70a_z$

$$D_{2n} = D_{1n} \quad \longrightarrow \quad \varepsilon_2 E_{2n} = \varepsilon_1 E_{1n}$$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{2.5}{4}(70a_z) = 43.75a_z$$

$$E_2 = -30a_x + 50a_y + 43.75a_z$$

$$D_2 = \varepsilon_o \varepsilon_r E_2 = 4x\frac{10^{-9}}{36\pi}(-30,50,43.75) = \underline{\underline{-1.061a_x + 1.768a_y + 1.547a_z \ \text{nC/m}^2}}$$

(b) $$P_2 = \varepsilon_o \chi_{e2} E_2 = 3x\frac{10^{-9}}{36\pi}(-30,50,43.75) = \underline{\underline{0.7958a_x + 1.326a_y + 1.161a_z \ \text{nC/m}^2}}$$

(c) $E_1 \bullet a_z = E_1 \cos\theta_n$

$$\cos\theta_n = \frac{70}{\sqrt{30^2 + 50^2 + 70^2}} = 0.7683 \quad \longrightarrow \quad \underline{\underline{\theta_n = 39.79^o}}$$

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5.29 Two homogeneous dielectric regions 1 ($\rho \leq 4$ cm) and 2 ($\rho \geq 4$ cm) have dielectric constants 3.5 and 1.5, respectively. If $\mathbf{D}_2 = 12\mathbf{a}_\rho - 6\mathbf{a}_\phi + 9\mathbf{a}_z$ nC/m^2, calculate: (a) $\mathbf{E}_1$ and $\mathbf{D}_1$, (b) $\mathbf{P}_2$ and ρ_{pv2}, (c) the energy density for each region.

The Solution:

Prob. 5.29 (a) $D_{2n} = 12a_\rho = D_{1n}, \qquad D_{2t} = -6a_\phi - 9a_z$

$$E_{2t} = E_{2t} \longrightarrow \frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$

$$D_{1t} = \frac{\varepsilon_1}{\varepsilon_2} D_{2t} = \frac{3.5\varepsilon_o}{1.5\varepsilon_o}(-6a_\phi + 9a_z) = -14a_\phi + 21a_z$$

$$\underline{\underline{D_1 = 12a_\rho - 14a_\phi + 21a_z \text{ nC/m}^2}}$$

$$E_1 = D_1/\varepsilon_1 = \frac{(12,-14,21)x10^{-9}}{3.5x\dfrac{10^{-9}}{36\pi}} = \underline{\underline{387.8a_\rho - 452.4a_\phi + 678.6a_z \text{ V/m}}}$$

(b) $$P_2 = \varepsilon_o \chi_{e2} E_2 = 0.5\varepsilon_o \frac{D_2}{\varepsilon_2} = \frac{0.5\varepsilon_o}{1.5\varepsilon_o}(12,-6,9) = \underline{\underline{4a_\rho - 2a_\phi + 3a_z \text{ nC/m}^2}}$$

$$\rho_{v2} = \nabla \bullet P_2 = 0$$

(c) $$w_{E1} = \frac{1}{2} D_1 \bullet E_1 = \frac{1}{2}\frac{D_1 \bullet D_1}{\varepsilon_o \varepsilon_{r1}} = \frac{1}{2}\frac{(12^2 + 14^2 + 21^2)x10^{-18}}{3.5x\dfrac{10^{-9}}{36\pi}} = \underline{\underline{12.62 \text{ mJ/m}^2}}$$

$$w_{E1} = \frac{1}{2}\frac{D_2 \bullet D_2}{\varepsilon_o \varepsilon_{r2}} = \frac{1}{2}\frac{(12^2 + 6^2 + 9^2)x10^{-18}}{5x\dfrac{10^{-9}}{36\pi}} = \underline{\underline{9.839 \text{ mJ/m}^2}}$$