Strict fuzzy sets and strict fuzzy probability

Article *in* International Journal of Applied Mathematics and Statistics · February 2016

1 author:

Youssef Abed
Canadian International College

International Journal of Applied Mathematics and Statistics,
Int. J. Appl. Math. Stat.; Vol. 56; Issue No. 2; Year 2017,
ISSN 0973-1377 (Print), ISSN 0973-7545 (Online)

Strict Fuzzy Sets and Strict Fuzzy Probability

Youssef Prince Abed[1], **Abdul Hadi Nabih Ahmed**[2] and **Hamiden Abd Elwahed Khalifa**[3]

[1] Department of Basic Science
Canadian International Collage
[1]Email: Youssefabed@windowslive.com

[12] Department of Mathematical Statistics & [3] Department of Operation Research
Institute of Statistical Studies and Research, Cairo University
[2]Email: drhadi2010@gmail.com & [3]Email: hamiden_2008@yahoo.com

ABSTRACT

Fuzzy sets have been introduced by Zadeh (1965) as a generalization of classical set theory. Each element in fuzzy belongs to the set by certain degree which is specified by membership function. From this stand point, an extension of classical probability theory is reached "fuzzy probability". However, the definition of membership function and fuzzy set operations are not strict enough to attain the law of non-contradiction and the law of exclude the middle. In this paper, we will strictly define membership function and consequently redefining fuzzy probability such that, each element has a degree of belongness to the set and still satisfying such desirable laws.

Keywords: Fuzzy sets, Fuzzy probability, Strict fuzzy Sets, Strict fuzzy probability, Maximum least strict fuzz likelihood function, Strict fuzzy expectation.

Mathematics Subject Classification: 94D99, 60A86

1. INTRODUCTION

The concept of fuzzy sets and fuzzy probability introduced by Zadeh (1965) and (1986) respectively has become increasingly applied in many fields. However, there are some theoretical considerations of fuzzy set theory and fuzzy probability that needs revision. Using the operations introduced by Zadeh for fuzzy sets and fuzzy probability, we will show that two fundamental laws of thought will not hold.

The law of excluded the middle (everything is either be or not to be), this correspond in set theory to the union of a set and it's complement has to produce universal set. While, for fuzzy sets the union operation on the fuzzy set A and A^c is defined as

$$\mu_{A\cup A^c}(\omega) = Max\{\mu_A(\omega), \mu_{A^c}(\omega)\} \neq \mu_\Omega(\omega), \qquad (1)$$

where $\mu_\Omega(\omega)$ is membership function defined on Ω (universal set).

Another law of thought is the law of non-contradiction (nothing can be and not to be), an element has to belong to as set or it's complement but not both, while, the intersection operation on fuzzy set A and it's complement A^c is defined by

$$\mu_{A\cap A^c}(\omega) = Min\{\mu_A(\omega), \mu_{A^c}(\omega)\} \neq \mu_\phi(\omega), \tag{2}$$

where $\mu_\phi(\omega)$ denotes that each element in the empty set ϕ has a membership value of zero.

Using the definition of fuzzy sets and it's operations, Zadeh (1986) introduced the concept of fuzzy probability or probability of an event given that each element belongs to this event by certain degree. However, the nonexistence of strict boarders between fuzzy set and it's complement results the following inequalities

$$P(A \cup A^c) = \int_\Omega Max\{\mu_A(\omega), 1 - \mu_A(\omega)\}dP \neq P(\Omega). \tag{3}$$

and

$$P(A \cap A^c) = \int_\Omega Min\{\mu_A(\omega), 1 - \mu_A(\omega)\}dP \neq P(\phi). \tag{4}$$

There were many other attempts to link probability theory and fuzzy sets. See, Singpurwalla and Booker (2004) introduced a new definition of fuzzy probability by conditioning on membership function. In their paper they showed that $P(A \cup A^c) = P(\Omega)$ and $P(A \cap A^c) = P(\phi)$. However, their new definition is based on classical operation of fuzzy sets. In other words, although $P(A \cup A^c) = P(\Omega)$ and $P(A \cap A^c) = P(\phi)$, we have the inequalities of $\mu_{A\cup A^c}(\omega) \neq \mu_\Omega(\omega)$ and $\mu_{A\cap A^c}(\omega) \neq \mu_\phi(\omega)$. What we need is to redefine membership function and operations on fuzzy sets in such way so we can satisfy previous laws.

This paper will include two major sections, in first section we introduce the concept and properties of strict fuzzy sets and in second section, based on the definition of strict fuzzy sets we introduce strict fuzzy probability and it's properties.

2. STRICT FUZZY SETS

Let Ω be universal set, we define membership function on Ω such that, $\mu_\Omega(\omega): \Omega \rightarrow [0,1]$. Now, we define strict fuzzy subset A form Ω with charctarizing membership of

$$\mu_A(\omega) = I_A(\omega)\mu_\Omega(\omega) \quad \forall\, \omega \in \Omega, \tag{5}$$

where $\mu_A(\omega)$ reflects the degree of membership of ω in A, while

$$I_A(\omega) = \begin{cases} 1 & if\ \omega \in A; \mu_\Omega(\omega) > 0 \\ 0 & if\ \omega \in A; \mu_\Omega(\omega) = 0. \end{cases} \tag{6}$$

The notation $\omega \in A; \mu_\Omega(\omega) > 0$ means that, elements in A that have membership values greater than zero. While, the notation $\omega \in A; \mu_\Omega(\omega) = 0$ denotes the elements in A that have membership values equal to zero (elements that are outside the set A) . It is important to know that, we are defining strict boarders of a set A using indicator function $I_A(\omega)$. So, the set A will be shown to be disjoint with it's complement. However, classical representation of membership function has no strict boarders between a set and it's complement. Below figure will show the difference between two definitions.

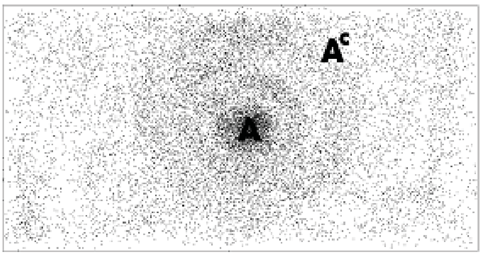

Figure (2.1) shows the degree of membership of each element in the set A and it's complement (the darker the point, the higher the degree of membership), some elements can belong to A and A^c.

Figure (2.1): Fuzzy set and it's complement.

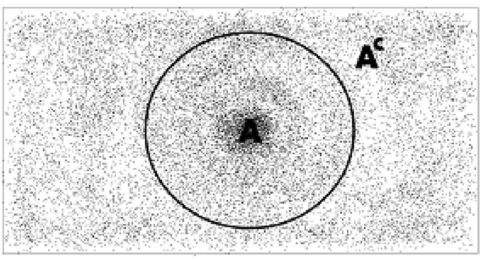

Figure (2.2) shows that each element is in strict fuzzy set A is only in the set. In other words, there is a strict boarder between the set A and it's complement.

Figure (2.2): Strict fuzzy set and it's complement.

In next section we introduce some operations on strict fuzzy sets.

2.1 Basic Operation on Strict-fuzzy Sets

Based on our definition of strict fuzzy membership function, we will define operations on strict fuzzy sets in such way that we can satisfy desirable set properties.

2.1.1 Empty set Strict Fuzzy Sets:

A strict fuzzy set A is said to be empty if each element ω belong to A by membership value of zero, formally

$$\mu_A(\omega) = 0 \quad \forall\, \omega \in \Omega. \tag{7}$$

Previous definition is similar to the definition in classical set theory that is, if all elements do not belong to a set, then it is empty.

2.1.2 Intersection of Strict Fuzzy Sets:

The intersection of two strict fuzzy sets A and B is defined using indicator function of both sets as

$$\begin{aligned}\mu_{A\cap B}(\omega) &= Min\{\mu_A(\omega), \mu_B(\omega)\} \\ &= Min\{I_A(\omega), I_B(\omega)\}\mu_\Omega(\omega),\end{aligned} \tag{8}$$

here we strictly associate each element in the set $A \cap B$ by membership value. It is considerable to note that the minimum operation is independent from the membership function $\mu_\Omega(\omega)$.

2.1.3 Union of Strict Fuzzy Sets:

The union of strict fuzzy sets A and B is defined by

$$\begin{aligned}\mu_{A\cup B}(\omega) &= Max\{\mu_A(\omega), \mu_B(\omega)\} \\ &= Max\{I_A(\omega), I_B(\omega)\}\mu_\Omega(\omega).\end{aligned} \tag{9}$$

Similarly, the maximum operation is independent of membership of universal set.

2.1.4 Complement of Strict Fuzzy Sets:

The complement of a strict fuzzy set A is all the elements not belonging to the set A with corresponding membership function

$$\begin{aligned}\mu_{A^c}(\omega) &= \mu_\Omega(\omega) - \mu_A(\omega) \\ &= \big(1 - I_A(\omega)\big)\mu_\Omega(\omega),\end{aligned} \tag{10}$$

this definition of the membership function of the complement is strict enough to disjoin a strict fuzzy set from it's complement.

Example: Let our strict universal fuzzy set $\Omega = \{1/0.2, 2/0.5, 3/0, 4/0.9, 5/0.7, 6/0.2\}$, we have to not note that, the element 3 is not a member of the set Ω, consider a subset A containing $\{2, 3, 4\}$, then based on our definition of strict membership function in equation (5), strict fuzzy set $A = \{\,2/0.5, 3/0, 4/0.9\}$, similarly consider strict fuzzy subset $B = \{3/0, 4/0.9, 5/0.7, 6/0.2\}$. Then, the union of both sets will be $A \cup B = \{\,2/0.5, 3/0, 4/0.9, 5/0.7, 6/0.2\}$ and the intersection is $A \cap B = \{\,3/0, 4/0.9\}$ finally, the complement of strict fuzzy A is $A^c = \{1/0.2, 5/0.7, 6/0.2\}$ .It can be verified that $A \cup A^c$ is Ω and $A \cap A^c$ yields the empty set.

2.1.5 Strict Fuzzy Relation:

Consider elementary element ω_i in the strict fuzzy set A_i for $i = 1,2, \dots, n$ and define the set $A = A_1 \times A_2, \dots, A_n$. Then, each element of A is order pair taking the form $(\omega_1, \omega_2, \dots, \omega_n)$. Hence, membership function can be defined on A as $\mu_A(\omega_1, \omega_2, \dots, \omega_n)\colon A \to [0,1]$ based on classical indicator function $I_A(\omega_1, \omega_2, \dots, \omega_n)$.

2.2 Properties of Strict Fuzzy Sets

In this section, some important properties of strict fuzzy sets that coincide with classical set properties will be shown.

2.2.1 Equality of Strict fuzzy sets:

Two strict fuzzy sets A and B are equal if the memberships of both sets are equal for all elements in Ω, namely

$$\mu_A(\omega) = \mu_B(\omega) \quad \forall\, \omega \in \Omega, \tag{11}$$

this is the condition for the equality of strict fuzzy sets.

2.2.2 Containment of strict fuzzy sets:

Claim 2.2.2: A strict fuzzy set A is contained in B if and only if

$$\mu_A(\omega) \leq \mu_B(\omega) \quad \forall\, \omega \in \Omega. \tag{12}$$

Proof: If $\exists\ \omega \in B \cap A^c$, then $I_A(\omega) \leq I_B(\omega)$ and since $\mu_\Omega(\omega) \geq 0$, we can reach $\mu_A(\omega) \leq \mu_B(\omega)$. Similarly, if $\mu_A(\omega) \leq \mu_B(\omega)$, then each $\omega \in B$ also belong to A, then $A \subseteq B$.

2.2.3 Associative and Commutative Properties of Intersection and Union:

It follows from the properties of the minimum and the maximum that associative and commutative properties hold for the union and intersection of strict fuzzy sets.

2.2.4 The Law of Exclude the Middle:

Each element in strict fuzzy set is either in the set or outside the set. So, if A is strict fuzzy set, then

$$\mu_{A\cup A^c}(\omega) = Max\{\mu_A(\omega), \mu_{A^c}(\omega)\} = \mu_\Omega(\omega). \tag{13}$$

It is important to note that, the union of strict fuzzy set and it's complement yields the strict universal fuzzy set and not classical universal set (crisp universal set) but if membership function values of strict fuzzy universal set are equal to zeros and ones for all elements, then we have classical universal set.

2.2.5 The law of non-contradiction:

Each element cannot be in strict fuzzy set and it's complement. So, if A is strict fuzzy set, then

$$\mu_{A\cap A^c}(\omega) = Min\{\mu_A(\omega), \mu_{A^c}(\omega)\} = \mu_\phi(\omega), \tag{14}$$

where $\mu_\phi(\omega)$ denote that each element ω in ϕ has a membership of zero, which is required.

2.2.6 Demorgan Law's:

Claim 2.2.6: Using our definition and properties above, strict fuzzy set obeys Demorgan laws for the union and intersection

$$\mu_{(A\cap B)^c}(\omega) = \mu_{A^c\cup B^c}(\omega). \\ \mu_{(A\cup B)^c}(\omega) = \mu_{A^c\cap B^c}(\omega). \tag{15}$$

Proof: by our definition of intersection in (8) and the complement in (10), we can write

$$\begin{aligned} \mu_{(A\cap B)^c}(\omega) &= \{1 - Min\{\mu_A(\omega), \mu_B(\omega)\}\}\mu_\Omega(\omega) \\ &= \begin{cases} \mu_{A^c}(\omega) & if\ I_{A^c}(\omega) > I_{B^c}(\omega) \\ \mu_{B^c}(\omega) & if\ I_{A^c}(\omega) < I_{B^c}(\omega) \\ \mu_{A^c}(\omega)\ or\ \mu_{B^c}(\omega) & if\ I_{A^c}(\omega) = I_{B^c}(\omega) \end{cases} \\ &= Max\{\mu_{A^c}(\omega), \mu_{B^c}(\omega)\}. \end{aligned} \tag{16}$$

While for intersection, using the definition of union in (9) and complement in (10), we can clearly write

$$\begin{aligned} \mu_{(A\cup B)^c}(\omega) &= \{1 - Max\{\mu_A(\omega), \mu_B(\omega)\}\}\mu_\Omega(\omega) \\ &= \begin{cases} \mu_{A^c}(\omega) & if\ I_{A^c}(\omega) < I_{B^c}(\omega) \\ \mu_{B^c}(\omega) & if\ I_{A^c}(\omega) > I_{B^c}(\omega) \\ \mu_{A^c}(\omega)\ or\ \mu_{B^c}(\omega) & if\ I_{A^c}(\omega) = I_{B^c}(\omega) \end{cases} \\ &= Min\{\mu_{A^c}(\omega), \mu_{B^c}(\omega)\}. \end{aligned} \tag{17}$$

Which is our needed result.

3. STRICT FUZZY PROBABILITY

Consider a probability space (Ω, B,P), where $\Omega = R^n$ is the sample space, B is Borel sigma algebra and P is a probability measure on B such that $P: B \to [0,1]$. Let $x \in R^n$ and let $\mu_\Omega(x)$ be a Borel measurable function, then, we can define the probability of a subset A (an event) bye Lesbuge-Stejles integral as

$$P(A) = \int_{R^n} I_A(x)\mu_\Omega(x)\, dP. \tag{18}$$

Now if membership function $\mu_\Omega(x)$ is a simple function taking the form of $\sum_{i=1}^n a_i B_i$, where $a_i \geq 0$ and $B_i \in$ B, then the integral defined by $\int_{R^n} I_A(x)\mu_\Omega(x)\, dP = \sum_{i=1}^n a_i P(B_i)$. While, if $\mu_\Omega(x)$ is non-negative function the integral will be $\int_{R^n} I_A(x)\mu_\Omega(x)\, dP = sup_{g\in S(\mu_\Omega(x))} \int_{R^n} g\, dP$ for $S(\mu_\Omega(x))$ to be collection of simple functions such that, $g(x) \leq \mu_\Omega(x)\ \forall\, x \in \Omega$.

3.1 Properties of Strict Fuzzy Probability

It is important to show how our new definition of membership function and strict fuzzy probability will help to attain the laws of classical probability theory but in more general settings.

3.1.1 Exhaustive Strict Fuzzy Event:

Claim 3.1.1: The probability of x belonging strict fuzzy set A or it's complement is the same as the probability of x belonging to the universal set

$$P(A \cup A^c) = P(\Omega). \tag{19}$$

Proof: from our definition in equation (9) we can write

$$P(A \cup A^c) = \int_\Omega Max\{\mu_A(x), \mu_{A^c}(x)\}dP, \tag{20}$$

using equation (13)

$$P(A \cup A^c) = \int_\Omega \mu_\Omega(x)\, dP = P(\Omega) \leq 1 \;\; \forall \;\; x \in \Omega. \tag{21}$$

Which is our desired result.

3.1.2 Mutually exclusive Strict Fuzzy events:

Claim 3.1.2: The probability of x belonging to the set A and it's complement is equal to the probability of the empty set

$$P(A \cap A^c) = P(\phi). \tag{22}$$

Proof:

$$P(A \cap A^c) = \int_\Omega Min\{\mu_A(x), \mu_{A^c}(x)\}dP, \tag{23}$$

by equation (14), we can write

$$P(A \cap A^c) = \int_\Omega \mu_\phi(x)\, dP = P(\phi) = 0 \;\; \forall \;\; x \in \Omega. \tag{24}$$

As required.

3.1.3 Monotonicity of Strict Fuzzy Probability:

Claim 3.1.3: If $A \subseteq B$, then the probability satisfies $P(A) \leq P(B)$

Proof:
From equation (12), we know that if $A \subseteq B$, then $\mu_A(x) \leq \mu_B(x)$. Now, we have two cases: case number 1 is that, $\mu_A(x)$ and $\mu_B(x)$ are simple functions. We can write $\mu_B(x) = \mu_A(x) + g$, where g is a simple function.

Using linearity property of the integral defined in (18), we can write

$$\int_{\Omega} \mu_B(x)\, dP = \int_{\Omega} \mu_A(x)\, dP + \int_{\Omega} g\, dP \geq \int_{\Omega} \mu_A(x)\, dP. \tag{25}$$

Case 2: both functions $\mu_A(x)$ and $\mu_B(x)$ are non-negative functions. Let $S(\mu_A(x))$ be collection of simple functions such that, $g(x) \leq \mu_A(x) \leq \mu_B(x)$, then, $sup_{g\in S(\mu_A(x))} \int_{R^n} g\, dP \leq sup_{g\in S(\mu_B(x))} \int_{R^n} g\, dP$ this implies that $\int_{\Omega} \mu_A(x)\, dP \leq \int_{\Omega} \mu_B(x)\, dP$.

3.1.4 Probability of the Union of Strict fuzzy Events:

Claim 3.1.14: For two Strict fuzzy events A and B, the probability of the union of two strict fuzzy sets is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{26}$$

Proof:

$$\begin{aligned} P(A \cup B) &= \int_{\Omega} Max\{\mu_A(x), \mu_B(x)\} dP \\ &= \int_{\Omega} (\mu_A(x) + \mu_B(x) - Min\{\mu_A(x), \mu_{A^c}(x)\})\, dP, \end{aligned} \tag{27}$$

by linearity property of the integral we can write

$$P(A \cup B)\ = P(A) + P(B) - P, \tag{28}$$

this is the same result as in classical probability theory.

In general, if we have n strict fuzzy sets $A_1, A_2, \dots, A_n$, Then, the union can be written as

$$\begin{aligned} P(\bigcup_{i=1}^{n} A_i) = \int_{\Omega} Max_i\{\mu_{\ i}(x)\}\, dP \ &= \sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i \cap A_j) \\ &+ \sum_{i<j<K}^{n} P(A_i \cap A_j \cap A_K) - \cdots + (-1)^{n-1}\, P\left(\bigcap_{i=1}^{n} A_i\right), \end{aligned} \tag{29}$$

which is same expression of union of n events in classical probability theory.

3.1.5 Independence of Strict Fuzzy Events:

Consider we have n strict fuzzy events $A_1, A_2, \dots, A_n$, we say they are independent if

$$\mu_{\cap_{i=1}^{n} A_i}(x) = \prod_{i=1}^{n} \mu_{A_i}(x). \tag{30}$$

Then, clearly we can reach the familiar law of the probability of independent events, as

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i). \tag{31}$$

Based on this definition, we can define strict fuzzy conditional probability.

3.1.6 Conditional Strict Fuzzy Probability:

We can define the conditional strict fuzzy probability for x belonging to the strict fuzzy set A give information that it belongs to the strict fuzzy set B by

$$P(x \in A|x \in B) = \frac{\int_{\Omega} Min\{\mu_A(x), \mu_B(x)\}dP}{\int_{\Omega} \mu_B(x)dP}, \tag{32}$$

provided that $\int_{\Omega} \mu_B(x)dP$ is different from zero.

3.1.7 Strict Fuzzy Total Probability and Strict Fuzzy Bayes Theorem

Consider we have n strict fuzzy events (sets) $A_1, A_2, \dots, A_n$, such that $Min\{\mu_{A_i}(x), \mu_{A_j}(x)\} = 0 \ \ \forall i \neq j$ and $Max_{1\leq i\leq n}\{\mu_{A_i}(x)\} = \mu_B(x)$, then

$$\begin{aligned} P(B) &= \int_{\Omega} Max_i\{\mu_{A_i}(x)\}\, dP \\ &= \sum_{i=1}^{n} \int_{\Omega} \mu_{A_i}(x)\, dP. \end{aligned} \tag{33}$$

Then, the familiar Bayes formula can be reached as

$$P(x \in A|x \in B) = \frac{P(x \in B|x \in A)P(A)}{P(B)}, \tag{34}$$

provided that $P(B)$ is different from zero.

3.1.8 Strict Fuzzy Boole Inequality:

Claim 3.1.8: We will show that strict fuzzy probability satisfies Boole's inequality, defined by

$$P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i). \tag{35}$$

We will provide the proof for this statement.

We can define membership function for strict fuzzy sets $B_1, B_2, \dots, B_n$ as

$$\begin{gathered} \mu_{B_1}(x) = \mu_{A_1}(x) \\ \mu_{B_2}(x) = \mu_{A_2}(x) - \mu_{A_1}(x) \\ \mu_{B_3}(x) = \mu_{A_3}(x) - Min\{\mu_{A_1}(x), \mu_{A_2}(x)\} \\ . \\ . \\ . \\ \mu_{B_n}(x) = \mu_{A_n}(x) - Min_{1\le i\le n-1}\{\mu_{A_i}(x)\}. \end{gathered} \tag{36}$$

Lemma 3.1.8.1: we need to proof that $Min_i\{\mu_{B_i}(x)\} = 0$ to reach our crucial result

Proof:

Let $k > j$, then we can write

$$\mu_{B_j}(x) = (I_{A_j}(x) - Min_{1\le m\le n-1}\{I_{A_m}(x)\})\mu_\Omega(x), \tag{37}$$

similarly, the membership of the set B_k is

$$\mu_{B_k}(x) = (I_{A_k}(x) - Min_{1\le l\le n-1}\{I_{A_l}(x)\})\mu_\Omega(x), \tag{38}$$

now, if

$$I_{A_k}(x) - Min_{1\le l\le n-1}\{I_{A_l}(x)\} = 1, \tag{39}$$

then,

$$I_{A_j}(x) - Min_{1\le m\le n-1}\{I_{A_m}(x)\} = 0, \tag{40}$$

and we can finally reach

$$Min\left\{\mu_{B_j}(x), \mu_{B_k}(x)\right\} = 0. \tag{41}$$

This is our desired result.

Lemma 3.1.8.2: Another lemma has to be proved, that is $Max_i\{\mu_{B_i}(x)\} = Max_i\{\mu_{A_i}(x)\}$ for proving our theorem.

Proof:

$$\begin{aligned} Max_i\{\mu_{B_i}(x)\} &= Max_i\{I_{A_i}(x) - Min_{1\le j\le n-1}\{I_{A_j}(x)\}\}\mu_\Omega(x) \quad \forall\, i > j \\ &= Max_i\{I_{A_i}(x)\}\mu_\Omega(x). \end{aligned} \tag{42}$$

Which is required.

Now since $B_i \subseteq A_i$, then using previous lemma we can conclude

$$P(\bigcup_{i=1}^{\infty} B_i) \leq \sum_{i=1}^{\infty} P(A_i). \qquad (43)$$

This is classical probability theory result.

3.2 All Possible Strict Fuzzy Density Functions

In the literature of probability and statistics joint probability density functions are always used and consequently joint fuzzy probability density functions. It is useful to show how can we define all possible density functions and some of it's properties. Consider we are interested in knowing whether an outcome x will lie in a strict fuzzy set A_1 or A_2. Then, we can partition strict universal set Ω into strict fuzzy sets A_1, A_2, $A_3 = A_1 \cap A_2$ and $A_4 = (A_1 \cup A_2)^c$. Then, the "least strict fuzzy density function" is to be the density function for the union of these disjoint sets which is the same as the density function for Ω, defined by

$$f_\Omega(x) = Max_{1 \leq i \leq 4}\{\mu_{A_i}(x)\}f(x) = \sum_{i=1}^{4} \mu_{A_i}(x)f(x). \qquad (44)$$

Combination	Operation	Combination	Operation	Combination	Operation	Combination	Operation
$i = 1$	Material nonimplication	$i = 1,2$	Exclusive Disjunction	$i = 2,4$	Negation of A_i	$i = 1,3,4$	Converse Implication
$i = 2$	Converse nonimplication	$i = 1,3$	A_i	$i = 3,4$	Biconditional	$i = 2,3,4$	Material Implication
$i = 3$	Conjunction	$i = 1,4$	Negation of A_m	$i = 1,2,3$	Disjunction	$i = 1,2,3,4$	Tautology (Ω)
$i = 4$	Joint Denial	$i = 2,3$	A_m	$i = 1,2,4$	Alternative denial	None	False

From equation (44) $i \in \{1, 2, 3, 4\}$, each possible subset of this set corresponds to a density function. Below table will show all possible density function for two sets A_1 and A_2.

Table 3.2.1. All possible density functions

It is clear that for two sets we have 2^4 possible density function. In general if we have n sets we will have 2^n possible density functions. Now, by defining least density function some interesting properties will unfold.

Boundness of Maximum least strict fuzzy likelihood: Generally speaking, if we had observed $x \in R^n$ with joint pdf $f(x;\theta)$ where $\theta \in R^n$and we are interested whether x lieing in at least $A_i \quad \forall i = 1,2,\dots,n$ such that, $Min_i\{\mu_{A_i}(x)\} = 0$ then, the density is to be defined by

$$f_{\cup_{i=1}^{n} A_i}(x;\theta) = Max_i\{\mu_{A_i}(x)\}f(x;\theta), \tag{45}$$

then, the Maximum least Strict Fuzzy Likelihood is defined by

$$sup_\theta f_{\cup_{i=1}^{n} A_i}(x;\theta), \tag{46}$$

and since $\cup_{i=1}^{n} A_i \subseteq \Omega$ and $f_{\cup_{i=1}^{n} A_i}(x;\theta) \leq f_\Omega(x;\theta)$ then,

$$Sup_\theta\{f_{\cup_{i=1}^{n} A_i}(x;\theta)\} \leq sup_\theta\{f_\Omega(x;\theta)\}, \tag{47}$$

which is an upper bound for Maximum least strict fuzzy likelihood.

Boundness of strict fuzzy expectation: if we consider n strict fuzzy sets $A_1, A_2, \dots, A_n$, the "strict fuzzy least expectation" for any function $g(x)$ is defined by

$$E[g(x)] = \int_\Omega g(x) f_{\cup_{i=1}^{n} A_i}(x;\theta)\, dx, \tag{48}$$

if $g(x)$ is non-negative function, then

$$\int_\Omega g(x) f_{\cup_{i=1}^{n} A_i}(x;\theta)\, dx \leq \int_\Omega g(x) f_\Omega(x;\theta)\, dx, \tag{49}$$

which is an upper bound for strict fuzzy least expectation.

4. CONCLUSION

Throughout the paper, we introduce a strict definition of membership function and operation on fuzzy set. It has been shown that from these definitions we can preserve the law of exclude the middle, the law of non-contradiction and other important set properties.

A consequent result is that, strict fuzzy probability is properly defined such that any element can belong to either a set or it's complement and an event and it's complement are mutually exclusive. Also, we have shown that total probability and Bayes formula are derivable under new fuzzy probability definition. Another important concept that was introduced is the concept of “least strict fuzzy density function”, that allow the definition of all possible fuzzy densities corresponding to all possible set operations.

It is clear that uncertainty plays important role in science; we hope that the introduction of strict fuzzy sets and strict fuzzy probability can endorse wide range of applications.

5. REFERENCES

1. Folland. B. (1999). Real Analysis: Modern Techniques and Their Applications, 2nd Edition, A Wiley Interscience Series of Texts Monographs, and Tracts.

2. Grimmett. G. and Stirzaker. D. (2001). Probability and Random Processes, 3rd edition, Oxford University Press.

3. Grinstead, C. and Snell, J. (1997). "Introduction to Probability, 2nd edition". American Mathematical Society, Providence.

4. Singpurwalla, N. and Booker, J. (2004). membership functions and probability measures of fuzzy sets. Journal of the American Statistical Association, Taylor & Francis, (99): 867-877.

6. Yurisuhov and Kelbert, M. (2014). "Basic Probability and Statistics, 2nd edition". Cambridge university press.

7. Zadeh, L. (1965). fuzzy sets. Journal of Information and Control, Elsvier, (8): 338-353.

8. Zadeh, L. (1968). probability measure of fuzzy event. Journal of Mathematical Analysis and Application, Academic Press, (23): 421-427.