Research Article

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Statistical inference of constant-stress partially accelerated life tests under type II generalized hybrid censored data from Burr III distribution

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Abstract: Accelerated life tests (ALTs) involve subjecting units to more extreme conditions than normal to reduce the duration of testing. These tests, whether fully accelerated or partially accelerated, are crucial in life testing research as they help save both time and money. When results from ALT cannot be extrapolated to normal circumstances, partial ALTs are performed. This study introduces a constant-stress partial ALT, which relies on a generalized Type-II hybrid censoring scheme and assumes that the lifetimes of units under the specified conditions follow the Burr III distribution. The estimates of parameters and accelerated factor of the Burr III distribution are obtained under normal use settings by applying the maximum likelihood and Bayesian techniques. Bayesian estimators are generated using symmetric and asymmetric loss functions through the Monte Carlo Markov Chain approach. Additionally, credible intervals and asymptotic confidence intervals are constructed. Simulation research is carried out using different censoring techniques and sample sizes in order to compare the suggested methodologies. Next, two data sets are analyzed to show the value of the proposed approaches. The study's conclusion contains a summary of its main conclusions.

Keywords: constant-stress partially accelerated life tests, Burr type III distribution, maximum-likelihood estimation, asymptotic confidence intervals, Bayesian estimation, Markov chain Monte Carlo

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Notation

n	total number of experiment items, where $n = n_1 + n_2$
T_1 & T_2	termination time of the experiment, predetermined by the experimenter, where $T_1 < T_2$
$y_{r:n}$	the time of the rth failure observed from n items under GTI-HCS
$y_{s:n}$	the time of the sth failure observed from n items under GTII-HCS
r & s	the failure number obtained at time T_1 and T_2 under GTI-HCS and GTII-HCS, respectively
y_{1i}	the items allocated at use condition in CSPALT, under GTII-HCS, where, $i = 1, \ldots, n_1$
y_{2i}	the items allocated at accelerated condition, in CSPALT, under GTII-HCS, where $i = 1, \ldots, n_2$
r_1	the failed items observed from n_1 before the censoring time h_1, in case of used condition under GTII-HCS
r_2	the failed items observed from n_2 units before the censoring time h_2, in case of accelerated condition, under GTII-HCS
d_{11} & d_{21}	the failure number obtained at time T_{11} and T_{21}, respectively, at used condition in CSPALT under GTII-HCS
d_{12} & d_{22}	the failure number obtained at time T_{12} and T_{22}, respectively, at accelerated condition in CSPALT under GTII-HCS

D_1 & D_2	the number of total failures in the experiment up to time h_1 & h_2, respectively, in CSPALT under GTII-HCS
T_{11} & T_{12}	time points predetermined by the experimenter in used condition of CSPALT under GTII-HCS
T_{21} & T_{22}	time points predetermined by the experimenter in accelerated condition of CSPALT under GTII-HCS

Abbreviations

Abias	absolute Bias
AD	Anderson-Darling
AIC	Akaike information criterion
Asy-CIs	asymptotic confidence intervals
BCIs	Bayesian credible intervals
BEs	Bayesian estimates
BIC	Bayesian information criterion
BIIID	Burr III distribution
CAIC	corrected AIC
CDF	cumulative distribution function
CIs	confidence intervals
CP	coverage probability
CSPALT	constant-stress partial accelerated life testing
CVM	Cramer-von Mises
G1 & G2	first group and second group, respectively
GTI−HCS	generalized type I – hybrid censoring sampling
GTII−HCS	generalized type II – hybrid censoring sampling
HCS	hybrid censored sampling
HF	hazard rate function
HQIC	Hannan–Quinn information criterion
K-S	Kolmogorov–Smirnov
KSD	K-S distance
KSPV	K-S *p*-value
LACI	length of asymptotic confidence intervals
LCCI	length of Bayesian credible intervals
LXF	linear exponential loss function
MCMC	Markov chain Monte Carlo
MH	Metropolis-Hastings
MLE	maximum likelihood estimate
PALT	partial accelerated life test
PP	probability-probability
QQ	quantile-quantile
StEr	standard error
TTT	total time on test
V-CM	variance–covariance matrix

1 Introduction

In investigations of reliability and life testing, censored data are crucial when it becomes impossible to obtain all the data needed for the experiment due to factors like expense or schedule. Type-I and Type-II censoring are two of the most common types of censoring. A hybrid censored sampling (HCS) approach, which combines Type-I and Type-II censoring schemes, was proposed by Epstein [1]. Suppose that $Y_{1:n} < Y_{2:n} <\ldots< Y_{n:n}$ denote the ordered failure times of the experimental units. In Type-I HCS (TI−HCS), the life-testing experiment ends at time $T_1 = \min(Y_{r:n}, T)$, where $Y_{r:n}$ represents the time of the rth failure out of n items, and T is the maximum time point for the test, $r \in (1, 2, \ldots, n)$, and $T \in (0, \infty)$. The test continues until a predefined number of units fail, at which point the experiment is terminated. Since the maximum amount of time allowed for the test under this system is T, there is a chance that relatively few failures in this HCS strategy will happen before time T_1. As a result of this, Childs *et al.* [2] suggested a new HCS that ensures a fixed number of failures, known as Type-II HCS (TII−HCS). In this situation, the life test ends at time $T_2 = \max(Y_{r:n}, T)$, where $r \in (1, 2, \ldots, n)$, and $T \in (0, \infty)$. Chandrasekar *et al.* [3] enhanced these censored sampling strategies by establishing two extensions of this kind, known as generalized TI−HCS (GTI−HCS) and generalized TII−HCS (GTII−HCS). In the context of GTI−HCS, $s, r \in (1, 2, \ldots, n)$ and $T \in (0, \infty)$ are fixed so that $s < r$. The experiment ends at $Y_{s:n}$ if the sth failure happens after time T. Thus, a minimal number s of failures is ensured in GTI−HCS. In GTII−HCS, one fixes $r \in (1, 2, \ldots, n)$ and $T_1, T_2 \in (0, \infty)$ so that $T_1 < T_2$. The experiment ends at time T_1 if the rth failure happens before that time T_1. If the rth failure happens between T_1 and T_2, the experiment is ended at $Y_{r:n}$. Lastly, the experiment ends at time T_2 if the rth failure happens after time T_2.

Rapid technological advancement, consumer demand for consistent products, and competitive markets have all placed pressure on producers to supply consistent, quality products. It is extremely difficult to observe the failure time for complicated, high-reliability devices during life testing, such as lasers, optical fibers, semiconductors, metal fatigue, electric cables, and insulating materials. Due to this, accelerated life testing (ALT) or partial ALT (PALT) is recommended in the industrial sector to gather the essential failure data quickly enough to determine the failure's link to external stress factors. Test items in ALTs are only tested at accelerated circumstances, or pressures higher than normal, in order to cause an early failure. A physically acceptable statistical model then extrapolates the data gathered under such accelerated settings to predict the lifespan distribution under typical use conditions. This test might save a significant amount of money, time, labor, and resources. There are several ways to administer stress, such as step stress, progressive stress, and continuous stress (see [4]).

The PALT is the most appropriate test to use, especially when test items are administered under both normal and higher-than-normal stress levels [5–8]. It is also the most crucial technique to apply when calculating the acceleration factor and projecting the accelerated test results to the situation. PALT integrates both standard life and ALT. As a result, PALT makes sense for calculating the acceleration factor ($\beta > 1$), which is defined as “the ratio of the hazard rate at the accelerated condition to that at normal conditions.” PALT may be broadly classified into two categories: step-stress PALT and constant-stress PALT (CSPALT). In step-stress PALT, after a certain number of failures or at a predetermined period, the test conditions for the remaining objects in the experiment change from normal use to higher stress. In CSPALT, every group of test units is subjected to different usage and accelerated conditions, which is the primary focus of this work. CSPALTs were examined by several authors, such as [9,10] for the Pareto distribution, [11] inverse Weibull distribution, [12,13] Gompartz distribution, [14] Weibull distribution, [15] Kumaraswamy distribution, [16] Lomax distribution, [17] weighted Lomax distribution, and [18] linear exponential distribution. For more studies, refer to [19–23].

To fit different failure life time data, Burr [24] created 12 cumulative distribution functions (CDFs). A lot of attention has recently been paid to the Burr type-III distribution (BIIID) among others since it can accept various hazard lifetime data. It can also accurately approximate a wide range of popular distributions for fitting lifetime data, including the gamma, Weibull, and log-normal distributions. Many scientific domains have employed the BIIID. Examples of these include modeling forestry-related events [25], fracture roughness [26], actuarial literature [27], meteorological literature [28], reliability theory [29,30], and operational risk [31], among others. The probability density function (PDF) of the BIIID is given by:

$$f_1(y) = \vartheta\lambda y^{-(\lambda+1)}(1 + y^{-\lambda})^{-\vartheta-1}; \quad y > 0, \tag{1}$$

where $\vartheta, \lambda > 0$, are the shape parameters. The CDF of the BIIID is

$$F_1(y) = (1 + y^{-\lambda})^{-\vartheta}; \quad y > 0. \tag{2}$$

The hazard rate function (HF) of the BIIID is given by:

$$\tau_1(y) = \frac{\vartheta\lambda y^{-(\lambda+1)}(1 + y^{-\lambda})^{-\vartheta-1}}{1 - (1 + y^{-\lambda})^{-\vartheta}}. \tag{3}$$

Numerous scholars have examined inferences for the parameters of the BIIID. A Bayesian estimator for the BIIID based on double censoring was examined by Abd-Elfattah and Alharbey [32]. Kim and Kim [33] discussed the Bayesian and classical estimation methodologies of the BIIID parameters based on dual generalized order statistics. The BIIID’s estimate and prediction issues were addressed using TII censored and progressive Type II hybrid censored samples by Altindag *et al.* [34] and Gamchi *et al.* [35], respectively,. The unified hybrid censored sample was used by Panahi [36] to develop the statistical inference of a BIIID. Hassan *et al.* [37] addressed the estimation of the lifetime performance index for BIIID based on progressive censoring scheme. Dutta and Kayal [38] provided estimation and prediction for the BIIID using unified progressive hybrid censoring. Bayesian and non-Bayesian inferences of the BIIID under joint progressive TII censoring were examined by Hassen *et al.* [39].

So far, there have been no articles specifically focused on estimating parameters of the BIIID in CSPALT with GTII-HCS data. This article aims to address this gap, given the flexibility and versatility of the GTII-HCS, along with the widespread application of BIIID in modeling lifetime data. Consequently, this paper’s primary goals are as follows:

1) Using the GTII-HCS, the classical (maximum likelihood) estimation method will be utilized to derive the point estimators and asymptotic confidence intervals (Asy-CIs) for the parameters and acceleration factor in CSPALT model.
2) The Bayes procedure will be applied to obtain Bayesian estimators and the Bayesian credible intervals (BCIs) of the BIIID parameters. By assuming independent gamma priors for the parameters, we can compute Bayes estimates under different loss functions.
3) The Markov chain Monte Carlo (MCMC) technique will be employed, implementing Gibbs sampling within the Metropolis Hastings (MH) framework, to generate samples from the posterior distributions.
4) A simulation study will be conducted to assess the efficiency of the estimators based on several precision measures.
5) To examine two real datasets to show the applicability of the proposed estimators.

The format of the paper is as follows: Section 2 presents the model description along with the assumptions made. In Section 3, we discuss the maximum likelihood (ML) and Asy-CI estimators for the model parameters and acceleration factor. Section 4 focuses on the Bayesian and BCI estimators for the model parameters and the acceleration factor. A simulation study to evaluate the performance of these estimates for the specified model is conducted in Section 5. The analysis of actual data is detailed in Section 6, while Section 7 concludes with some key findings.

2 Model description and assumption

This section gives a description of the model and its assumptions.

2.1 Model description

Suppose that total test items n are divided into two groups. Under used settings, n_1 randomly selected test items from n total are included in the first group (G1). The remaining $n_2 = n - n_1$ test items in the second group (G2) are subjected to accelerated operating conditions. The items in each group are tested using a GTII-HCS. Let the lifetimes $y_{\ell i}$, $i = 1, \ldots, n_\ell$, $\ell = 1, 2$ denoted two GTII−HSC, where y_{1i} , $i = 1, \ldots, n_1$ be the lifetimes of items in G1 (normal use condition) and $y_{2:i}$, $i = 1, \ldots, n_2$ be the lifetimes of items in G2 (accelerated condition). The experiment ends at time $T_{\ell 1}$ if the r_ℓth failure occurs before time $T_{\ell 1}$. If the r_ℓth failure occurs between $T_{\ell 1}$ and $T_{\ell 2}$, the experiment ends at $y_{\ell r_\ell}$. Lastly, the experiment ends at time $T_{\ell 2}$ if the r_ℓth failure occurs after time $T_{\ell 2}$.

Suppose that $y_{1i} = (y_{11} < \ldots < y_{1n_1})$, $i = 1, 2, \ldots, n_1$, an ordered sample of size n_1, are tested under used condition. Under the GTII-HCS, one of the three observation categories can be obtained:

Case 1U: $y_{11} < \ldots < y_{1r_1} < \ldots < y_{1d_{11}} < T_{11}$ if $y_{1d_{11}} < T_{11}$,
Case 2U: $y_{11} < \ldots < y_{1d_{11}} < \ldots < y_{1r_1}$ if $T_{11} < y_{1r_1} < T_{12}$,
Case 3U: $y_{11} < \ldots < y_{1d_{12}} < \ldots < T_{12}$ if $y_{1r_1} > T_{12}$.

Note that d_{11} and d_{12} are, respectively, the failure numbers obtained at time T_{11} and T_{12}. Also, T_{11} and T_{12} are time points determined by the experimenter according to how the experiment should continue based on the information about the product. Also, the likelihood function in CSPALT, under GTII-HCS has the following form:

$$L_1 = \frac{n_1!}{(n_1 - D_1)!}\prod_{i=1}^{D_1} f_1(y_{1i})[1 - F_1(h_1)]^{n_1 - D_1},$$

where D_1 represents the total number of failures in the experiment up to time h_1 under normal use conditions, and its value is determined by:

$$(D_1, h_1) = \begin{cases} (d_{11}, T_{11}) & \text{case 1U} \\ (r_1, y_{1r_1}) & \text{case 2U} \\ (d_{12}, T_{12}) & \text{case 3U.} \end{cases}$$

Similarly, suppose that $y_{2i} = (y_{21} < \ldots < y_{2n_2})$, $i = 1, 2, \ldots, n_2$, an ordered sample of size n_2, are tested under accelerated condition. Under GTII-HCS, one of the three observation categories can be obtained:

Case 1A: $y_{21} < \ldots < y_{2r_2} < \ldots < y_{2d_{21}} < T_{21}$ if $y_{2d_{21}} < T_{21}$,
Case 2A: $y_{21} < \ldots < y_{2d_{21}} < \ldots < y_{2r_2}$ if $T_{21} < y_{2r_2} < T_{22}$,
Case 3A: $y_{22} < \ldots < y_{2d_{22}} < \ldots < T_{22}$ if $y_{2r_2} > T_{22}$.

Note that d_{21} and d_{22} are, respectively, the number of failures observed at time T_{21} and T_{22}. Also, T_{21} and T_{22} are time points determined by the experimenter according to how the experiment should continue based on the information about the product. Also, the likelihood function in CSPALT, under GTII-HCS has the following form:

$$L_2 = \frac{n_2!}{(n_2 - D_2)!}\prod_{i=1}^{D_2} f_2(y_{2i})[1 - F_2(h_2)]^{n_2 - D_2},$$

where D_2 is the number of total failures in the experiment up to time h_2, under accelerated condition, and its value is determined by:

$$(D_2, h_2) = \begin{cases} (d_{21}, T_{21}) & \text{case 1A} \\ (r_2, y_{2r_2}) & \text{case 2A} \\ (d_{22}, T_{22}) & \text{case 3A.} \end{cases}$$

2.2 Model assumptions

Under normal operating conditions, an item's lifetime is assumed to follow the BIIID, with PDF, CDF, and HF obtained by Eqs. (1)–(3). On the other hand, when evaluating an item in accelerated conditions, its HF is represented as $h_2(y) = \rho h_1(y)$, where the acceleration factor is $\rho > 1$. So, the HF, CDF, and PDF of the BIIID, for $y > 0$, ϑ, $\lambda > 0$, and $\rho > 1$, under the accelerated condition are as follows:

$$\tau_2(y) = \frac{\vartheta\rho\lambda y^{-(\lambda+1)}(1 + y^{-\lambda})^{-\vartheta-1}}{1 - (1 + y^{-\lambda})^{-\vartheta}},$$

$$F_2(y) = 1 - [1 - (1 + y^{-\lambda})^{-\vartheta}]^{\rho}, \quad y > 0, \tag{4}$$

and,

$$f_2(y) = \rho\lambda\vartheta y^{-\lambda-1}(1 + y^{-\lambda})^{-\vartheta-1}[1 - (1 + y^{-\lambda})^{-\vartheta}]^{\rho-1}, \quad y > 0. \tag{5}$$

3 ML estimators

This section provides the ML technique to generate the point estimators of the unknown parameters λ, ϑ, and acceleration factor ρ based on the GTII−HCS with CSPALT. Furthermore, the Asy-CIs of the model parameters and acceleration factor are determined.

Suppose that $y_{1i} = (y_{11} < \ldots < y_{1n_1})$, $i = 1, \ldots, n_1$ be n_1 GTII−HCS drawn from BIIID with PDF (1) and CDF (2) under normal usage conditions. Also, assume that $y_{2i} \equiv (y_{21} < \ldots < y_{2n_2})$, $i = 1, 2, \ldots, n_2$ be n_2 GTII−HCS drawn from BIIID with PDF(5) and CDF(4) under accelerated conditions. Hence, the likelihood function in CSPALT for the BIIID under the GTII-HCS is obtained by combining the likelihood functions L_1 and L_2 as follows:

$$L(y|\vartheta,\lambda,\rho)=\prod_{\ell=1}^{2}\frac{n_\ell!}{(n_\ell-D_\ell)!}\prod_{i=1}^{D_\ell}f_\ell(y_{\ell i})[1-F_\ell(h_\ell)]^{n_\ell-D_\ell},\qquad(6)$$

where d_ℓ, $\ell=1,\ 2$ is the number of total failures in the experiment up to time h_ℓ and their values are given by:

$$(D_\ell,h_\ell)=\begin{cases}(d_{\ell 1},T_{\ell 1}) & \text{case } 1\\ (r_\ell,y_{\ell r_\ell}) & \text{case } 2\\ (d_{\ell 2},T_{\ell 2}) & \text{case } 3.\end{cases}$$

Then, inserting Eqs. (1), (2), (4), (5) in (6) gives:

$$L(y|\zeta)=\vartheta^{D_1+D_2}\lambda^{D_1+D_2}\rho^{D_2}\prod_{\ell=1}^{2}\frac{n_\ell!}{(n_\ell-D_\ell)!}\prod_{i=1}^{D_\ell}y_{\ell i}^{-(\lambda+1)}(1+y_{\ell i}^{-\lambda})^{-\vartheta-1}$$
$$[1-(1+y_{\ell i}^{-\lambda})^{-\vartheta}]^{\rho^{\ell-1}-1}[1-(1+h_\ell^{-\lambda})^{-\vartheta}]^{\rho^{\ell-1}(n_\ell-D_\ell)},$$

Table 1: Points estimation by ML and Bayesian methods for parameters: $\lambda=0.5$, $\vartheta=0.6$, $\rho=0.6$

$(T_{11},T_{12}),(T_{21},T_{22})$	r/n	n		ML		SLF		LXF ($q=-1.25$)		LXF ($q=1.25$)	
				Abias	MSE	Abias	MSE	Abias	MSE	Abias	MSE
(0.2, 0.5), (5, 20)	0.6	20, 15	λ	0.0327	0.2650	0.0391	0.1132	0.0327	0.1067	0.0206	0.0956
			ϑ	0.0332	0.2585	0.0419	0.1427	0.0331	0.1364	0.0164	0.1258
			ρ	0.0580	0.3162	0.0715	0.2003	0.0580	0.1858	0.0326	0.1617
		50, 40	λ	0.0108	0.1454	0.0130	0.0589	0.0108	0.0577	0.0064	0.0556
			ϑ	0.0151	0.1626	0.0186	0.0752	0.0151	0.0736	0.0082	0.0708
			ρ	0.0228	0.1897	0.0279	0.1047	0.0228	0.1012	0.0130	0.0952
		100, 120	λ	0.0062	0.0883	0.0068	0.0447	0.0061	0.0444	0.0047	0.0438
			ϑ	0.0045	0.1097	0.0055	0.0606	0.0044	0.0603	0.0023	0.0598
			ρ	0.0079	0.1210	0.0091	0.0669	0.0079	0.0664	0.0053	0.0655
		200, 150	λ	0.0034	0.0670	0.0038	0.0329	0.0034	0.0328	0.0026	0.0326
			ϑ	0.0026	0.0840	0.0032	0.0441	0.0026	0.0440	0.0015	0.0438
			ρ	0.0011	0.0902	0.0019	0.0531	0.0011	0.0529	0.0004	0.0527
(0.2, 0.5), (5, 20)	0.8	20, 15	λ	0.0325	0.2492	0.0384	0.1108	0.0324	0.1049	0.0210	0.0948
			ϑ	0.0310	0.2553	0.0394	0.1284	0.0310	0.1226	0.0149	0.1131
			ρ	0.0497	0.3087	0.0625	0.1712	0.0497	0.1590	0.0258	0.1389
		50, 40	λ	0.0138	0.1363	0.0160	0.0516	0.0138	0.0597	0.0096	0.0574
			ϑ	0.0117	0.1602	0.0150	0.0747	0.0117	0.0752	0.0052	0.0728
			ρ	0.0261	0.1791	0.0270	0.1021	0.0260	0.1055	0.0164	0.0993
		100, 120	λ	0.0053	0.0778	0.0060	0.0446	0.0053	0.0444	0.0040	0.0439
			ϑ	0.0054	0.1048	0.0065	0.0592	0.0054	0.0589	0.0033	0.0583
			ρ	0.0119	0.1117	0.0132	0.0656	0.0119	0.0650	0.0093	0.0640
		200, 150	λ	0.0030	0.0599	0.0035	0.0311	0.0031	0.0310	0.0023	0.0309
			ϑ	0.0019	0.0764	0.0025	0.0427	0.0019	0.0426	0.0008	0.0424
			ρ	0.0044	0.0871	0.0052	0.0520	0.0044	0.0518	0.0028	0.0514
(1, 3), (14, 20)	0.8	20, 15	λ	0.0206	0.1769	0.0246	0.0846	0.0206	0.0816	0.0129	0.0764
			ϑ	0.0323	0.2028	0.0390	0.1132	0.0323	0.1085	0.0194	0.1007
			ρ	0.0504	0.2692	0.0619	0.1725	0.0504	0.1617	0.0289	0.1435
		50, 40	λ	0.0094	0.1049	0.0108	0.0470	0.0093	0.0462	0.0063	0.0449
			ϑ	0.0122	0.1267	0.0148	0.0651	0.0122	0.0639	0.0071	0.0619
			ρ	0.0228	0.1637	0.0270	0.0998	0.0227	0.0967	0.0143	0.0912
		100, 120	λ	0.0053	0.0648	0.0058	0.0338	0.0053	0.0336	0.0043	0.0332
			ϑ	0.0043	0.0893	0.0052	0.0481	0.0043	0.0478	0.0024	0.0474
			ρ	0.0054	0.0998	0.0066	0.0615	0.0054	0.0612	0.0031	0.0605
		200, 150	λ	0.0032	0.0474	0.0035	0.0256	0.0032	0.0255	0.0026	0.0254
			ϑ	0.0006	0.0635	0.0010	0.0359	0.0006	0.0359	0.0004	0.0357
			ρ	0.0040	0.0768	0.0047	0.0480	0.0040	0.0478	0.0025	0.0474

where $\zeta \equiv (\lambda, \vartheta, \rho)$. The natural logarithms of likelihood function is

$$l^{\bullet} = (D_1 + D_2)(\log \vartheta + \log \lambda) + D_2 \log \rho - (\lambda + 1)\left[\sum_{i=1}^{D_1} \log y_{1i} + \sum_{i=1}^{D_2} \log y_{2i}\right] - (\vartheta + 1)\left[\sum_{i=1}^{D_1} \log(1 + y_{1i}^{-\lambda}) + \sum_{i=1}^{D_2} \log(1 + y_{2i}^{-\lambda})\right] + (\rho - 1)\sum_{i=1}^{D_2} \log[\upsilon_1(y_{2i}, \lambda, \vartheta)] + (n_1 - D_1) \times \log[\upsilon_2(h_1, \lambda, \vartheta)] + (n_2 - D_2)\rho \log[\upsilon_3(h_2, \lambda, \vartheta)],$$

where, $\upsilon_1(y_{2i}, \lambda, \vartheta) = [1 - (1 + y_{2i}^{-\lambda})^{-\vartheta}]$, $\upsilon_2(h_1, \lambda, \vartheta) = [1 - (1 + h_1^{-\lambda})^{-\vartheta}]$, and $\upsilon_3(h_2, \lambda, \vartheta) = [1 - (1 + h_2^{-\lambda})^{-\vartheta}]$.

The ML estimators $\hat{\lambda}$, $\hat{\vartheta}$, and $\hat{\rho}$, of λ, ϑ, and ρ are produced by simultaneously solving the following normal equations:

$$\frac{\partial l^{\bullet}}{\partial \lambda} = \frac{D_1 + D_2}{\lambda} + \sum_{i=1}^{D_1} \frac{(\vartheta + 1)\log y_{1i}}{1 + y_{1i}^{\lambda}} + \sum_{i=1}^{D_2} \frac{(\vartheta + 1)\log y_{2i}}{1 + y_{2i}^{\lambda}} + \sum_{i=1}^{D_2} \frac{(\rho - 1)\upsilon'_{1(\lambda)}}{\upsilon_1(y_{2i}, \lambda, \vartheta)} - \sum_{i=1}^{D_1} \log y_{1i} - \sum_{i=1}^{D_2} \log y_{2i} + \frac{(n_1 - D_1)\upsilon'_{2(\lambda)}}{\upsilon_2(h_1, \lambda, \vartheta)} + \frac{(n_2 - D_2)\upsilon'_{3(\lambda)}}{\upsilon_3(h_2, \lambda, \vartheta)} = 0, \tag{7}$$

Table 2: Intervals estimation by ML and Bayesian methods for parameters: $\lambda = 0.5$, $\vartheta = 0.6$, $\rho = 0.6$

$(T_{11}, T_{12}), (T_{21}, T_{22})$	r/n	N		ML		SLF		LXF ($q = -1.25$)		LXF ($q = 1.25$)	
				LACI	CP	LCCI	CP	LCCI	CP	LCCI	CP
(0.2, 0.5), (5, 20)	0.6	20, 15	λ	0.3986	94.80%	0.4168	94.80%	0.3984	94.80%	0.3660	94.10%
			ϑ	0.5190	95.10%	0.5351	94.10%	0.5190	94.10%	0.4890	95.10%
			ρ	0.6926	94.59%	0.7336	94.80%	0.6922	94.60%	0.6212	95.00%
		50, 40	λ	0.2223	94.99%	0.2253	94.90%	0.2222	95.00%	0.2166	95.00%
			ϑ	0.2825	96.00%	0.2859	95.29%	0.2823	95.60%	0.2758	95.48%
			ρ	0.3871	95.70%	0.3959	95.50%	0.3869	95.60%	0.3700	95.40%
		100, 120	λ	0.1725	95.40%	0.1733	95.50%	0.1725	95.40%	0.1709	95.40%
			ϑ	0.2359	96.69%	0.2365	95.47%	0.2358	95.70%	0.2344	95.70%
			ρ	0.2587	96.30%	0.2599	95.83%	0.2586	95.83%	0.2560	95.50%
		200, 150	λ	0.1280	96.09%	0.1283	95.61%	0.1280	95.60%	0.1274	95.60%
			ϑ	0.1724	96.79%	0.1727	95.80%	0.1723	95.80%	0.1716	95.80%
			ρ	0.2077	96.52%	0.2081	96.30%	0.2076	95.90%	0.2066	95.90%
(0.2, 0.5), (5, 20)	0.8	20, 15	λ	0.3916	94.70%	0.4074	94.79%	0.3915	94.70%	0.3627	95.70%
			ϑ	0.4655	95.10%	0.4793	95.00%	0.4652	95.10%	0.4398	94.70%
			ρ	0.5928	95.40%	0.6252	94.80%	0.5925	94.40%	0.5355	94.90%
		50, 40	λ	0.2276	94.99%	0.2306	94.90%	0.2277	95.00%	0.2221	95.80%
			ϑ	0.2917	95.40%	0.2851	95.20%	0.2915	95.40%	0.2849	95.20%
			ρ	0.4010	95.59%	0.3900	94.90%	0.4010	94.60%	0.3840	95.10%
		100, 120	λ	0.1728	95.19%	0.1734	95.39%	0.1728	95.60%	0.1715	95.90%
			ϑ	0.2300	95.95%	0.2307	95.50%	0.2298	95.70%	0.2281	95.40%
			ρ	0.2507	96.69%	0.2519	95.68%	0.2507	94.70%	0.2483	95.50%
		200, 150	λ	0.1211	95.89%	0.1214	95.90%	0.1211	95.90%	0.1207	96.90%
			ϑ	0.1670	96.60%	0.1673	95.70%	0.1670	95.90%	0.1663	95.60%
			ρ	0.2025	96.99%	0.2030	95.80%	0.2024	95.00%	0.2013	95.90%
(1, 3), (14, 20)	0.8	20, 15	λ	0.3099	94.20%	0.3176	94.80%	0.3098	94.60%	0.2953	94.10%
			ϑ	0.4066	95.50%	0.4167	95.20%	0.4065	95.50%	0.3877	95.60%
			ρ	0.6029	94.99%	0.6314	94.50%	0.6026	95.00%	0.5512	95.30%
		50, 40	λ	0.1776	94.50%	0.1794	95.25%	0.1776	95.50%	0.1743	95.40%
			ϑ	0.2462	95.70%	0.2486	95.70%	0.2460	95.70%	0.2412	95.70%
			ρ	0.3685	95.40%	0.3768	95.40%	0.3685	95.40%	0.3533	95.50%
		100, 120	λ	0.1300	95.40%	0.1304	95.40%	0.1300	95.74%	0.1293	95.80%
			ϑ	0.1868	95.79%	0.1874	95.80%	0.1868	95.90%	0.1856	95.80%
			ρ	0.2390	95.95%	0.2399	95.90%	0.2389	95.90%	0.2369	95.90%
		200, 150	λ	0.0994	96.40%	0.0995	95.90%	0.0994	96.40%	0.0990	96.40%
			ϑ	0.1406	96.79%	0.1409	95.97%	0.1407	96.80%	0.1402	96.80%
			ρ	0.1868	96.20%	0.1872	96.30%	0.1867	96.20%	0.1858	96.10%

$$\frac{\partial l^{\bullet}}{\partial \vartheta} = \frac{D_1 + D_2}{\vartheta} - \sum_{i=1}^{D_1} \log(1 + y_{1i}^{-\lambda}) - \sum_{i=1}^{D_2} \log(1 + y_{2i}^{-\lambda}) + \sum_{i=1}^{D_2} \frac{(\rho - 1)\upsilon'_{1(\vartheta)}}{\upsilon_1(y_{2i}, \lambda, \vartheta)} + \frac{(n_1 - D_1)\upsilon'_{2(\vartheta)}}{\upsilon_2(h_1, \lambda, \vartheta)} + \frac{(n_2 - D_2)\rho\upsilon'_{3(\vartheta)}}{\upsilon_3(h_2, \lambda, \vartheta)} = 0, \tag{8}$$

$$\frac{\partial l^{\bullet}}{\partial \rho} = \frac{D_2}{\rho} + \sum_{i=1}^{D_2} \log[\upsilon_1(y_{2i}, \lambda, \vartheta)] + (n_2 - D_2)\log[\upsilon_3(h_2, \lambda, \vartheta)] = 0, \tag{9}$$

where

$$\upsilon'_{1(\lambda)} = \frac{\partial \upsilon_1(y_{2i}, \lambda, \vartheta)}{\partial \lambda} = -\vartheta(1 + y_{2i}^{-\lambda})^{-\vartheta-1} y_{2i}^{-\lambda} \log y_{2i},$$

$$\upsilon'_{2(\lambda)} = \frac{\partial \upsilon_2(h_1, \lambda, \vartheta)}{\partial \lambda} = -\vartheta(1 + h_1^{-\lambda})^{-\vartheta-1} h_1^{-\lambda} \log h_1,$$

$$\upsilon'_{3(\lambda)} = \frac{\partial \upsilon_3(h_2, \lambda, \vartheta)}{\partial \lambda} = -\vartheta(1 + h_2^{-\lambda})^{-\vartheta-1} h_2^{-\lambda} \log h_2,$$

$$\upsilon'_{1(\vartheta)} = \frac{\partial \upsilon_1(y_{2i}, \lambda, \vartheta)}{\partial \vartheta} = (1 + y_{2i}^{-\lambda})^{-\vartheta} \log(1 + y_{2i}^{-\lambda}),$$

Table 3: Points estimation by ML and Bayesian methods for parameters: $\lambda = 2,\ \vartheta = 0.6,\ \rho = 0.6$

$(T_{11}, T_{12}), (T_{21}, T_{22})$	**r/n**	**n**		**ML**		**SLF**		**LXF ($q = -1.25$)**		**LXF ($q = 1.25$)**	
				Abias	**MSE**	**Abias**	**MSE**	**Abias**	**MSE**	**Abias**	**MSE**
(0.8, 1.8), (2.2, 5)	0.6	20, 15	λ	0.0114	0.5361	0.0190	0.3146	0.0158	0.3035	0.0669	0.2965
			ϑ	0.0427	0.1951	0.0485	0.1201	0.0425	0.1151	0.0309	0.1064
			ρ	0.0487	0.2431	0.0596	0.1779	0.0486	0.1658	0.0281	0.1461
		50, 40	λ	0.0106	0.3677	0.0014	0.2248	0.0150	0.2221	0.0418	0.2203
			ϑ	0.0204	0.1246	0.0228	0.0774	0.0202	0.0759	0.0153	0.0733
			ρ	0.0251	0.1545	0.0293	0.0970	0.0251	0.0942	0.0170	0.0894
		100, 120	λ	0.0032	0.2474	0.0005	0.1296	0.0032	0.1294	0.0084	0.1293
			ϑ	0.0067	0.0881	0.0075	0.0487	0.0067	0.0484	0.0050	0.0480
			ρ	0.0081	0.0967	0.0093	0.0572	0.0081	0.0568	0.0059	0.0561
		200, 150	λ	0.0028	0.2005	0.0004	0.0958	0.0028	0.0957	0.0056	0.0957
			ϑ	0.0028	0.0665	0.0032	0.0353	0.0028	0.0352	0.0019	0.0350
			ρ	0.0031	0.0795	0.0037	0.0473	0.0030	0.0471	0.0016	0.0468
(0.8, 1.8), (2.2, 5)	0.8	20, 15	λ	0.0053	0.4384	0.0257	0.3066	0.0068	0.2958	0.0643	0.2884
			ϑ	0.0417	0.1820	0.0476	0.1204	0.0417	0.1159	0.0303	0.1080
			ρ	0.0450	0.2352	0.0605	0.1623	0.0502	0.1525	0.0308	0.1360
		50, 40	λ	0.0046	0.3403	0.0188	0.2028	0.0061	0.1989	0.0188	0.1945
			ϑ	0.0115	0.1126	0.0137	0.0683	0.0114	0.0674	0.0070	0.0657
			ρ	0.0244	0.1541	0.0283	0.0916	0.0244	0.0889	0.0167	0.0841
		100, 120	λ	0.0011	0.2256	0.0038	0.1197	0.0012	0.1192	0.0040	0.1186
			ϑ	0.0066	0.0848	0.0080	0.0462	0.0072	0.0460	0.0056	0.0455
			ρ	0.0077	0.0948	0.0089	0.0570	0.0078	0.0565	0.0056	0.0558
		200, 150	λ	0.0010	0.1816	0.0032	0.0915	0.0011	0.0914	0.0007	0.0912
			ϑ	0.0022	0.0630	0.0034	0.0341	0.0029	0.0340	0.0021	0.0339
			ρ	0.0029	0.0779	0.0051	0.0447	0.0044	0.0445	0.0030	0.0442
(1.2, 3), (2.2, 5)	0.8	20, 15	λ	0.0191	0.3993	0.0034	0.2487	0.0185	0.2409	0.0608	0.2362
			ϑ	0.0376	0.1696	0.0422	0.1031	0.0376	0.0987	0.0286	0.0909
			ρ	0.0299	0.2053	0.0378	0.1322	0.0298	0.1230	0.0147	0.1076
		50, 40	λ	0.0086	0.3219	0.0206	0.1952	0.0088	0.1917	0.0145	0.1876
			ϑ	0.0155	0.1101	0.0176	0.0627	0.0154	0.0616	0.0112	0.0596
			ρ	0.0198	0.1463	0.0234	0.0832	0.0198	0.0810	0.0128	0.0771
		100, 120	λ	0.0021	0.2282	0.0047	0.1162	0.0022	0.1160	0.0028	0.1160
			ϑ	0.0064	0.0798	0.0072	0.0447	0.0064	0.0444	0.0049	0.0440
			ρ	0.0068	0.0890	0.0078	0.0537	0.0067	0.0533	0.0046	0.0526
		200, 150	λ	0.0042	0.1792	0.0028	0.0892	0.0041	0.0892	0.0067	0.0892
			ϑ	0.0032	0.0597	0.0036	0.0335	0.0032	0.0334	0.0024	0.0332
			ρ	0.0030	0.0734	0.0037	0.0449	0.0030	0.0448	0.0016	0.0444

$$\upsilon'_{2(\vartheta)} = \frac{\partial \upsilon_2(h_1, \lambda, \vartheta)}{\partial \vartheta} = (1 + h_1^{-\lambda})^{-\vartheta} \log(1 + h_1^{-\lambda}),$$
$$\upsilon'_{3(\vartheta)} = \frac{\partial \upsilon_3(h_2, \lambda, \vartheta)}{\partial \vartheta} = (1 + h_2^{-\lambda})^{-\vartheta} \log(1 + h_2^{-\lambda}).$$

One may determine the ML estimator of acceleration factor ρ as a function of the parameters λ and ϑ by using (9) and for fixed value, as seen below.

$$\hat{\gamma}(\rho) = \frac{-D_2}{\sum_{i=1}^{D_2} \log[\upsilon_1(y_{2i}, \lambda, \vartheta)] + (n_2 - D_2)\log[\upsilon_3(h_2, \lambda, \vartheta)]}. \quad (10)$$

Then inserting Eq. (10) in (7) and (8) and solving them numerically, we get the ML estimators $\hat{\lambda}$ and $\hat{\vartheta}$. It is important to note that these equations cannot be solved analytically. Some numerical methods, such as the Newton-Raphson technique, can be used to get the necessary estimate. The ML estimator $\hat{\rho}$ of ρ is then produced by inserting $\hat{\lambda}$ and $\hat{\vartheta}$ in Eq. (10).

It might be more useful to find a range of values that, with a given probability, include the unknown parameters rather than point estimates for them. Interval estimates are the name given to these ranges. The Asy-CIs of the unknown parameters $\zeta = (\lambda, \vartheta, \rho)^T$ are constructed here by employing the asymptotic characteristics of the ML estimators. Using the large sample theory, we can determine that the asymptotic distribution of the ML estimators $\hat{\zeta} = (\hat{\lambda}, \hat{\vartheta}, \hat{\rho})^T$ is a normal distribution with a variance–covariance matrix (VC-M) of $I^{-1}(\zeta)$ and a mean of ζ. To estimate $I^{-1}(\zeta)$, where the

Table 4: Intervals estimation by ML and Bayesian methods for parameters: $\lambda = 2,\ \vartheta = 0.6,\ \rho = 0.6$

$(T_{11}, T_{12}), (T_{21}, T_{22})$	r/n	n		ML		SLF		LXF ($q = -1.25$)		LXF ($q = 1.25$)	
				LACI	CP	LCCI	CP	LCCI	CP	LCCI	CP
(0.8, 1.8), (2.2, 5)	0.6	20, 15	λ	1.1878	94.19%	1.2314	94.40%	1.1895	94.30%	1.1330	94.20%
			ϑ	0.4194	94.79%	0.4308	94.70%	0.4196	94.80%	0.3993	94.90%
			ρ	0.6219	95.00%	0.6576	94.60%	0.6216	94.60%	0.5622	94.60%
		50, 40	λ	0.8676	94.40%	0.8815	95.40%	0.8692	95.40%	0.8485	95.40%
			ϑ	0.2868	95.30%	0.2902	95.20%	0.2870	95.30%	0.2810	95.40%
			ρ	0.3563	95.20%	0.3625	95.20%	0.3561	95.20%	0.3441	95.20%
		100, 120	λ	0.5076	95.40%	0.5081	95.70%	0.5074	95.90%	0.5061	95.83%
			ϑ	0.1883	95.90%	0.1887	96.00%	0.1882	95.90%	0.1871	95.90%
			ρ	0.2207	95.50%	0.2215	95.80%	0.2206	95.39%	0.2189	95.91%
		200, 150	λ	0.3754	95.89%	0.3757	95.90%	0.3753	96.39%	0.3745	95.97%
			ϑ	0.1376	95.99%	0.1378	96.29%	0.1376	96.49%	0.1371	96.70%
			ρ	0.1844	96.79%	0.1850	96.19%	0.1844	95.80%	0.1833	96.80%
(0.8, 1.8), (2.2, 5)	0.8	20, 15	λ	1.1604	94.99%	1.1982	94.90%	1.1601	94.90%	1.1026	94.20%
			ϑ	0.4243	94.99%	0.4336	95.10%	0.4241	95.00%	0.4066	95.30%
			ρ	0.5651	95.20%	0.5906	95.10%	0.5649	95.20%	0.5195	95.37%
		50, 40	λ	0.7799	95.40%	0.7918	95.40%	0.7796	95.40%	0.7591	95.00%
			ϑ	0.2603	95.50%	0.2626	95.40%	0.2603	95.50%	0.2560	95.50%
			ρ	0.3354	95.80%	0.3417	95.80%	0.3354	95.80%	0.3235	95.70%
		100, 120	λ	0.4679	95.59%	0.4692	95.63%	0.4677	95.60%	0.4650	95.40%
			ϑ	0.1781	95.95%	0.1786	95.63%	0.1781	95.73%	0.1771	95.83%
			ρ	0.2197	95.95%	0.2207	95.94%	0.2197	95.94%	0.2177	95.84%
		200, 150	λ	0.3577	96.99%	0.3588	96.00%	0.3584	95.90%	0.3575	96.10%
			ϑ	0.1330	96.29%	0.1332	95.90%	0.1330	95.90%	0.1326	95.90%
			ρ	0.1737	96.40%	0.1741	96.50%	0.1736	96.30%	0.1728	95.90%
(1.2, 3), (2.2, 5)	0.8	20, 15	λ	0.9397	94.39%	0.9752	94.40%	0.9418	94.30%	0.8952	94.60%
			ϑ	0.3580	94.89%	0.3688	95.00%	0.3580	94.90%	0.3383	94.80%
			ρ	0.4680	95.70%	0.4968	96.00%	0.4679	95.70%	0.4179	95.30%
		50, 40	λ	0.7508	96.10%	0.7612	96.10%	0.7509	96.10%	0.7334	96.00%
			ϑ	0.2337	94.99%	0.2359	95.20%	0.2338	95.00%	0.2297	95.20%
			ρ	0.3080	95.20%	0.3131	95.20%	0.3078	95.20%	0.2981	95.20%
		100, 120	λ	0.4552	95.50%	0.4554	95.30%	0.4550	95.40%	0.4547	95.30%
			ϑ	0.1724	95.60%	0.1729	95.60%	0.1724	95.60%	0.1714	95.70%
			ρ	0.2072	95.30%	0.2083	95.00%	0.2073	95.30%	0.2054	95.20%
		200, 150	λ	0.3493	95.40%	0.3496	95.50%	0.3493	95.40%	0.3489	95.20%
			ϑ	0.1303	95.40%	0.1304	95.50%	0.1303	95.40%	0.1299	95.40%
			ρ	0.1752	96.20%	0.1757	96.20%	0.1751	96.20%	0.1741	96.30%

asymptotic VC-M $I^{-1}(\hat{\zeta})$ is employed, which may be generated by inverting the observed Fisher information matrix. The asymptotic VC-M in this instance is as follows:

$$I^{-1}(\hat{\zeta}) = \begin{pmatrix} \frac{-\partial^2 l^{\bullet}}{\partial\lambda^2} & \frac{-\partial^2 l^{\bullet}}{\partial\lambda\partial\vartheta} & \frac{-\partial^2 l^{\bullet}}{\partial\lambda\partial\rho} \\ & \frac{-\partial^2 l^{\bullet}}{\partial\vartheta^2} & \frac{-\partial^2 l^{\bullet}}{\partial\vartheta\partial\rho} \\ & & \frac{-\partial^2 l^{\bullet}}{\partial\rho^2} \end{pmatrix}^{-1}_{\hat{\zeta}=(\hat{\lambda},\hat{\vartheta},\hat{\delta})} = \begin{pmatrix} \hat{\sigma}_{11}^2 & \hat{\sigma}_{12}^2 & \hat{\sigma}_{13}^2 \\ & \hat{\sigma}_{22}^2 & \hat{\sigma}_{23}^2 \\ & & \hat{\sigma}_{33}^2 \end{pmatrix}.$$

In Appendix 1, the second partial derivatives are provided. Asymptotic normality of the ML estimators allows for the construction of the $100(1-\delta)\%$ Asy-CIs of λ, ϑ, and ρ, respectively.

$$\hat{\lambda} \pm z_{\delta/2}\hat{\sigma}_{11}, \quad \hat{\vartheta} \pm z_{\delta/2}\hat{\sigma}_{22}, \quad \hat{\rho} \pm z_{\delta/2}\hat{\sigma}_{33},$$

where $z_{\delta/2}$ is the upper $\delta/2$ th percentile point of the standard normal distribution.

4 Bayesian estimation

In this part, the Bayes estimators for the unknown parameters λ, ϑ, and the acceleration factor ρ of the BIIID are presented. It is assumed that the independent parameter ρ

Table 5: Points estimation by ML and Bayesian methods for parameters: $\lambda = 2,\ \vartheta = 1.6,\ \rho = 1.5$

$(T_{11}, T_{12}), (T_{21}, T_{22})$	r/n	n		ML		SLF		LXF ($q = -1.25$)		LXF ($q = 1.25$)	
				Abias	MSE	Abias	MSE	Abias	MSE	Abias	MSE
(1.4, 3.5), (1.5, 4)	0.6	20, 15	λ	0.0209	0.3769	0.0117	0.2413	0.0210	0.1400	0.0389	0.1435
			ϑ	0.0317	0.3435	0.0399	0.1551	0.0316	0.1465	0.0157	0.1336
			ρ	0.0401	0.4721	0.0028	0.1766	0.0401	0.1763	0.0632	0.1833
		50, 40	λ	0.0050	0.2487	0.0021	0.1255	0.0049	0.1244	0.0174	0.1239
			ϑ	0.0088	0.2070	0.0137	0.1053	0.0086	0.1035	0.0014	0.1010
			ρ	0.0114	0.3284	0.0024	0.1758	0.0116	0.1747	0.0297	0.1716
		100, 120	λ	0.0043	0.1692	0.0020	0.1023	0.0041	0.1021	0.0084	0.1021
			ϑ	0.0032	0.1532	0.0051	0.0910	0.0032	0.0906	0.0004	0.0899
			ρ	0.0032	0.2317	0.0024	0.1205	0.0034	0.1200	0.0013	0.1191
		200, 150	λ	0.0022	0.1319	0.0010	0.0780	0.0022	0.0780	0.0045	0.0780
			ϑ	0.0028	0.1134	0.0039	0.0690	0.0028	0.0689	0.0003	0.0687
			ρ	0.0031	0.1957	0.0024	0.0939	0.0031	0.0938	0.0007	0.0937
(1.4, 3.5), (1.5, 4)	0.8	20, 15	λ	0.0104	0.4120	0.0272	0.2008	0.0101	0.1935	0.0231	0.1859
			ϑ	0.0382	0.3738	0.0535	0.1499	0.0386	0.1887	0.0101	0.1738
			ρ	0.0233	0.5195	0.0039	0.1259	0.0221	0.2486	0.0709	0.2445
		50, 40	λ	0.0091	0.2390	0.0226	0.1234	0.0091	0.1312	0.0081	0.1275
			ϑ	0.0138	0.2080	0.0205	0.1019	0.0138	0.1171	0.0007	0.1140
			ρ	0.0066	0.3260	0.0037	0.1162	0.0064	0.1979	0.0323	0.1949
		100, 120	λ	0.0046	0.1510	0.0066	0.0977	0.0045	0.0974	0.0043	0.0969
			ϑ	0.0003	0.1465	0.0021	0.0896	0.0031	0.0894	0.0006	0.0892
			ρ	0.0033	0.2368	0.0036	0.1124	0.0034	0.1236	0.0020	0.1231
		200, 150	λ	0.0001	0.1101	0.0010	0.0726	0.0009	0.0726	0.0022	0.0724
			ϑ	0.0002	0.1055	0.0010	0.0650	0.0019	0.0649	0.0004	0.0649
			ρ	0.0004	0.1881	0.0021	0.0937	0.0007	0.0935	0.0018	0.0932
(1.8, 4), (1.9, 4.5)	0.8	20, 15	λ	0.0047	0.3412	0.0045	0.1337	0.0049	0.1303	0.0230	0.1293
			ϑ	0.0228	0.3294	0.0313	0.1415	0.0229	0.1344	0.0069	0.1244
			ρ	0.0471	0.4379	0.0336	0.1176	0.0464	0.1776	0.0703	0.1879
		50, 40	λ	0.0007	0.2282	0.0045	0.1028	0.0046	0.1018	0.0112	0.1013
			ϑ	0.0123	0.2068	0.0168	0.1032	0.0122	0.1010	0.0033	0.0979
			ρ	0.0200	0.3188	0.0118	0.1052	0.0199	0.1503	0.0358	0.1507
		100, 120	λ	0.0005	0.1461	0.0029	0.0915	0.0042	0.0915	0.0086	0.0914
			ϑ	0.0020	0.1457	0.0038	0.0844	0.0020	0.0841	0.0015	0.0836
			ρ	0.0077	0.2141	0.0053	0.1014	0.0077	0.1139	0.0124	0.1138
		200, 150	λ	0.0001	0.1012	0.0023	0.0695	0.0013	0.0695	0.0008	0.0694
			ϑ	0.0009	0.1006	0.0017	0.0630	0.0010	0.0630	0.0013	0.0630
			ρ	0.0055	0.1784	0.0041	0.0893	0.0054	0.0894	0.0080	0.0895

has a truncated gamma distribution $\rho - 1\sim$ gamma (a_3, b_3), and that the random variables λ and ϑ have independent gamma prior distributions, $\lambda\sim$ Gamma $(a_1,\ b_1)$, and $\vartheta\sim$ Gamma $(a_2,\ b_2)$. The joint prior of parameters and acceleration factor is as follows:

$$\pi(\zeta) \propto \lambda^{a_1-1}\vartheta^{a_2-1}(\rho-1)^{a_3-1}e^{-(b_1\lambda+b_2\vartheta+b_3(\rho-1))}, \qquad (11)$$

where, $a_1,\ a_2,\ a_3,\ b_1,\ b_2$, and b_3 are the hyper-parameters. Interestingly, the gamma prior density for the acceleration factor ρ was first used by DeGroot and Goel [40]. In the Bayesian analysis, independent gamma priors have been used because of the gamma distribution's remarkable flexibility (see, for instance, Dey *et al.* [41] and Kundu and Howlader [42]). The likelihood function (6) and the joint prior distribution (11) are used to get the joint posterior distribution of $\lambda,\ \vartheta$, and ρ using Bayes theorem.

$$\begin{aligned}\pi(\zeta|\underset{\rightarrow}{y}) = {} & M\lambda^{a_1+D_1+D_2-1}\vartheta^{a_2+D_1+D_2-1}(\rho-1)^{a_3+D_2-1}\\ & \times e^{-(b_1\lambda+b_2\vartheta+b_3(\rho-1))}\prod_{i=1}^{D_1}y_{1i}^{-(\lambda+1)}(1+y_{1i}^{-\lambda})^{-\vartheta-1}\\ & \times\prod_{i=1}^{D_2}y_{2i}^{-(\lambda+1)}(1+y_{2i}^{-\lambda})^{-\vartheta-1}[1\\ & -(1+y_{1i}^{-\lambda})^{-\vartheta}][1-(1+y_{2i}^{-\lambda})^{-\vartheta}]^{(\rho-1)-1}\\ & \times[1-(1+h_1^{-\lambda})^{-\vartheta}]^{(n_1-D_1)}[1-(1+h_2^{-\lambda})^{-\vartheta}]^{(\rho-1)(n_2-D_2)},\end{aligned} \qquad (12)$$

where,

Table 6: Intervals estimation by ML and Bayesian methods for parameters: $\lambda = 2,\ \vartheta = 1.6,\ \rho = 1.5$

$(T_{11}, T_{12}), (T_{21}, T_{22})$	r/n	n		ML		SLF		LXF (q = −1.25)		LXF (q = 1.25)	
				LACI	CP	LCCI	CP	LCCI	CP	LCCI	CP
(1.4, 3.5), (1.5, 4)	0.6	20, 15	λ	0.5428	93.59%	0.5521	93.50%	0.5427	93.60%	0.5419	93.80%
			ϑ	0.5613	93.09%	0.5877	93.10%	0.5612	93.10%	0.5204	92.60%
			ρ	0.6734	92.09%	0.6840	92.80%	0.6731	92.10%	0.6748	92.10%
		50, 40	λ	0.4875	93.89%	0.4921	94.20%	0.4873	93.90%	0.4810	94.70%
			ϑ	0.4036	93.49%	0.4097	93.20%	0.4044	93.40%	0.3962	93.60%
			ρ	0.6833	94.59%	0.6796	95.00%	0.6684	94.60%	0.6629	95.00%
		100, 120	λ	0.3999	94.59%	0.4009	94.70%	0.4003	94.60%	0.3991	95.00%
			ϑ	0.3551	95.10%	0.3562	95.30%	0.3550	95.10%	0.3525	95.20%
			ρ	0.4701	95.50%	0.4722	95.50%	0.4704	95.50%	0.4671	95.30%
		200, 150	λ	0.3058	95.29%	0.3059	95.30%	0.3057	94.93%	0.3054	94.30%
			ϑ	0.2701	95.89%	0.2703	95.90%	0.2701	95.90%	0.2696	94.90%
			ρ	0.3678	95.95%	0.3682	95.60%	0.3676	95.95%	0.3666	95.40%
(1.4, 3.5), (1.5, 4)	0.8	20, 15	λ	0.7570	94.19%	0.7801	94.50%	0.7577	94.20%	0.7236	94.70%
			ϑ	0.7225	94.89%	0.7505	94.80%	0.7242	94.80%	0.6803	94.60%
			ρ	0.9618	94.29%	1.0144	94.40%	0.9711	94.50%	0.9176	94.20%
		50, 40	λ	0.5113	95.70%	0.5179	95.80%	0.5114	95.70%	0.5001	95.80%
			ϑ	0.4562	94.69%	0.4613	94.80%	0.4559	94.70%	0.4470	94.50%
			ρ	0.7755	94.59%	0.7896	94.50%	0.7757	94.60%	0.7537	94.60%
		100, 120	λ	0.3815	94.69%	0.3823	94.70%	0.3815	94.70%	0.3801	94.60%
			ϑ	0.3509	95.10%	0.3513	95.50%	0.3507	95.10%	0.3496	95.20%
			ρ	0.4846	94.89%	0.4853	94.70%	0.4844	94.90%	0.4827	95.00%
		200, 150	λ	0.2847	94.49%	0.2849	94.50%	0.2846	94.50%	0.2840	94.50%
			ϑ	0.2543	96.10%	0.2547	96.10%	0.2545	96.10%	0.2541	96.00%
			ρ	0.3658	95.20%	0.3673	95.10%	0.3667	95.20%	0.3655	95.20%
(1.8, 4), (1.9, 4.5)	0.8	20, 15	λ	0.5111	91.47%	0.5241	91.80%	0.5105	91.50%	0.4989	91.80%
			ϑ	0.5198	92.47%	0.5413	92.90%	0.5192	92.50%	0.4870	92.10%
			ρ	0.6695	93.07%	0.6785	93.20%	0.6722	93.10%	0.6834	93.10%
		50, 40	λ	0.3993	93.19%	0.4027	93.40%	0.3991	93.20%	0.3950	93.40%
			ϑ	0.3936	92.79%	0.3992	92.80%	0.3934	92.70%	0.3839	92.30%
			ρ	0.5848	92.99%	0.5924	93.20%	0.5844	93.00%	0.5742	93.20%
		100, 120	λ	0.3583	94.59%	0.3589	94.50%	0.3582	94.60%	0.3570	94.70%
			ϑ	0.3298	94.69%	0.3307	94.70%	0.3297	94.70%	0.3279	94.50%
			ρ	0.4460	94.39%	0.4472	94.40%	0.4458	94.40%	0.4435	94.60%
		200, 150	λ	0.2725	94.79%	0.2726	94.80%	0.2724	94.80%	0.2721	94.70%
			ϑ	0.2472	94.89%	0.2473	94.90%	0.2471	94.90%	0.2468	94.80%
			ρ	0.3498	94.59%	0.3500	94.50%	0.3499	94.70%	0.3497	94.70%

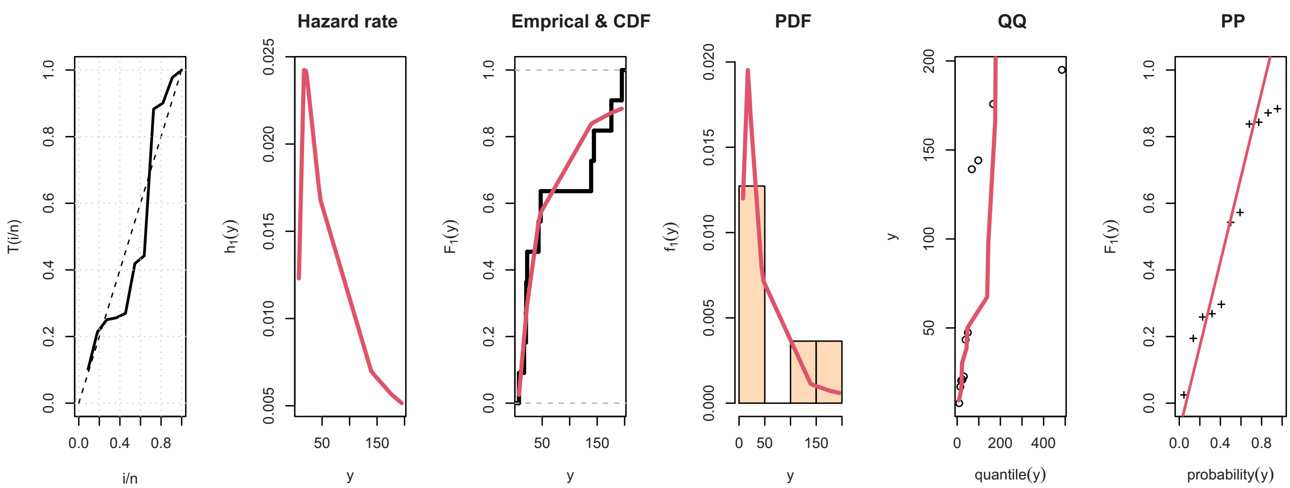

Figure 1: Some fitting plots of the BIIID for oil breakdown data with 30 kV.

$$M^{-1} = \int_0^\infty\int_0^\infty\int_0^\infty \pi(\zeta|\underline{y})\mathrm{d}\lambda\mathrm{d}\vartheta\mathrm{d}\rho.$$

According to the Bayes method, selecting a loss function that matches each potential estimator will help determine which estimator is the best. In this case, estimates are obtained for two distinct categories of loss functions: symmetric and asymmetric loss functions. The first kind will be represented by the squared error loss function (SLF), and the second type will be illustrated by the use of the linear exponential loss function (LXF). The SLF is inappropriate when either an overestimation or an underestimating takes place. As an alternate choice to estimate the parameters in this situation, LXF can be used. When there is more substantial overestimation than underestimation, and vice versa, the LXF is helpful. Bayes estimates of the function $U(\zeta) = U(\lambda, \vartheta, \rho)$, under SLF and LXF, are given, respectively, by:

$$\tilde{U}_{\mathrm{SLF}}(\zeta) = E(U(\zeta)),$$

and

$$\tilde{U}_{\mathrm{LXF}}(\zeta) = \frac{-1}{q}\ln\left[E(e^{-qU(\zeta)})\right], \quad q \neq 0,$$

where the parameter q represents the sign that indicates the direction of asymmetry. The integral given by Eq. (12) cannot generally be obtained in a closed form. Here, the MCMC method is used to generate samples from the posterior distributions, and the Bayesian estimators for each parameter and acceleration factor are then computed. Making a decision from several MCMC plans that are offered might be difficult. Notable subclasses of MCMC

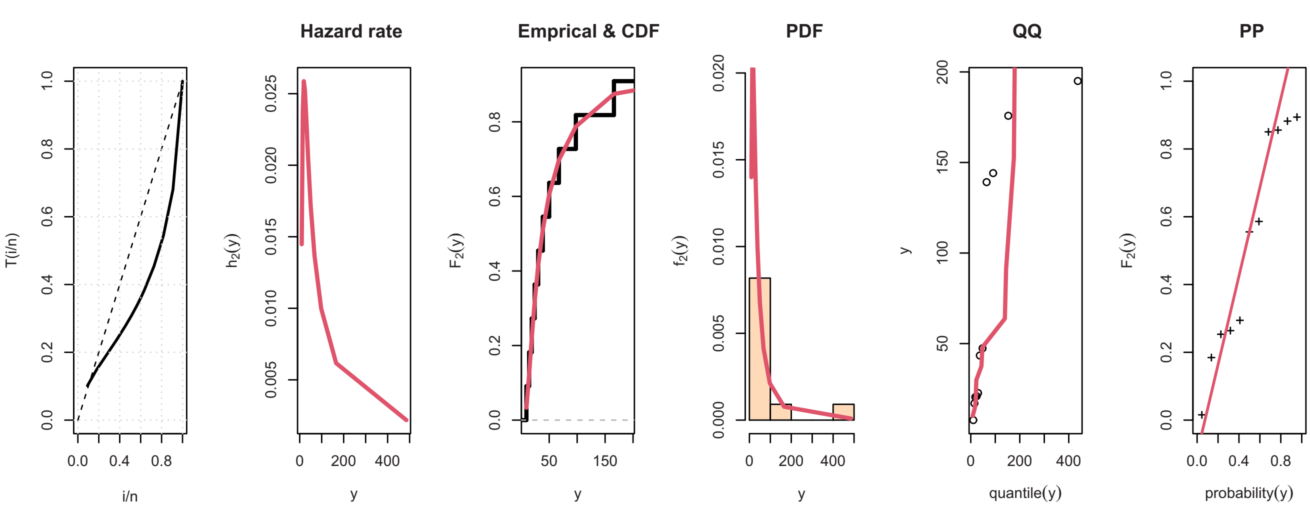

Figure 2: Some fitting plots of the BIIID for oil breakdown data with 32 kV.

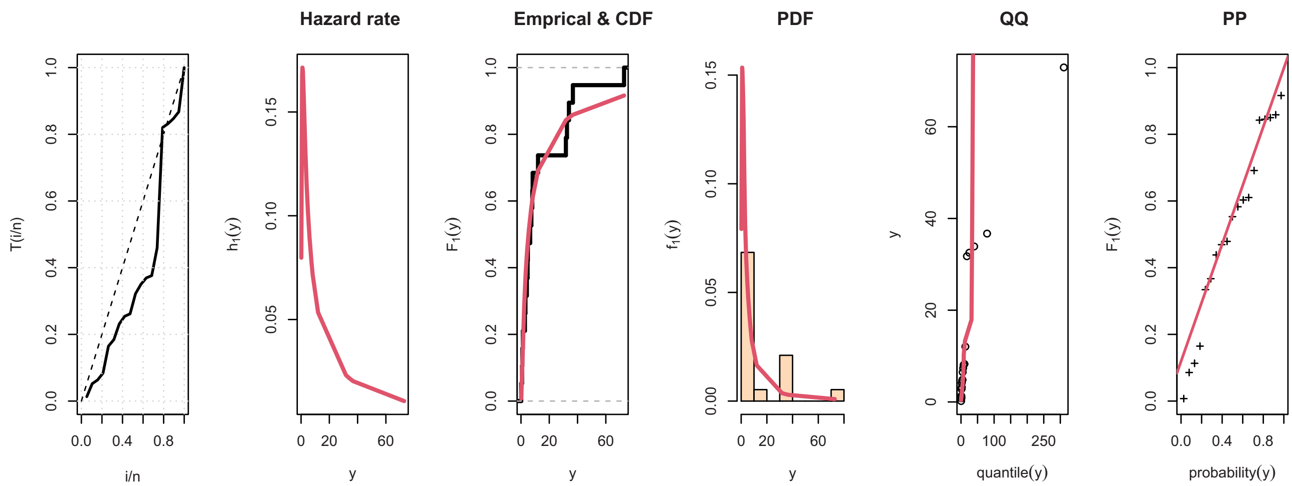

Figure 3: Some fitting plots of the BIIID for oil breakdown data with 34 kV.

methods include Gibbs sampling and the more general Metropolis inside Gibbs samplers. With the MCMC methodology, we can always achieve an appropriate parameter interval estimate since we generate the probability intervals based on the empirical posterior distribution, which gives it an edge over the ML method. This is commonly unavailable while using an ML estimator. The MCMC samples may actually be used to completely quantify the posterior uncertainty about the parameters λ, ϑ, and ρ. This also applies to any function that depends on the inputs by employing a kernel estimate of the posterior distribution. The following are λ, ϑ, and ρ's conditional posterior densities: The conditional posterior densities of λ, ϑ, and ρ are as follows:

$$\pi_1(\lambda|\vartheta,\rho,\underset{\rightarrow}{y}) \propto \lambda^{a_1+D_1+D_2-1}\prod_{i=1}^{D_\ell} y_{\ell i}^{-(\lambda+1)}(1+y_{\ell i}^{-\lambda})^{-\vartheta-1} \times [1-(1+y_{\ell i}^{-\lambda})^{-\vartheta}]^{(\rho^{\ell-1}-1)-1}[[1-(1+h_\ell^{-\lambda})^{-\vartheta}]^{(\rho^{\ell-1}-1)}]^{n_\ell-D_\ell},$$

$$\pi_2(\vartheta|\lambda,\rho,\underset{\rightarrow}{y}) \propto \vartheta^{a_2+D_1+D_2-1}\prod_{i=1}^{D_\ell}(1+y_{\ell i}^{-\lambda})^{-\vartheta-1} \times [1-(1+y_{\ell i}^{-\lambda})^{-\vartheta}]^{(\rho^{\ell-1}-1)-1}[[1-(1+h_\ell^{-\lambda})^{-\vartheta}]^{(\rho^{\ell-1}-1)}]^{n_\ell-D_\ell},$$

and

$$\pi_3(\rho|\vartheta,\rho,\underset{\rightarrow}{y}) \propto (\rho-1)^{a_3+D_2-1}\prod_{i=1}^{D_\ell}[1-(1+y_{\ell i}^{-\lambda})^{-\vartheta}]^{(\rho^{\ell-1}-1)-1} \times [1-(1+h_\ell^{-\lambda})^{-\vartheta}]^{(\rho^{\ell-1}-1)(n_\ell-D_\ell)}.$$

The following MH-within-Gibbs sampling steps can be used to obtain samples of λ, ϑ, and ρ.

Step 1: Set the initial values $\lambda^{(0)}=\hat{\lambda}$, $\vartheta^{(0)}=\hat{\vartheta}$ and $\rho^{(0)}=\hat{\rho}$.

Step 2: Set $I=1$.

Step 3: Generate λ^* from a normal distribution with a mean $(\hat{\lambda})$ and variance $(V_{\hat{\lambda}})$, represented as $N(\hat{\lambda},\ V_{\hat{\lambda}})$. Similarly, generate ϑ^* from the normal distribution with a mean $(\hat{\vartheta})$ and variance $(V_{\hat{\theta}})$, represented as $N(\hat{\vartheta},\ V_{\hat{\vartheta}})$.

Table 7: The KS statistics and some statistical measures for the oil breakdown, assuming complete data

kV		Estimates	StEr	AIC	CAIC	BIC	HQIC	CVM	AD	KSD	PVKS
30	λ	1.0703	0.2388	121.0621	122.5621	121.8579	120.5604	0.0648	0.4582	0.2015	0.6925
	ϑ	34.8903	25.6433								
32	λ	0.0811	0.0836	137.4472	139.6290	139.5714	137.4246	0.0406	0.3104	0.1407	0.8885
	ϑ	15.3974	14.0132								
	ρ	6737.773	65473.326								
34	λ	0.8260	0.1332	143.0185	143.7685	144.9074	143.3382	0.0604	0.4150	0.1235	0.9002
	ϑ	3.0668	0.7256								
36	λ	1.0538	0.8830	77.4611	79.6429	79.5853	77.4385	0.0356	0.2168	0.1083	0.9864
	ϑ	2.9483	2.3280								
	ρ	1.5475	2.5319								

Also, generate ρ^* from the normal distribution with a mean $(\hat{\rho})$ and variance $(V_{\hat{\rho}})$, represented as $N(\hat{\rho},\ V_{\hat{\rho}})$.

Step 4: Obtain $\hbar_\lambda = \min\left[1, \frac{\pi(\lambda^* \mid \vartheta^{(I-1)}, \rho^{(I-1)}, \underline{y})}{\pi(\lambda^{(I-1)} \mid \vartheta^{(I-1)}, \rho^{(I-1)}, \underline{y})}\right]$, $\hbar_\vartheta = \min\left[1, \frac{\pi(\vartheta^* \mid \lambda^{(I-1)}, \rho^{(I-1)}, \underline{y})}{\pi(\vartheta^{(I-1)} \mid \lambda^{(I-1)}, \rho^{(I-1)}, \underline{y})}\right]$, and $\hbar_\rho = \min\left[1, \frac{\pi(\rho^* \mid \lambda^{(I-1)}, \vartheta^{(I-1)}, \underline{y})}{\pi(\rho^{(I-1)} \mid \lambda^{(I-1)}, \vartheta^{(I-1)}, \underline{y})}\right]$.

Step 5: Generate samples U_j; $j = 1, 2, 3$ from the uniform $U(0, 1)$ distribution.

Step 6: If $U_1 \le \hbar_\lambda, U_2 \le \hbar_\vartheta$, and $U_3 \le \hbar_\rho$, then set $\lambda^{(I)} = \lambda^*$, $\vartheta^{(I)} = \vartheta^*$, $\rho^{(I)} = \rho^*$; otherwise $\lambda^{(I)} = \lambda^{(I-1)}$, $\vartheta^{(I)} = \vartheta^{(I-1)}$, and $\rho^{(I)} = \rho^{(I-1)}$, respectively.

Step 7: Set $I = I + 1$.

Step 8: Repeat steps 3–7 B times and obtain $\lambda^{(I)}$, $\vartheta^{(I)}$, and $\rho^{(I)}$, for $I = 1, 2, \ldots, B$.

5 Simulation study

This section explores the model's performance through simulations. First, we present an illustrative example using a simulated dataset. Subsequently, we conduct a more extensive Monte Carlo simulation study. All results are accompanied by detailed discussions to explain their significance. It is important to note that finding exact solutions for the model parameters (mentioned earlier) is mathematically challenging due to the system of nonlinear equations involved. In our case, with three parameters, we were dealing with three complex equations. Therefore, we employed numerical methods to estimate the desired parameters. Since we were able to derive the first and second derivatives of the objective functions (functions used for parameter estimation), we employed Newton's method to find the ML estimates (MLEs). Convergence issues can arise in optimization processes if researchers don't pick good starting values. To address this, we randomly generated initial values close to the actual model parameters. For brevity, we won't delve into the details of Newton's method, which is known for its efficient convergence properties.

This section focuses on a simulation study designed to compare the performance of different parameter estimation methods and their corresponding confidence intervals in Tables 1–6. We'll be evaluating two types of estimates: MLEs and Bayesian estimates (BEs). The comparison will involve two key metrics:

- Mean squared error (MSE) and absolute bias (Abias): These metrics will be used to assess the accuracy of the estimates themselves. Lower MSE and Abias indicate estimates that are closer to the true parameter values on average.
- Average length and coverage probability (CP): We'll compare the average lengths of two types of CIs: Asy-CIs (LACI) and BCIs (LCCI). Additionally, we'll examine their CP. Ideally, CIs should have an average length that captures the true parameter value within a certain confidence level (*e.g.*, 95%) most of the time.

The following procedure outlines how we'll conduct the simulation study:

Step 1: Assign values for $n_1, n_2, r_1, r_2, T_{11}, T_{12}, T_{21}$, and T_{22}.

Step 2: Using the given prior parameters λ, ϑ, and ρ, generate λ, and ϑ from the gamma distributions (a_1, b_1), and (a_2, b_2) while ρ has a truncated gamma distribution $\rho - 1\sim$ gamma (a_3, b_3).

Step 3: Generate two random samples from the CDFs $F_1(y)$ and $F_2(y)$, as defined in Eqs. (2) and (4), and apply the GTII−HCS's technique to obtain the two samples.

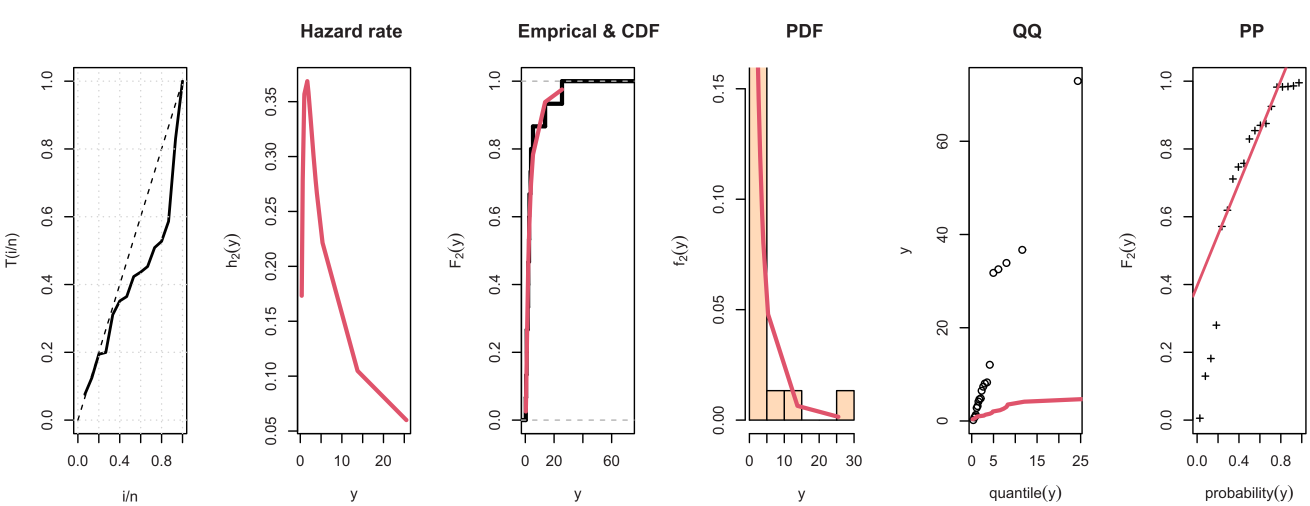

Figure 4: Some fitting plots of the BIIID for oil breakdown data with 36 kV.

Table 8: MLEs and BEs for Dataset I

r_1, r_2	$(T_{11}, T_{12}), (T_{21}, T_{22})$		Estimates	StEr	Lower	Upper	LCI	D_1, h_1	D_2, h_2
7, 8	(40, 100), (12, 40)	ML	0.4525	0.0871	0.2818	0.6233	0.3415	7, 47.3	8, 13.95
			4.2424	0.9846	2.3126	6.1722	3.8596		
			2.0583	1.1426	−0.1812	4.2977	4.4789		
		Bayesian	0.4629	0.0831	0.3152	0.6382	0.3229		
			4.2653	0.8722	2.5826	6.0238	3.4412		
			1.9395	0.8465	0.5368	3.6748	3.1381		
7, 12	(40, 100), (12, 40)	ML	0.4464	0.0874	0.2752	0.6177	0.3425	7, 47.3	10, 40
			4.0723	0.9138	2.2812	5.8634	3.5821		
			1.7981	0.9022	0.0299	3.5664	3.5366		
		Bayesian	0.4500	0.0840	0.2900	0.6124	0.3224		
			4.1798	0.8274	2.6655	5.9024	3.2369		
			1.9486	0.7603	0.6512	3.4412	2.7901		
10, 12	(40, 100), (12, 40)	ML	0.4218	0.0822	0.2608	0.5829	0.3221	7, 100	10, 40
			4.1317	0.9222	2.3241	5.9392	3.6151		
			1.9832	1.0050	0.0133	3.9530	3.9396		
		Bayesian	0.4312	0.0785	0.2872	0.5991	0.3118		
			4.1480	0.8279	2.6167	5.8130	3.1962		
			2.0193	0.8072	0.6469	3.5833	2.9363		
10, 12	(50, 120), (20, 60)	ML	0.4161	0.0808	0.2578	0.5745	0.3166	7, 100	10, 40
			4.1491	0.9089	2.3677	5.9305	3.5628		
			2.0343	0.9998	0.0747	3.9939	3.9192		
		Bayesian	0.4203	0.0747	0.2781	0.5677	0.2897		
			4.2227	0.8335	2.7154	5.9073	3.1919		
			2.1427	0.8283	0.7292	3.8388	3.1096		

Step 4: Solve the nonlinear Eqs. (7) and (8) using the Newton-Raphson technique to obtain the MLEs of the parameters λ, ϑ, and ρ.

Step 5: Establish the Asy-CIs using asymptotic VC-M of the estimators.

Step 6: Use the MH algorithm to generate an iterative sequence of 11,000 random samples with N = 11,000 and M = 1,000.

Step 7: Calculate the BEs of λ, ϑ, and ρ based on the LXF and SLF.

Step 8: Repeat Steps 2 to 8 1,000 times for various sample sizes and censoring schemes and calculate the MSEs and Abias of all the estimates.

Tables 1, 3 and 5 present the MSEs and Abias of the MLEs and BEs for the parameters λ, ϑ, and ρ and are evaluated against the SLF and LXF (with varying q values) loss functions. Tables 2, 4 and 6 show the LCI and the corresponding 95% CPs derived from the asymptotic distributions of MLEs and the credible intervals. The prior parameters were determined using the elective hyperparameter method proposed by Dey *et al.* [41].

The analysis of data in Tables 1–6 reveals several trends regarding the performance of different estimation methods:

1) Accuracy improves with sample size: As the number of samples (n) increases, the MSEs and Abias generally decrease for estimation methods. This indicates that larger datasets lead to more accurate estimates of the underlying parameters.

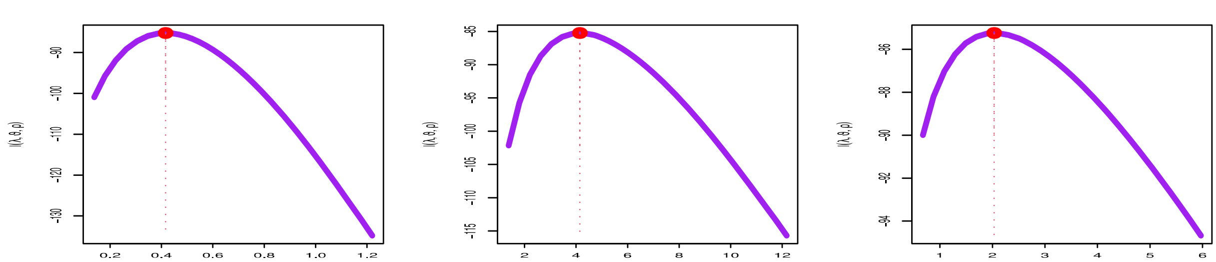

Figure 5: Likelihood profile plots for parameters: r_1 = 10, r_2 = 12, (T_{11}, T_{12}) = (50, 120), and (T_{21}, T_{22}) = (20, 60).

Table 9: MLE and BE for Dataset II

r_1, r_2	(T_{11}, T_{12}), (T_{21}, T_{22})		Estimates	StEr	Lower	Upper	LCI	D_1, h_1	D_2, h_2
12, 10	(10, 21), (2.5, 4)	ML	0.7775	0.1252	0.5320	1.0229	0.4910	14, 21	8, 4
			3.1911	0.6394	1.9380	4.4442	2.5063		
			1.5007	0.7065	0.1160	2.8854	2.7694		
		Bayesian	0.7754	0.1211	0.5591	1.0322	0.4730		
			3.2650	0.5967	2.1395	4.4449	2.3054		
			1.6506	0.6704	0.4787	3.0498	2.5710		
12, 10	(10, 15), (2.5, 3.5)	ML	0.8035	0.1307	0.5474	1.0596	0.5122	14, 15	7, 3.5
			3.1292	0.6362	1.8822	4.3762	2.4940		
			1.3326	0.6547	0.0494	2.6158	2.5664		
		Bayesian	0.7961	0.1239	0.5674	1.0421	0.4747		
			3.2614	0.6275	2.1560	4.5741	2.4181		
			1.5375	0.6494	0.3383	2.8957	2.5575		
5, 5	(5, 8), (2.5, 3.5)	ML	0.7962	0.1340	0.5336	1.0587	0.5251	11, 8	10, 3.5
			3.3263	0.6435	2.0650	4.5877	2.5227		
			2.3940	1.1121	0.2144	4.5736	4.3593		
		Bayesian	0.7991	0.1308	0.5562	1.0614	0.5053		
			3.3603	0.6117	2.1467	4.5444	2.3976		
			2.4509	0.9853	0.8368	4.4390	3.6022		
10, 10	(5, 8), (2.5, 3.5)	ML	0.8074	0.1391	0.5348	1.0800	0.5451	10, 6.5	10, 2.9
			3.4245	0.6650	2.1212	4.7279	2.6067		
			2.7017	1.2933	0.1669	5.2365	5.0696		
		Bayesian	0.8122	0.1363	0.5631	1.0908	0.5277		
			3.4904	0.6047	2.3413	4.6321	2.2908		
			2.8277	1.1439	0.8727	5.0707	4.1979		

2) Observed failure time matters: For a fixed sample size, using datasets with higher observed failure times results in lower MSEs and Abias for the estimated parameters. This suggests that data with more extreme values can improve estimation accuracy.
3) Bayesian estimates with LXF: The BEs obtained using the LXF (with $q = 1.25$) consistently outperform other methods in terms of MSEs and Abias. This suggests that the LXF might be a particularly effective choice for this specific scenario.
4) Credible intervals offer advantages: When comparing the LCI and CP of different CIs, credible intervals show smaller LCCI and CPs closer to the desired level (95%) compared to normal approximation CIs, which LACI and CP. This suggests that BCIs might be a more reliable choice for uncertainty quantification in this setting.

6 Data analysis

In this section, two real datasets for oil breakdown time analysis are introduced to show how the model using ML and Bayesian estimation methods works in practice based on real data from Nelson [4]. This study analyzes oil breakdown times for insulating fluid under different stress levels.

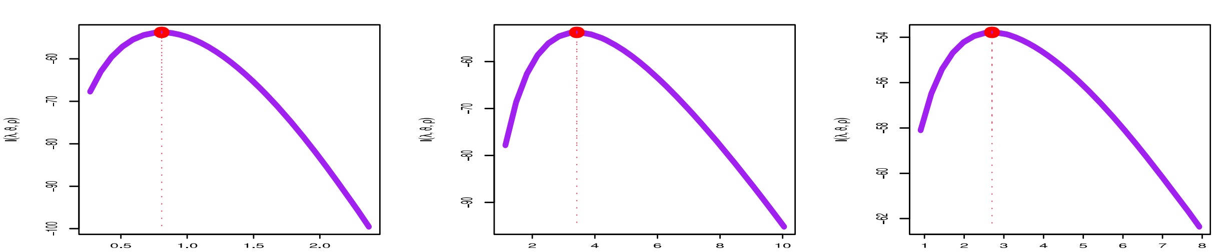

Figure 6: Likelihood profile plots for parameters: $r_1 = 10$, $r_2 = 10$, $(T_{11}, T_{12}) = (5, 8)$, and $(T_{21}, T_{22}) = (2.5, 3.5)$.

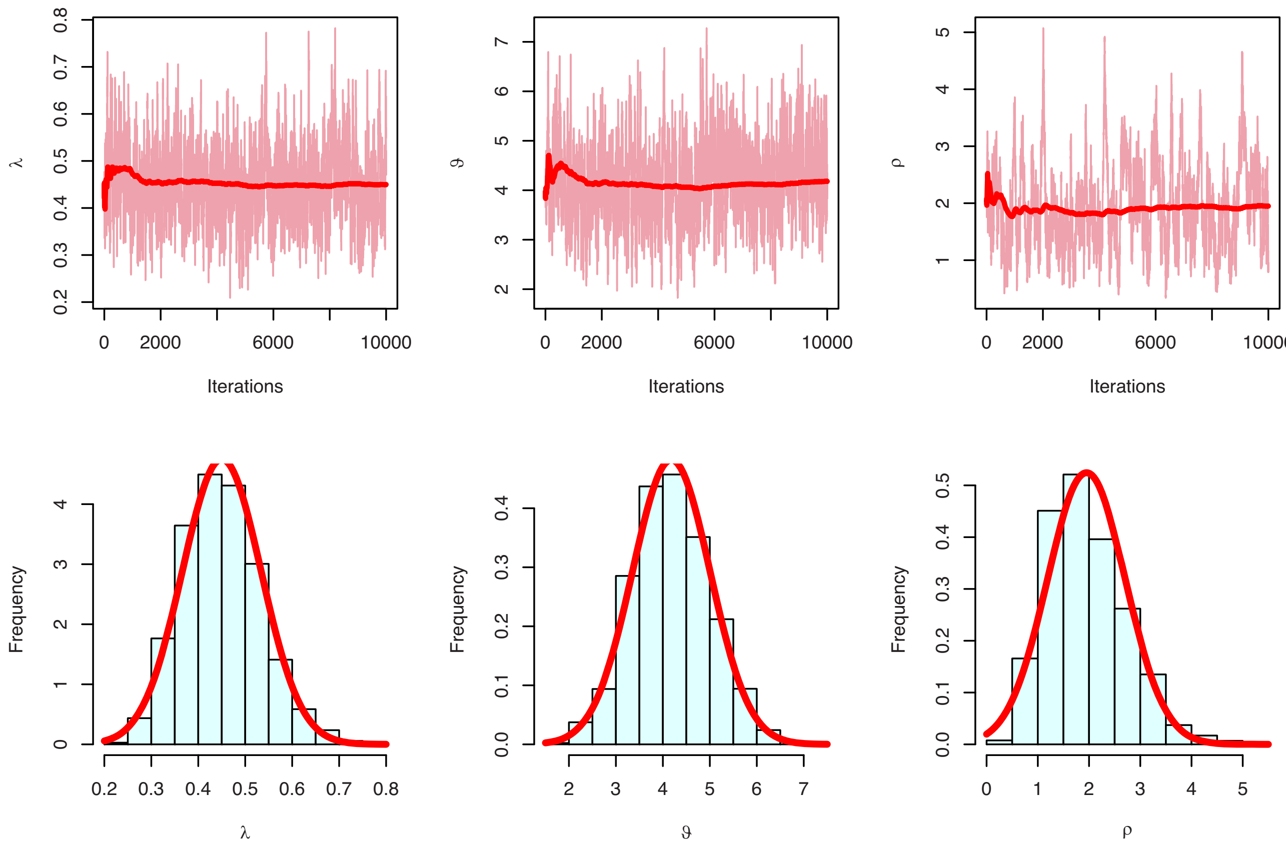

Figure 7: MCMC plots $r_1 = 10$, $r_2 = 12$, $(T_{11}, T_{12}) = (50, 120)$, and $(T_{21}, T_{22}) = (20, 60)$.

Dataset I: This dataset comprises breakdown times measured under various constant high voltage levels. For illustrative purposes, the data at 30 kV is assumed to represent normal use conditions, while the data at 32 kV is considered accelerated data. The 30 kV dataset is considered representative of normal stress conditions as follows: 7.74, 17.05, 20.46, 21.02, 22.66, 43.40, 47.30, 139.07, 144.12, 175.88, and 194.90. While the 32 kV dataset is considered representative accelerated stress conditions as follows: 0.27, 0.40, 0.69, 0.79, 2.75, 3.91, 9.88, 13.95, 15.93, 27.80, 53.24, 82.85, 89.29, 100.58, and 215.10.

Dataset II: This dataset focuses on breakdown times at two specific stress levels: 34 kV and 36 kV. The 34 kV dataset is considered representative of normal stress conditions as follows: 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, and 72.89. While the 36 kV data is considered representative accelerated stress conditions as follows: 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.9, 3.67, 3.99, 5.35, 13.77, and 25.5.

To analyze these datasets, the goodness-of-fit of the BIIID was initially assessed using the one-sample Kolmogorov–Smirnov (K-S) test. Table 7 presents the MLEs and their standard errors (StEr) under used and accelerated conditions. It also includes the K-S distance (KSD) with the corresponding p-value (PVKS) and some of the most famous statistical measures, such as Akaike's information criterion (AIC), Bayesian information criterion (BIC), corrected AIC (CAIC), Hannan–Quinn information criterion (HQIC), Anderson–Darling (AD), and Cramer–von Mises (CVM) calculated based on the MLEs for each dataset. The goodness-of-fit test suggests that, irrespective of the estimator used, the BIIID is an appropriate life model for the analyzed datasets. Figures 1–4 illustrate the total time on test plots compared to the HF plot, the empirical CDF *versus* the fitted CDF, the histogram with the PDF line, as well as the quantile-quantile and probability-probability plots of the BIIID for each dataset, respectively.

Table 8 discusses MLE and BE for dataset I, where normal use condition (30 kV) and accelerated stress condition (32 kV) with different values of times and different censored sizes. We are concluding that the BE has the smallest value of StEr and a smaller LCI than MLE. We estimated the D_1, h_1, D_2, and D_2 GTII–HCS based on CSPALT. Figure 5 discusses likelihood profile plots for parameters with $r_1 = 10$, $r_2 = 12$, $(T_{11}, T_{12}) = (50, 120)$, and $(T_{21},$

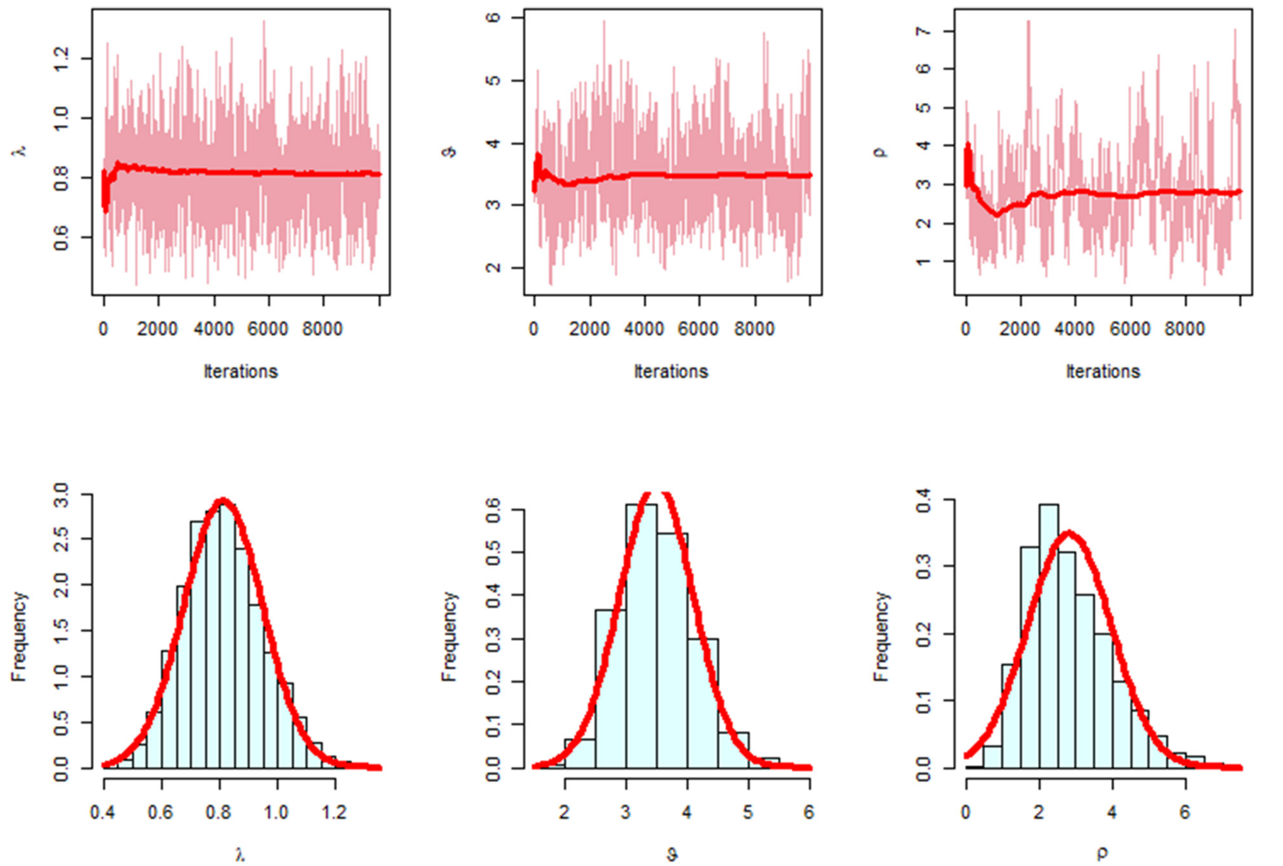

Figure 8: MCMC plots for $r_1 = 10$, $r_2 = 10$, $(T_{11}, T_{12}) = (5, 8)$, and $(T_{21}, T_{22}) = (2.5, 3.5)$.

$T_{22}) = (20, 60)$ to confirm the likelihood estimates have maximum and unique solutions.

Table 9 discusses MLE and BE for dataset II, where normal use condition (34 kV) and accelerated stress condition (36 kV) with different values of times and different censored sizes. We are concluding that the BE has the smallest value of StEr and a smaller LCI than MLE. We estimated D_1, h_1, D_2, and h_2 GTII−HCS based on CSPALT. Figure 6 discusses likelihood profile plots for parameters with $r_1 = 10$, $r_2 = 10$, $(T_{11}, T_{12}) = (5, 8)$, and $(T_{21}, T_{22}) = (2.5, 3.5)$ to confirm the likelihood estimates have maximum and unique solutions.

Figures 7 and 8 show the results of an MCMC simulation for estimating the posterior distribution of two parameters. The MCMC is a computational method used to sample from complex probability distributions. Top row: Presents trace plots for three parameters, visualizing the values generated by the MCMC simulation across iterations. Ideally, trace plots should display a lack of trend and reach stationarity (a stable distribution) after an initial "burn-in" period where the initial samples are discarded. Bottom row: Shows the posterior densities of the three parameters, illustrating the probability distribution of each parameter after incorporating the data and prior information into the MCMC analysis.

7 Concluding remarks

In order to reduce the testing length, units are subjected to more demanding conditions during ALTs. ALTs or PALTs are crucial for life testing research since they save money and time. PALTs are carried out in situations when the findings of ALT cannot be extended to regular conditions. The CSPALT proposed in this work is based on the premise that units' lifespan under usage conditions follows the BIIID and is based on a GTII-HCS. Under typical use conditions, the ML approach is used to derive the estimates of the BIIID's parameters and accelerated factor. Using the MCMC method, the Bayesian estimates are generated based on symmetric and asymmetric loss functions. Additionally, Asy-CIs and BCIs have been produced. To evaluate the proposed methodology, simulation studies with varying censoring strategies and sample sizes have been conducted. Our conclusion from the simulation dataset is that, in terms of MSEs and Abias, the BEs derived with the LXF (q = 1.25) regularly beat alternative approaches. It would appear from this that the LXF may be an especially good option in this case. Bayesian credible intervals display reduced average length and coverage probabilities nearer the intended level (95%) when compared with the corresponding Asy-CIs. In this context, it appears that BCIs might be a more reliable choice for quantifying uncertainty. Two data sets were then examined to demonstrate the effectiveness of the suggested strategies.

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Appendix 1

The elements of the Fisher information matrix are given by:

$$\frac{\partial^2 l^\bullet}{\partial\lambda^2} = -\frac{D_1 + D_2}{\lambda^2} - \sum_{i=1}^{D_1} \frac{(\vartheta + 1)y_{1i}^{\lambda}(\log y_{1i})^2}{(1 + y_{1i}^{\lambda})^2} - \sum_{i=1}^{D_2} \frac{(\vartheta + 1)y_{2i}^{\lambda}(\log y_{2i})^2}{(1 + y_{2i}^{\lambda})^2} + \sum_{i=1}^{D_2} \frac{\upsilon''_{1(\lambda)}(\rho - 1)}{\upsilon_1(y_{2i}, \lambda, \vartheta)} - \sum_{i=1}^{D_2} \frac{(\rho - 1)\upsilon'^2_{1(\lambda)}}{(\upsilon_1(y_{2i}, \lambda, \vartheta))^2} + \frac{(n_1 - D_1)\upsilon''_{2(\lambda)}}{\upsilon_2(h_1, \lambda, \vartheta)} - \frac{(n_1 - D_1)\upsilon'^2_{2(\lambda)}}{(\upsilon_2(h_1, \lambda, \vartheta))^2} - \frac{(n_2 - D_2)\upsilon'^2_{3(\lambda)}}{(\upsilon_3(h_2, \lambda, \vartheta))^2} + \frac{(n_2 - D_2)\upsilon''_{3(\lambda)}}{\upsilon_3(h_2, \lambda, \vartheta)},$$

$$\frac{\partial^2 l^\bullet}{\partial\vartheta^2} = -\frac{D_1 + D_2}{\vartheta^2} - \sum_{i=1}^{D_2} \frac{(\rho - 1)(1 + y_{2i}^{-\lambda})^{-\vartheta}(\log(1 + y_{2i}^{-\lambda}))^2}{\upsilon_1(y_{2i}, \lambda, \vartheta)} - \frac{(n_1 - D_1)(1 + h_1^{-\lambda})^{-\vartheta}(\log(1 + h_1^{-\lambda}))^2}{\upsilon_2(h_1, \lambda, \vartheta)} - \sum_{i=1}^{D_2} \frac{(\rho - 1)\upsilon'^2_{1(\vartheta)}}{(\upsilon_1(y_{2i}, \lambda, \vartheta))^2} - \frac{(n_1 - D_1)\upsilon'^2_{2(\vartheta)}}{\upsilon_2(h_1, \lambda, \vartheta)} - \frac{(n_2 - D_2)(1 + h_2^{-\lambda})^{-\vartheta}(\log(1 + h_2^{-\lambda}))^2}{\upsilon_3(h_2, \lambda, \vartheta)} - \frac{(n_2 - D_2)\upsilon'^2_{3(\vartheta)}}{\upsilon_3(h_2, \lambda, \vartheta)},$$

$$\frac{\partial^2 l^\bullet}{\partial\vartheta\partial\lambda} = \frac{\partial^2 l^\bullet}{\partial\lambda\partial\vartheta} = \sum_{i=1}^{D_1} \frac{\log y_{1i}}{1 + y_{1i}^{\lambda}} + \sum_{i=1}^{D_2} \frac{\log y_{2i}}{1 + y_{2i}^{\lambda}} + \sum_{i=1}^{D_2} \frac{(\rho - 1)\upsilon''_{1(\vartheta\lambda)}}{\upsilon_1(y_{2i}, \lambda, \vartheta)} - \sum_{i=1}^{D_2} \frac{(\rho - 1)\upsilon'_{1(\lambda)}\upsilon'_{1(\vartheta)}}{(\upsilon_1(y_{2i}, \lambda, \vartheta))^2} + \frac{(n_1 - D_1)\upsilon''_{2(\vartheta\lambda)}}{\upsilon_2(h_1, \lambda, \vartheta)} - \frac{(n_1 - D_1)\upsilon'_{2(\lambda)}\upsilon'_{2(\vartheta)}}{\upsilon_2(h_1, \lambda, \vartheta)} + \frac{(n_2 - D_2)\rho\upsilon''_{3(\vartheta\lambda)}}{\upsilon_3(h_2, \lambda, \vartheta)} - \frac{(n_2 - D_2)\upsilon'_{3(\lambda)}\upsilon'_{3(\vartheta)}}{\upsilon_3(h_2, \lambda, \vartheta)},$$

$$\frac{\partial^2 l^\bullet}{\partial\rho\partial\lambda} = \frac{\partial^2 l^\bullet}{\partial\lambda\partial\rho} = \sum_{i=1}^{D_2} \frac{\upsilon'_{1(\lambda)}}{[\upsilon_1(y_{2i}, \lambda, \vartheta)]} + \frac{(n_2 - D_2)\upsilon'_{3(\lambda)}}{\upsilon_3(h_2, \lambda, \vartheta)},$$

$$\frac{\partial^2 l^\bullet}{\partial\rho\partial\vartheta} = \frac{\partial^2 l^\bullet}{\partial\vartheta\partial\rho} = \sum_{i=1}^{D_2} \frac{\upsilon'_{1(\vartheta)}}{[\upsilon_1(y_{2i}, \lambda, \vartheta)]} + \frac{(n_2 - D_2)\upsilon'_{3(\vartheta)}}{\upsilon_3(h_2, \lambda, \vartheta)},$$

$$\frac{\partial^2 l^\bullet}{\partial\rho^2} = -\frac{D_2}{\rho^2}, \quad \upsilon''_{1(\lambda)} = \frac{\partial\upsilon'_{1(\lambda)}}{\partial\lambda} = \vartheta(\log y_{2i})^2 y_{2i}^{-\lambda}[(1 + y_{2i}^{-\lambda})^{-\vartheta-1} - (\vartheta + 1)y_{2i}^{-\lambda}(1 + y_{2i}^{-\lambda})^{-\vartheta-2}],$$

$$\upsilon''_{2(\lambda)} = \frac{\partial\upsilon'_{2(\lambda)}}{\partial\lambda} = \vartheta(\log h_1)^2 h_1^{-\lambda}[(1 + h_1^{-\lambda})^{-\vartheta-1} - (\vartheta + 1)h_1^{-\lambda}(1 + h_1^{-\lambda})^{-\vartheta-2}],$$

$$\upsilon''_{3(\lambda)} = \frac{\partial\upsilon'_{3(\lambda)}}{\partial\lambda} = \vartheta(\log h_2)^2 h_2^{-\lambda}[(1 + h_2^{-\lambda})^{-\vartheta-1} - (\vartheta + 1)h_2^{-\lambda}(1 + h_2^{-\lambda})^{-\vartheta-2}],$$

$$\upsilon''_{1(\vartheta\lambda)} = \frac{\partial\upsilon'_{1(\vartheta)}}{\partial\lambda} = y_{2i}^{-\lambda}\log y_{2i}(1 + y_{2i}^{-\lambda})^{-\vartheta-1}[\vartheta\log(1 + y_{2i}^{-\lambda}) - 1],$$

$$\upsilon''_{2(\vartheta\lambda)} = \frac{\partial\upsilon'_{2(\vartheta)}}{\partial\lambda} = h_1^{-\lambda}\log h_1(1 + h_1^{-\lambda})^{-\vartheta-1}[\vartheta\log(1 + h_1^{-\lambda}) - 1],$$

and

$$\upsilon''_{3(\vartheta\lambda)} = \frac{\partial\upsilon'_{3(\vartheta)}}{\partial\lambda} = h_2^{-\lambda}\log h_2(1 + h_2^{-\lambda})^{-\vartheta-1}[\vartheta\log(1 + h_2^{-\lambda}) - 1].$$