Scientific African 31 (2026) e03142

Contents lists available at ScienceDirect

Scientific African

journal homepage: www.elsevier.com/locate/sciaf

A comparative study of estimation methods for the New Sine Topp-Leone Fréchet distribution

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ARTICLE INFO

Editor name: Mohamed Fathy El-Amin Mousa

Keywords:
Sine Topp-Leone-G family
Stress–strength reliability
Monte Carlo simulation
Parameter estimation
Flexible distribution models

ABSTRACT

Modeling and fitting complex datasets are crucial domains in various scientific fields. This article presents the sine Topp-Leone Fréchet distribution, a new probability distribution obtained by integrating the Fréchet distribution with the sine Topp-Leone-G family. The suggested model's most noteworthy feature is its ability to accurately represent a broad variety of hazard rate patterns (such as upside-down, reversed J-shaped, decreasing, and increasing) with just three parameters. Its density may also display unimodal, reversed-J, and right-skewed shapes. We offer a number of statistical properties for the proposed distribution. Using sixteen different estimation approaches, we estimated the parameters of the suggested model and investigated a number of its distributional characteristics. We evaluate the validity of the suggested model and compare the efficacy of various parameter estimates using a Monte Carlo simulation. Numerical analysis shows that the Kolmogorov and second-order left-tail Anderson–Darling estimators exhibit superior efficiency and consistency in a variety of scenarios. They frequently receive the highest overall ranking based on the sum of all performance criteria and consistently produce estimates with low bias and mean squared error. Additionally, we evaluated the suggested distribution's adaptability using two real-world datasets, demonstrating its remarkable ability to precisely fit the data. According to our findings, the suggested distribution performs better in terms of fit quality than a cotangent Fréchet, odd exponential-logarithmic Fréchet, Topp-Leone Fréchet, odd-generalized exponential inverse Weibull, generalized inverse Weibull distribution, arctan inverse Weibull, new arctan Fréchet, and Fréchet distributions.

Introduction

In recent years, a vast amount of data has been produced worldwide in various application areas. Data heterogeneity and complexity have increased exponentially as a result of this phenomenon. Finding patterns in these datasets and drawing conclusions

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https://doi.org/10.1016/j.sciaf.2025.e03142
Received 27 September 2025; Received in revised form 21 November 2025; Accepted 17 December 2025

Available online 18 December 2025
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from them are necessary for various reasons, including planning and decision-making. To achieve this, suitable statistical models are required to make precise deductions from a given dataset. Recently generated families of distributions have enabled highly flexible modeling of real data. Adding new parameters to existing distributions enhances their capacity to more precisely describe tail shapes in addition to increasing their applicability to real-life phenomena. Several distributions and families of distributions have been produced by earlier research employing innovative techniques. These comprise, among many others, the type II generalized Topp-Leone (TL)-G family by [1], the heavy-tailed generalized TL-G by [2], the TL-G family by [3], the modified Kies flexible G-family by [4], the power Lindley-G family by [5], the TL Cauchy-G family by [6], the length-biased truncated Lomax-G family by [7], the sine-G (S-G) family by [8], the new TL-G family by [9], the extension of the odd inverse Weibull-G family by [10], the Harris-TL-G family by [11], new odd reparameterized exponential transformed-X family by [12], neutrosophic Gompertz-G by [13,14] and the odd inverted TL-G family by [15]. Some recent families related to trigonometric functions are the tangent TL-G family by [16], the sine alpha power-G family by [17], and the sine TL-G (STL-G) family by [18] along with related models [19–21].

Recently, Al-Babtain et al. [18] proposed an STL-G family with one shape parameter, a hybrid family that integrates the properties of the TL-G and S-G families. The TL-G family is regarded as an acceptable substitute for the exp-G family. Meanwhile, the S-G family is valued for the mathematical simplicity of its probabilistic functions, which helps avoid complex parameterization. The cumulative distribution function (CDF) and probability density function of the STL-G family are defined, respectively, by:

$$F(y;\delta,\boldsymbol{\Phi}) = \sin\left[\frac{\pi}{2}\left(1-\overline{G}(y;\boldsymbol{\Phi})^2\right)^{\delta}\right]; \quad y\in\mathbb{R}, \tag{1}$$

$$f(y;\delta,\boldsymbol{\Phi}) = \delta\pi g(y;\boldsymbol{\Phi})\overline{G}(y;\boldsymbol{\Phi})\left(1-\overline{G}(y;\boldsymbol{\Phi})^2\right)^{\delta-1}\cos\left[\frac{\pi}{2}\left(1-\overline{G}(y;\boldsymbol{\Phi})^2\right)^{\delta}\right]; \quad y\in\mathbb{R}, \tag{2}$$

where, $g(.)$, $G(.)$, and $\overline{G}(.) = 1 - G(.)$ are the PDF, CDF, and survival function (SF) of the baseline distribution with $\boldsymbol{\Phi}$ as the set of parameters. It can be mentioned that δ is the shape parameter.

An essential distribution created within the extreme value theory is the Fréchet distribution (FRD), also known as the inverse Weibull distribution (IWD). It can be applied to wind speeds [22], rainfall and air pollution [23], accelerated life testing [24], economics [25], earthquakes [26], reliability analysis [27], and information measures [28,29]. Generalizations and extensions of the FRD were created by numerous authors. For instance, Nadarajah and Kotz [30] investigated the exponentiated FRD with an extra shape parameter. The gamma extended FRD was established by Silva et al. [31]. Hassan et al. [32] introduced the odds generalized exponential-IWD. The truncated Weibull FRD using a truncated method was presented by [33]. The Kumaraswamy power FRD was proposed by Alsadat et al. [33] using the Kumaraswamy generator. The Weibull FRD was introduced by Afify et al. [34] utilizing the Weibull generator. Al Sobhi [35] proposed a modified Kies FRD by employing a modified Kies generator. The transmuted exponentiated FRD was proposed by Elbatal et al. [36] using the transmuted generator. The Gompertz FRD and its inverted version were suggested respectively by Oguntunde et al. [37] and Akarawak et al. [38]. The sine-FRD was presented by Aldahlan [39] by using the sine-G family. The CDF and PDF of the FRD, with scale parameter c and shape parameter, b are given by:

$$G(y) = e^{-cy^{-b}}; \quad c,b,y>0, \tag{3}$$

$$g(y) = cby^{-(b+1)}e^{-cy^{-b}}; \quad c,b,y>0. \tag{4}$$

The powerful motivation behind proposing the Sine Topp-Leone Fréchet Distribution (STL-FRD) is to bridge a critical gap between the standard models of extreme value theory and the intricate patterns found in modern real-world data. The simplicity of the classic FRD is also a drawback, even though it is a fundamental tool for simulating extreme events and maxima (such as wind speeds, financial crashes, and significant insurance claims). Its shape is relatively rigid, often failing to capture the complex, such as multi-modal tendencies, varied tail weights, and non-standard hazard rates present in contemporary datasets from fields like reliability engineering, survival analysis, and meteorology. The STL-FRD is a strategic enhancement intended to be a versatile model, not just a small improvement. The model incorporates a strong shape flexibility mechanism by combining the FRD with the STL-G family. This enables a broad range of density and hazard shapes that would previously require several different, simpler models to approximate to be produced by a single, compact model (with only three parameters). This motivation is powerfully validated by the following:

- A more adaptable variant of the FRD, the STL-FRD generates more flexible kurtosis than the baseline FRD and enhances the fitting of real-world data. There are some possible shapes for the HRF of the STL-FRD: decreasing, increasing, reversed J-shaped, and upside-down bathtub.
- There are several possible shapes for its density function, including unimodal, asymmetrical, and reversed-J shapes. The STL-FRD can also be used to model a wide range of data in the medical and engineering sciences. The better fit of this fact over competing distributions was shown by modeling two real datasets from both domains.
- The performance of different classical estimators of the STL-FRD for different sample sizes and parameter combinations is demonstrated. Consequently, the STL-FRD parameters are estimated using a range of estimation methods, including: the Cramér-von Mises estimates (CVMEs), the maximum product of spacing estimates (MPSEs), minimum spacing (MS) absolute-log distance estimates (MSALDEs), Anderson–Darling estimates (ADEs), MS squared log-distance estimates (MSSLDEs), weighted least squares estimates (WLSEs), MS squared distance estimates (MSSDEs), maximum likelihood estimates (MLEs), second-order left-tail ADEs (ADSOEs), the percentiles estimates (PCEs), the Kolmogorov estimates (KEs), right tail ADEs (RTADEs), ordinary LSEs (OLSEs), left-tail ADEs (LTADEs), MS Linex distance estimates (MSLNDE), and MS absolute distance estimates (MSADEs).

- To provide guidelines for choosing the best estimation technique, a Monte Carlo simulation study is carried out. The STL-FRD's adaptability allows it to fit two real datasets better than some other models.

The next sections of this work are scheduled as follows: The model formulation with main statistical functions is covered in Section "The Sine Topp-Leone-Fréchet Distribution Formulation". A number of mathematical properties, such as moments, quantile function (QF), moments, incomplete moment (INM), moment generating function (MGF), and stress–strength reliability (SSR), are presented in Section "Statistical Properties". Section "Estimation Methods" examines sixteen classical estimation methods for the STL-FRD parameters. Additionally, in Section "Numerical simulation", simulation studies are used to examine the efficacy of the suggested estimation tools, and in Section "Applications", we used two real data sets to evaluate the performance of the suggested STL-FRD model. Section "Conclusion" concludes the article.

The Sine Topp-Leone-Fréchet distribution formulation

This section provides an overview of the STL-FRD by introducing its fundamental expressions. Suppose that Y has the STL-FRD its CDF is generated by replacing CDF (3) of the FRD in the CDF (1) of the STL-G family as shown below:

$$F(y;\boldsymbol{\Phi}) = \sin\left[\frac{\pi}{2}\left(1-\left[1-e^{-cy^{-b}}\right]^{2}\right)^{\delta}\right]; \quad c,b,\delta,y>0, \tag{5}$$

where, $\boldsymbol{\Phi}=(b,c,\delta)^{T}$ is the set of parameters, c is the scale parameter, b, and δ are the shape parameters. The PDF of the STL-FRD is provided by inserting CDF (3) and PDF (4) in (2)

$$\begin{aligned} f(y;\boldsymbol{\Phi}) &= \delta\pi c b y^{-(b+1)} e^{-cy^{-b}}\left(1-e^{-cy^{-b}}\right)\left(1-\left(1-e^{-cy^{-b}}\right)^{2}\right)^{\delta-1} \\ &\times \cos\left[\frac{\pi}{2}\left(1-\left[1-e^{-cy^{-b}}\right]^{2}\right)^{\delta}\right]; \quad y,c,b,\delta>0. \end{aligned} \tag{6}$$

Plotting the form of the STL-FRD's PDF in Fig. 1 provides a visual representation of its behavior. Fig. 1 illustrates how the STL-FRD's PDF has distinct shapes. These shapes are decreasing, right-skewed, heavy-tailed, and unimodal. Fig. 1 depicts the 3D of PDF(6) and contour plots of the STL-FRD for selected parameter combinations. While the contour plots offer a more in-depth look at the PDF landscape, the 3D plots show the PDF's general shape. These illustrations shed light on the distribution's shape and behavior in various scenarios. For example, the plot (c = 1.5, b = 2) shows a unimodal PDF, with the peak occurring around δ = 1.25. The contour lines show that the PDF falls as we go away from the peak in any direction. The plot (c = 2.0, b = 1.2) shows a right skewed PDF, with the peak occurring around δ = 1.2. The contour lines show that the PDF falls as we go away from the peak in any direction. The reliability function, hazard rate function (HRF), and cumulative HRF are given by:

$$R(y;\boldsymbol{\Phi}) = 1-\sin\left[\frac{\pi}{2}\left(1-\left[1-e^{-cy^{-b}}\right]^{2}\right)^{\delta}\right], \tag{7}$$

$$h(y;\boldsymbol{\Phi}) = \frac{\delta\pi c b y^{-(b+1)} e^{-cy^{-b}}\left(1-e^{-cy^{-b}}\right)\left(1-\left(1-e^{-cy^{-b}}\right)^{2}\right)^{\delta-1}\cos\left[\frac{\pi}{2}\left(1-\left[1-e^{-cy^{-b}}\right]^{2}\right)^{\delta}\right]}{1-\sin\left[\frac{\pi}{2}\left(1-\left[1-e^{-cy^{-b}}\right]^{2}\right)^{\delta}\right]}, \tag{8}$$

and

$$Ch(y;\boldsymbol{\Phi}) = -\ln\left\{1-\sin\left[\frac{\pi}{2}\left(1-\left[1-e^{-cy^{-b}}\right]^{2}\right)^{\delta}\right]\right\}. \tag{9}$$

Plotting the form of the STL-FRD's HRF in Fig. 2 provides a visual representation of its behavior. Fig. 2 illustrates how the STL-FRD's HRF has distinct shapes. These shapes are decreasing or upside-down. Fig. 2 depicts the 3D HRF and contour plots of the STL-FRD for selected parameter combinations. For example, the plot (c = 1.5, b = 2) shows an upside-down HRF, with the peak occurring around δ = 1.1. The contour lines show that the PDF decreases as we go away from the peak in any direction. The plot (c = 2.0, b = 1.2) shows an upside-down HRF, with the peak occurring around δ = 1.2. The contour lines show that the PDF decreases as we go away from the peak in any direction.

Statistical properties

The QF, linear expansion of PDF, kth-moment, INM, and MGF are among the many statistical characteristics of the STL-FRD that we established in this section.

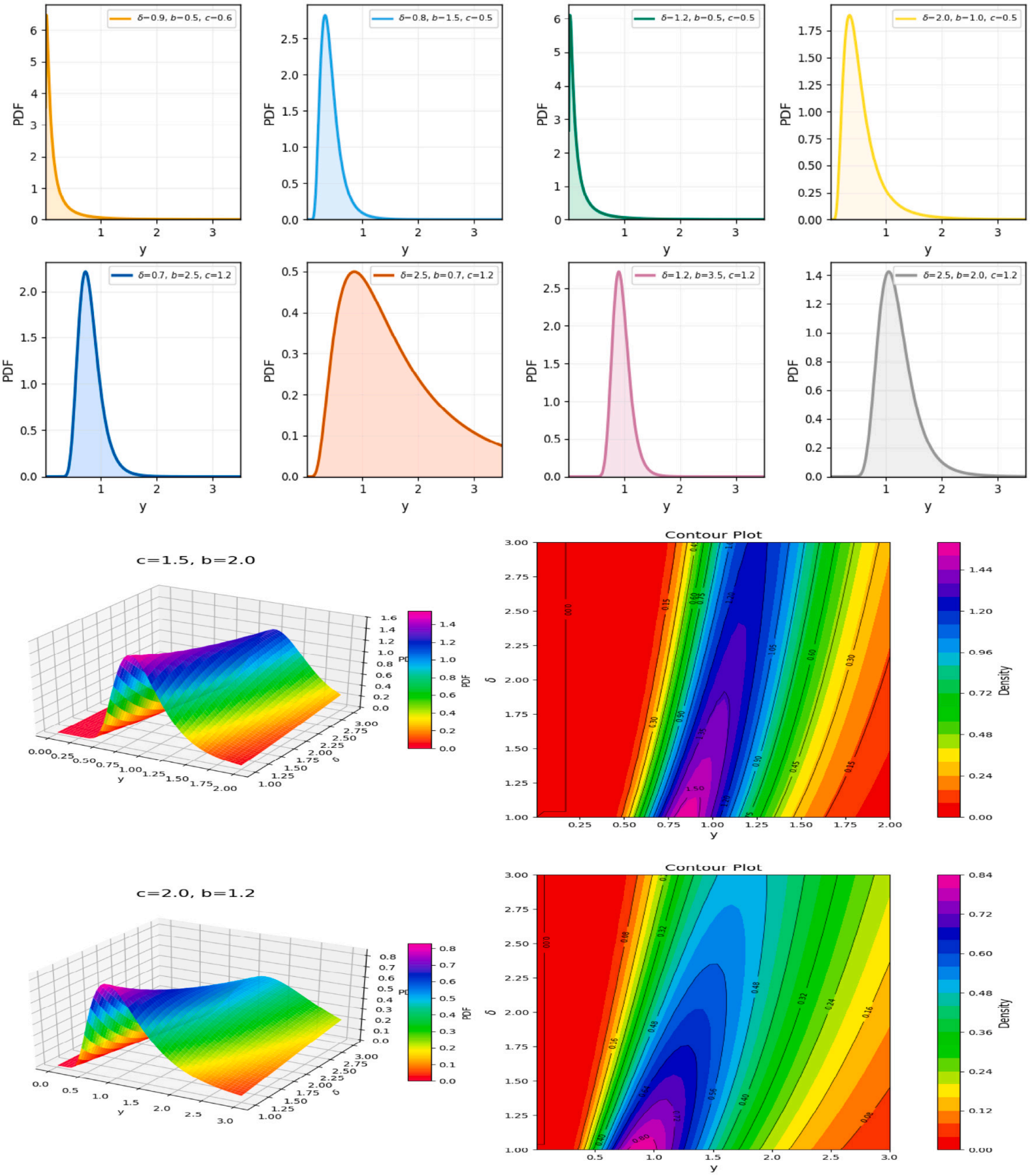

Fig. 1. 2D plots, 3D plots and contours of the PDF for the STL-FRD.

Linear expansion

For the sake of simplicity, we provide a linear representation of the PDF (6) by using the following cos expansion

$$\cos(d) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} d^{2j}, \tag{10}$$

in the PDF (6)

$$f(y;\boldsymbol{\Phi}) = \sum_{j_1=0}^{\infty} \frac{(-1)^{j_1}\,\delta c b}{(2j_1)!} \frac{\pi^{2j_1+1}\, y^{-(b+1)} e^{-cy^{-b}}}{2^{2j_1}} \left(1 - e^{-cy^{-b}}\right)\left(1 - \left(1 - e^{-cy^{-b}}\right)^2\right)^{\delta(2j_1+1)-1}. \tag{11}$$

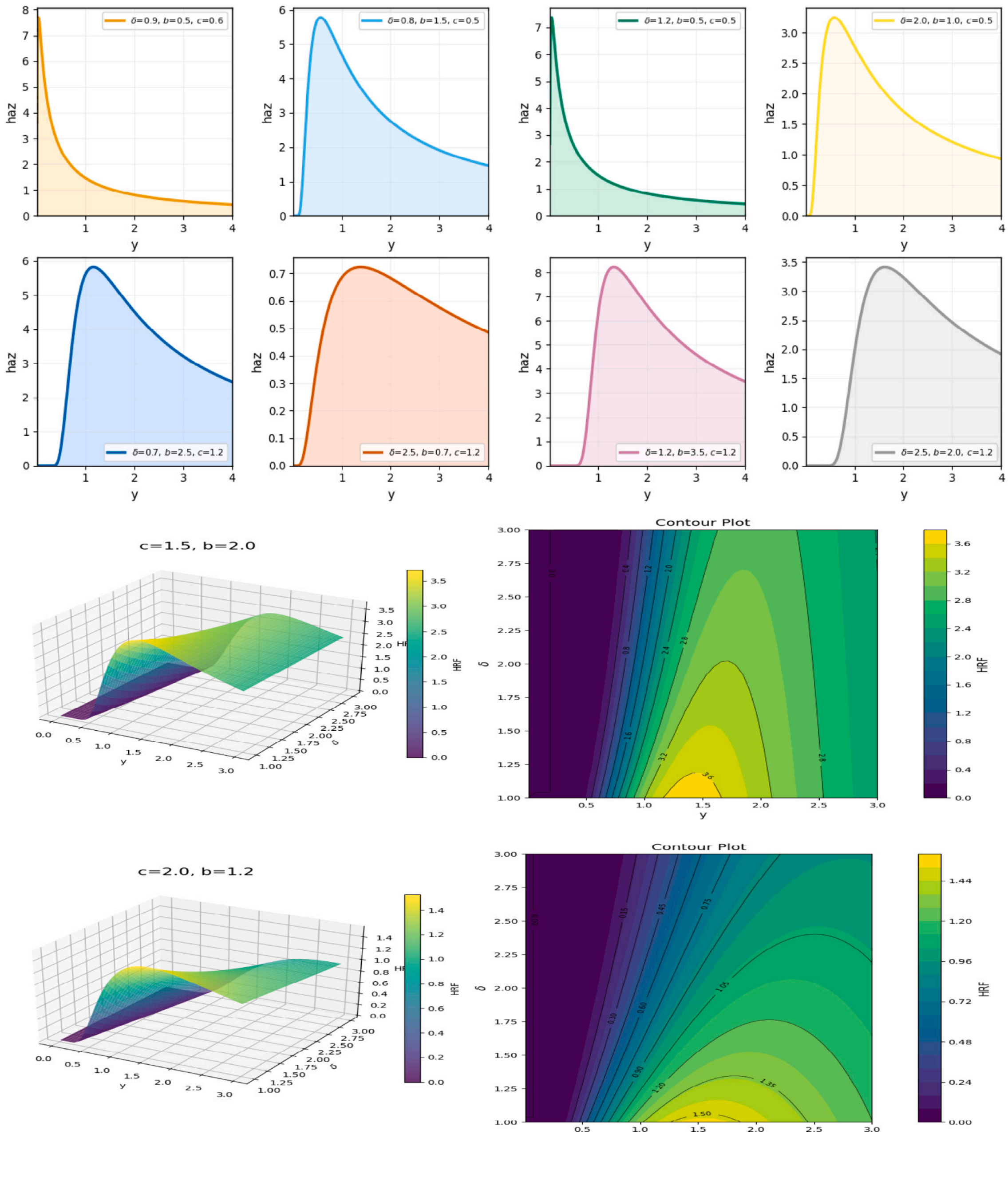

Fig. 2. 2D plots, 3D plots and contours of the HRF for the STL-FRD.

Using the following binomial expansion

$$(1-h)^z = \sum_{l=0}^{\infty} (-1)^{\ell} \binom{z}{\ell} h^{\ell}, \quad |h| < 1, z > 0, \tag{12}$$

in the last term of PDF (11) provides

$$f(y;\boldsymbol{\Phi}) = \sum_{j_1,j_2=0}^{\infty} \binom{\delta(2j_1+1)-1}{j_2} \frac{(-1)^{j_1+j_2}\,\delta cb}{(2j_1)!} \frac{\pi^{2j_1+1}\, y^{-(b+1)} e^{-cy^{-b}}}{2^{2j_1}} \left(1-e^{-cy^{-b}}\right)^{2j_2+1}. \tag{13}$$

Again by using the binomial expansion (12) in PDF (13) gives

$$f(y;\boldsymbol{\Phi}) = \sum_{j_1,j_2,j_3=0}^{\infty} \Delta_{j_1,j_2,j_3}\, cb\delta\, y^{-(b+1)} e^{-c(j_3+1)y^{-b}}, \tag{14}$$

where,

$$\Delta_{j_1,j_2,j_3} = \sum_{j_3=0}^{\infty} \binom{2j_2+1}{j_3}\binom{\delta(2j_1+1)-1}{j_2} \frac{(-1)^{j_1+j_2+j_3}\,\pi^{2j_1+1}}{2j_1!\,2^{2j_1}}.$$

Moments measures

Moments are fundamental for characterizing probability distributions. They can be used to determine key properties such as the mean, variance (σ^2), skewness, and kurtosis. Hence, we determine the STL-FRD's kth moment. The kth moment of STL-FRD has PDF (14) with parameters $c > 0, b > 0, \delta > 0$ is given by

$$\mu'_k = \int_0^{\infty} y^k f(y;\boldsymbol{\Phi})dy = \sum_{j_1,j_2,j_3=0}^{\infty} \Delta_{j_1,j_2,j_3}\, c\delta \int_0^{\infty} y^{k-(b+1)} e^{-c(j_3+1)y^{-b}} dy. \tag{15}$$

Using the transformation $z = c(j_3+1)\ y^{-b}$ in Eq. (15), and after simplification, the kth moment of the STL-FRD is given (for $b > k$) by:

$$\begin{aligned} \mu'_k &= \sum_{j_1,j_2,j_3=0}^{\infty} \Delta_{j_1,j_2,j_3}\, c\delta \int_0^{\infty} \left(\frac{z}{c(j_3+1)}\right)^{1-\frac{k}{b}} e^{-z} dz \\ &= \sum_{j_1,j_2,j_3=0}^{\infty} \frac{\Delta_{j_1,j_2,j_3}\,\delta c}{\left[c(j_3+1)\right]^{(1-k/b)}}\, \Gamma\left(1-\frac{k}{b}\right); \qquad b > k, \end{aligned} \tag{16}$$

where, $\Gamma(s) = \int_0^{\infty} v^{s-1} e^{-v} dv$ and $s = 1, 2, \ldots$ stands for the gamma function. Fig. 3 displays the 3D and contour plots of mean, variance, skewness, and kurtosis for the STL-FRD.

Another helpful method for producing the fundamental moments of any probability distribution is the MGF approach. Thus, the STL-FRD's expression is obtained (for $b > i$) as:

$$M_Y(t) = E(e^{ty}) = \sum_{i=0}^{\infty} \frac{t^i}{i!}\mu'_i = \sum_{i,j_1,j_2,j_3=0}^{\infty} \frac{t^i}{i!} \frac{\Delta_{(j_1,j_2,j_3)}\delta c}{\left[c(j_3+1)\right]^{\left(1-\frac{i}{b}\right)}}\, \Gamma\left(1-\tfrac{i}{b}\right); \quad b > i.$$

The kth INM of Y, if Y follows the STL-FRD, is given by

$$\mu'_y(s) = \int_0^{s} y^k f(y;\boldsymbol{\Phi})dy = \sum_{j_1,j_2,j_3=0}^{\infty} \Delta_{j_1,j_2,j_3}\, c\delta \int_0^{s} y^{k-(b+1)} e^{-c(j_3+1)y^{-b}} dy. \tag{17}$$

Using the transformation $z = c(j_3+1)\ y^{-b}$ in Eq. (17), and after simplification, the kth moment of the STL-FRD is given (for $b > k$) by:

$$\begin{aligned} \mu'_y(s) &= \sum_{j_1,j_2,j_3=0}^{\infty} \Delta_{j_1,j_2,j_3}\, c\delta \int_{c(j_3+1)}^{\infty} \left(\frac{z}{c(j_3+1)}\right)^{1-\frac{k}{b}} e^{-z} dz \\ &= \sum_{j_1,j_2,j_3=0}^{\infty} \frac{\Delta_{j_1,j_2,j_3}\,\delta c}{\left[c(j_3+1)\right]^{(1-k/b)}}\, \Gamma\left(1-\frac{k}{b}\right); \qquad b > k, \end{aligned}$$

where, $\Gamma(a,x) = \int_x^{\infty} y^{a-1} e^{-y} dy, a = 1, 2, \ldots$ is the upper incomplete gamma function. The Bonferroni and Lorenz curves, as well as mean deviations, are the primary uses of the first incomplete moment. In the fields of economics, reliability, demography, insurance, and medicine, these curves are extremely helpful.

Table 1 gives numerical values of some moment measures, such as mean(μ'_1), σ^2, standard deviation σ, skewness, kurtosis, and coefficient of variation (CV) for some chosen values of parameters. We conclude from this table that increasing δ or c leads to larger mean and σ^2 values, while the CV remains relatively stable. Across all parameter combinations, the skewness values (≈ 1.9–3.2) and the very high kurtosis (≈ 12–44) confirm that the STL-FRD is consistently right-skewed and strongly leptokurtic. This supports the analytical conclusion derived from the moment expressions in Eq. (15) and the MGF approach, indicating heavy-tailed behavior regardless of moderate changes in the shape parameter b.

Stress-strength reliability

The formula $R = P(Y_2 < Y_1)$ is defined as the probability that a component will complete a task under specified circumstances and at a given stress level, which is known as reliability under stress. The component fails when $Y_1 \le Y_2$, if Y_1 and Y_2 represent

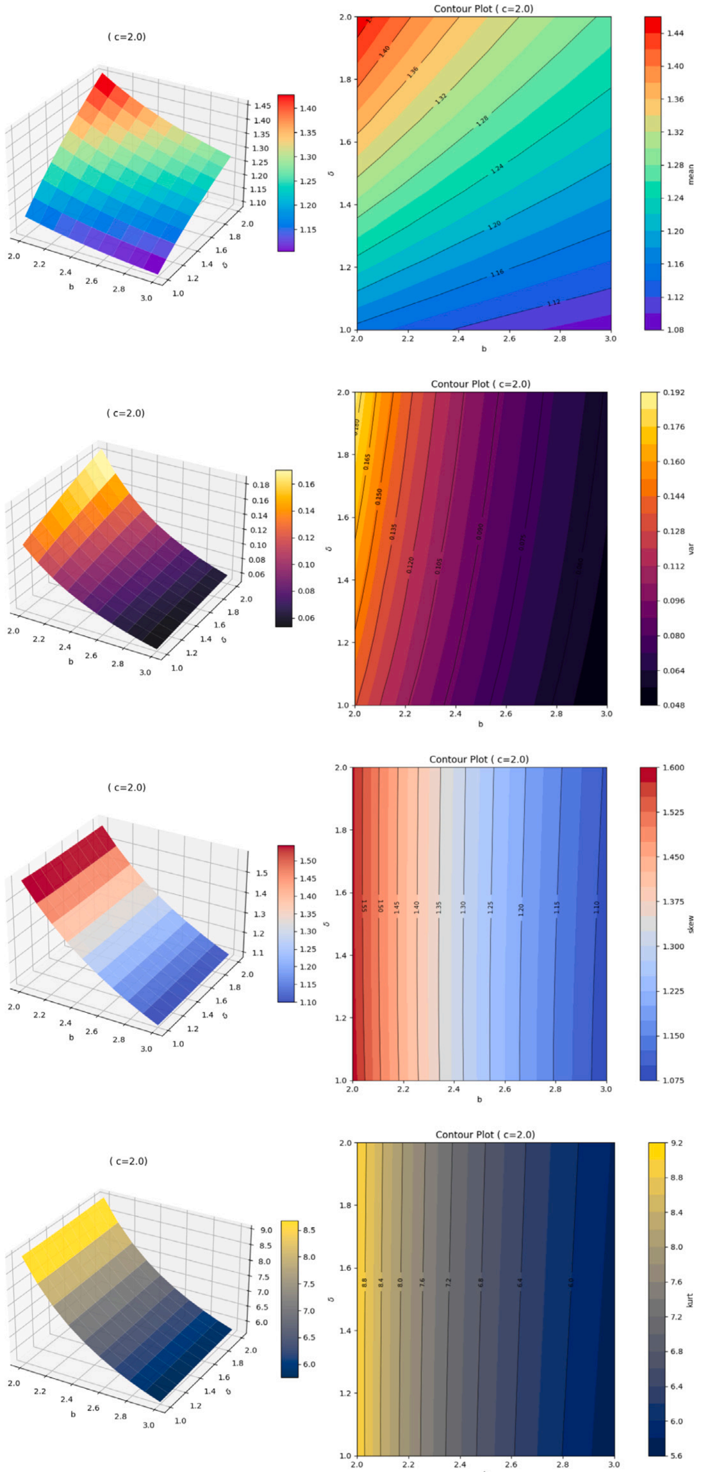

Fig. 3. The 3D and contour plots of mean, variance, skewness, and kurtosis for the STL-FRD at $c = 2$.

the component's strength and stress, respectively. The S-SR, R is estimated to be $P(Y_2 < Y_1)$. Suppose that the strength $Y_1 \sim$ STL-FRD(c_1, b, δ_1) and stress $Y_2 \sim$ STL-FRD(c_2, b, δ_2), then the S-SR is obtained as follows:

$$\begin{aligned} R = &\int_0^\infty \delta\pi c_1 b y^{-(b+1)} e^{-c_1 y^{-b}} (1 - e^{-c_1 y^{-b}}) \left(1 - (1 - e^{-c_1 y^{-b}})^2\right)^{\delta_1 - 1} \\ &\times \cos\left[\tfrac{\pi}{2}\left(1 - (1 - e^{-c_1 y^{-b}})^2\right)^{\delta_1}\right] \sin\left[\tfrac{\pi}{2}\left(1 - (1 - e^{-c_2 y^{-b}})^2\right)^{\delta_2}\right] dy. \end{aligned} \tag{18}$$

We apply expansion (3) and the subsequent expansion to solve Equation (18).

$$\sin(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)!} z^{2r+1},$$

Hence, the SSR is as follows:

$$\begin{aligned} R = &\sum_{l_1, l_2 = 0}^{\infty} \frac{(-1)^{l_1 + l_2} \delta\pi c_1 b}{(2l_2+1)! 2l_1!} \left(\frac{\pi}{2}\right)^{2l_1 + 2l_2 + 1} \int_0^\infty y^{-(b+1)} e^{-c_1 y^{-b}} \left(1 - e^{-c_1 y^{-b}}\right) \left(1 - \left[1 - e^{-c_1 y^{-b}}\right]^2\right)^{\delta_1(2l_1+1) - 1} dy \\ &\left(1 - \left[1 - e^{-c_2 y^{-b}}\right]^2\right)^{(2l_2+1)\delta_2}. \end{aligned} \tag{19}$$

Using the generalized binomial expansion (12) in last terms of (19) gives

$$\left(1 - \left[1 - e^{-c_1 y^{-b}}\right]^2\right)^{\delta_1(2l_1+1) - 1} = \sum_{l_3=0}^{\infty} (-1)^{l_3} \binom{\delta_1(2l_1+1) - 1}{l_3} \left[1 - e^{-c_1 y^{-b}}\right]^{2l_3}, \tag{20}$$

and

$$\left(1 - \left[1 - e^{-c_2 y^{-b}}\right]^2\right)^{\delta_2(2l_2+1)} = \sum_{l_4=0}^{\infty} (-1)^{l_4} \binom{\delta_2(2l_2+1)}{l_4} \left[1 - e^{-c_2 y^{-b}}\right]^{2l_4}. \tag{21}$$

Then by inserting Eqs. (20) and (21) in Eq. (19), we get

$$R = \sum_{l_1, l_2, l_3, l_4 = 0}^{\infty} B_{l_1, l_2, l_3, l_4} \int_0^\infty b\delta c_1 y^{-(b+1)} e^{-c_1 y^{-b}} \left[1 - e^{-c_2 y^{-b}}\right]^{2l_4} \left[1 - e^{-c_1 y^{-b}}\right]^{2l_3+1} dy, \tag{22}$$

where

$$B_{l_1, l_2, l_3, l_4} = \left(\tfrac{\pi}{2}\right)^{2l_1 + 2l_2 + 1} \binom{\delta_1(2l_1+1) - 1}{l_3} \binom{\delta_2(l_2+1)}{l_4} \frac{(-1)^{l_1 + l_2 + l_3 + l_4} \pi}{(2l_2+1)!\, 2l_1!}.$$

Using the binomial expansion in (22), then the S-SR of the STL-FRD is given by

$$R = \sum_{l_1, l_2, l_3, l_4, l_5, l_6 = 0}^{\infty} \frac{B^*_{l_m} \delta c_1}{c_1(1 + l_5) + c_2 l_6},$$

$$B^*_{l_m} = \sum_{l_5, l_6 = 0}^{\infty} \binom{2l_3 + 1}{l_5} \binom{2l_4}{l_6} (-1)^{l_5 + l_6} B_{l_1, l_2, l_3, l_4}, \quad m = 1, 2, 3, 4, 5, 6$$

Quantile function

The QF of a statistical distribution must be determined for a number of real-world uses, particularly for creating random numbers. We determine the QF of the STL-FRD using the inverse form of CDF (5), which is provided by

$$u = \sin\left[\tfrac{\pi}{2}\left(1 - \left(1 - e^{-cQ(u)^{-b}}\right)^2\right)^{\delta}\right], \quad 0 < u < 1. \tag{23}$$

After simplification, the QF of the STL-FRD is obtained as follows:

$$Q(u) = \left\{\tfrac{-1}{c} \ln\left[1 - \left(1 - \left(\tfrac{2}{\pi}\arcsin(u)\right)^{1/\delta}\right)^{0.5}\right]\right\}^{-1/b}, \quad 0 < u < 1. \tag{24}$$

Several distributional properties of the STL-FRD can be obtained by using the expression of QF given in Eq. (24). These distributional characteristics could include the following: (i) the median, also known as the second quartile, represented by $Q_{0.5}$, (ii) the first quartile, represented by $Q_{0.25}$, and (iii) the third quartile, represented by $Q_{0.75}$. The STL-FRD's additional distributional properties can be obtained by using these quartiles $(Q_{0.25}, Q_{0.5}, Q_{0.75})$. The following formula is used to create randomly generated data sets using the QF (25):

$$y_j = \left\{\tfrac{-1}{c} \ln\left[1 - \left(1 - \left(\tfrac{2}{\pi}\arcsin(u_j)\right)^{1/\delta}\right)^{0.5}\right]\right\}^{-1/b}, \quad j = 1, 2, \ldots, n. \tag{25}$$

Table 1
Results of moments and quantiles associated with the STL-FRD.

δ	c	b	Mean	σ^2	σ	skewness	kurtosis	CV	$Q_{0.25}$	$Q_{0.5}$	$Q_{0.75}$	$\ddot{S}$	$\ddot{K}$
0.8	3	1.2	1.1295	0.3495	0.5912	3.2360	44.3746	0.5234	0.7526	0.9929	1.3385	0.1797	1.3434
		1.5	1.0833	0.1850	0.4302	2.2324	16.7995	0.3971	0.7966	0.9943	1.2627	0.1516	1.3156
		1.7	1.0656	0.1336	0.3655	1.8982	12.2364	0.3430	0.8183	0.9950	1.2286	0.1384	1.3043
	3.5	1.2	1.2222	0.3982	0.6311	3.2255	44.1667	0.5163	0.8198	1.0769	1.4460	0.1790	1.3429
		1.5	1.1544	0.2046	0.4523	2.2305	16.8019	0.3918	0.8531	1.0610	1.3432	0.1514	1.3158
		1.7	1.1273	0.1456	0.3816	1.8988	12.2595	0.3385	0.8691	1.0537	1.2974	0.1382	1.3044
1.2	3	1.2	1.5835	0.6869	0.8288	3.2360	44.3309	0.5234	1.0552	1.3920	1.8766	0.1798	1.3433
		1.5	1.4195	0.3177	0.5637	2.2324	16.7995	0.3971	1.0439	1.3029	1.6546	0.1519	1.3158
		1.7	1.3526	0.2153	0.4640	1.8982	12.2364	0.3430	1.0386	1.2630	1.5595	0.1385	1.3042
	3.5	1.2	1.7136	0.7827	0.8847	3.2255	44.1208	0.5163	1.1494	1.5098	2.0273	0.1790	1.3429
		1.5	1.5127	0.3513	0.5927	2.2304	16.8019	0.3918	1.1178	1.3904	1.7601	0.1513	1.3157
		1.7	1.4309	0.2346	0.4844	1.8988	12.2595	0.3385	1.1033	1.3375	1.6469	0.1382	1.3045
1.5	3	1.2	1.9071	0.9964	0.9982	3.2360	44.3013	0.5234	1.2708	1.6765	2.2601	0.1799	1.3433
		1.5	1.6472	0.4278	0.6541	2.2324	16.7995	0.3971	1.2113	1.5119	1.9200	0.1518	1.3158
		1.7	1.5424	0.2799	0.5291	1.8982	12.2364	0.3430	1.1843	1.4401	1.7782	0.1385	1.3044
	3.5	1.2	2.0637	1.1354	1.0655	3.2255	44.0898	0.5163	1.3843	1.8183	2.4416	0.1790	1.3430
		1.5	1.7554	0.4730	0.6878	2.2304	16.8019	0.3918	1.2971	1.6134	2.0424	0.1513	1.3158
		1.7	1.6317	0.3050	0.5523	1.8988	12.2595	0.3385	1.2580	1.5251	1.8778	0.1382	1.3044

For the suggested STL-FRD, the Bowleys skewness [40] and Moors kurtosis [41] measures are given, respectively, by:

$$\ddot{S} = \frac{Q_{0.25} + Q_{0.75} - 2Q_{0.5}}{Q_{0.75} - Q_{0.25}}, \tag{26}$$

and,

$$\ddot{K} = \frac{(Q_{0.875} - Q_{0.625}) + (Q_{0.375} - Q_{0.125})}{Q_{0.75} - Q_{0.25}}. \tag{27}$$

Table 1 shows that all quartiles $Q_{0.25}, Q_{0.5}, Q_{0.75}$ increase systematically as δ they grow, confirming the direct scaling effect of this parameter on the distribution. The Bowley skewness ($\ddot{S} \approx 0.14 - 0.18$) remains small and positive in all settings, indicating a mild right skewness. Likewise, Moore's kurtosis ($\ddot{K} \approx 1.30 - 1.34$) is stable and slightly above 1, suggesting a distribution that is only moderately more peaked than the normal case. These quantile-based measures complement the classical moment results, reinforcing the conclusion that the STL-FRD retains a consistent, gently right-skewed shape with moderate tail thickness under different parameter combinations.

Estimation methods

This section outlines classical techniques for estimating the parameters b, c, and δ of the STL-FRD. The estimation process fundamentally involves optimizing an objective function, either through maximization or minimization, to derive efficient and statistically robust estimators.

Method of maximum likelihood

Maximum likelihood estimation [42] is the most prevalent technique for parameter estimation. It identifies the parameter values that maximize the likelihood function, which expresses the probability of observing the given sample data under the assumed STL-FRD. The appeal of this method lies in its well-established asymptotic properties, such as consistency and efficiency, making it a cornerstone of parametric inference.

Consider a simple random sample of size n drawn from the STL-FRD. The log-likelihood function for the model parameters is derived as:

$$\begin{aligned}\log L = (\delta - 1)\sum_{i=1}^{n} \log\left(1 - \left(e^{-cy_i^{-b}} - 1\right)^2\right) + \sum_{i=1}^{n} \log\left(\cos\left(\frac{1}{2}\pi\left(1 - \left(e^{-cy_i^{-b}} - 1\right)^2\right)^{\delta}\right)\right) - c\sum_{i=1}^{n} y_i^{-b} \\ + \sum_{i=1}^{n} \log\left(1 - e^{-cy_i^{-b}}\right) - (b+1)\sum_{i=1}^{n} \log\left(y_i\right) + n\log(\pi b c \delta).\end{aligned} \tag{28}$$

The complex, nonlinear form of Eq. (28) precludes an analytical solution via direct differentiation. Consequently, we resort to numerical optimization algorithms to compute the MLEs. The derivatives of Eq. (28) with respect to parameters δ, c, and b are determined as follows:

$$\frac{\partial l}{\partial \delta} = \frac{n}{\delta} - \sum_{i=1}^{n} \frac{1}{2}\pi\left(1 - \left(e^{-cy_i^{-b}} - 1\right)^2\right)^{\delta} \log\left(1 - \left(e^{-cy_i^{-b}} - 1\right)^2\right) \tan\left(\frac{1}{2}\pi\left(1 - \left(e^{-cy_i^{-b}} - 1\right)^2\right)^{\delta}\right)$$

$$+\sum_{i=1}^{n}\log\left(1-\left(e^{-cy_i^{-b}}-1\right)^2\right),$$

$$\frac{\partial l}{\partial c}=\frac{n}{c}+(\delta-1)\sum_{i=1}^{n}\frac{2y_i^{-b}e^{-2cy_i^{-b}}-2y_i^{-b}e^{-cy_i^{-b}}}{1-\left(e^{-cy_i^{-b}}-1\right)^2}+\sum_{i=1}^{n}\frac{y_i^{-b}e^{-cy_i^{-b}}}{1-e^{-cy_i^{-b}}}-\sum_{i=1}^{n}y_i^{-b}-\frac{1}{2}\pi\sum_{i=1}^{n}\Bigg(2\delta y_i^{-b}e^{-2cy_i^{-b}}$$
$$\left(1-\left(e^{-cy_i^{-b}}-1\right)^2\right)^{\delta-1}-2\delta y_i^{-b}e^{-cy_i^{-b}}\left(1-\left(e^{-cy_i^{-b}}-1\right)^2\right)^{\delta-1}\Bigg)\tan\left(\frac{1}{2}\pi\left(1-\left(e^{-cy_i^{-b}}-1\right)^2\right)^{\delta}\right),$$

$$\frac{\partial l}{\partial b}=\frac{n}{b}+c\sum_{i=1}^{n}y_i^{-b}\log\left(y_i\right)-\sum_{i=1}^{n}\log\left(y_i\right)+(\delta-1)\sum_{i=1}^{n}\frac{2cy_i^{-b}\log\left(y_i\right)e^{-cy_i^{-b}}-2cy_i^{-b}\log\left(y_i\right)e^{-2cy_i^{-b}}}{1-\left(e^{-cy_i^{-b}}-1\right)^2}$$
$$-\frac{1}{2}\pi\sum_{i=1}^{n}\Bigg(2c\delta y_i^{-b}\log\left(y_i\right)e^{-cy_i^{-b}}\left(1-\left(e^{-cy_i^{-b}}-1\right)^2\right)^{\delta-1}-2c\delta y_i^{-b}\log\left(y_i\right)e^{-2cy_i^{-b}}\left(1-\left(e^{-cy_i^{-b}}-1\right)^2\right)^{\delta-1}\Bigg)$$
$$\tan\left(\frac{1}{2}\pi\left(1-\left(e^{-cy_i^{-b}}-1\right)^2\right)^{\delta}\right)+\sum_{i=1}^{n}-\frac{cy_i^{-b}\log\left(y_i\right)e^{-cy_i^{-b}}}{1-e^{-cy_i^{-b}}}.$$

Anderson–Darling method

The AD method [43] is a minimum distance estimator known for its heightened sensitivity to discrepancies in the tails of a distribution. For the STL-FRD, we obtain estimators for b, c, and δ by minimizing the statistic:

$$A(\boldsymbol{y})=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\Delta_{i:n}+\log(1-\Delta_{n-i+1:n})\right],$$

where $\Delta_{i:n}=F(y_{i:n})=\sin\left(\frac{1}{2}\pi\left(1-\left(e^{-cy_{i:n}^{-b}}-1\right)^2\right)^{\delta}\right)$ is the CDF evaluated at the ith ordered sample.

Cramér–von Mises method

The CVM estimation [44] criterion minimizes an unweighted quadratic distance between the empirical and theoretical cumulative distribution functions, making it particularly effective for detecting deviations in the central region of the distribution. The estimators are derived by minimizing:

$$C(\boldsymbol{y})=\frac{1}{12n}+\sum_{i=1}^{n}\left(\Delta_{i:n}-\frac{2i-1}{2n}\right)^2.$$

Method of maximum product of spacings

The MPS estimation method was proposed by [45] and offers a robust alternative to the ML method, especially for distributions with complex boundaries. The technique maximizes the geometric mean of the spacings between consecutive order statistics. The objective function to maximize is:

$$\delta(\boldsymbol{y})=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\boldsymbol{\Phi}_i(\boldsymbol{y}),$$

where $\boldsymbol{\Phi}_i(\boldsymbol{y})=\Delta_{i:n}-\Delta_{i-1:n}$, with the conventions $\Delta_{0:n}=0$ and $\Delta_{n+1:n}=1$.

Ordinary and weighted least squares methods

The LS method [46] minimizes the sum of squared differences between the theoretical CDF and the empirical plotting positions:

$$LS(\boldsymbol{y})=\sum_{i=1}^{n}\left[\Delta_{i:n}-\frac{i}{n+1}\right]^2.$$

Its weighted version, called WLS [46] incorporates weights to account for the non-constant variance of the order statistics, yielding the minimization criterion:

$$WLS(\boldsymbol{y})=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\Delta_{i:n}-\frac{i}{n+1}\right]^2.$$

Percentile method

The percentile method [47] aligns empirical and theoretical quantiles. The estimators are found by minimizing the squared distance between the observed order statistics and the model quantiles:

$$PCE = \sum_{i=1}^{n} \left[y_{i:n} - Q(p_i) \right]^2, \quad \text{where } p_i = \frac{i}{n+1}.$$

Right and left tail Anderson–Darling methods

To emphasize specific tail behavior, two variants of the AD statistic are employed. The RTAD method [46] enhances sensitivity in the upper tail and is minimized:

$$R(\boldsymbol{y}) = \frac{n}{2} - 2\sum_{i=1}^{n} \Delta_{i:n} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log(1-\Delta_{i:n}).$$

Conversely, the LTAD method [48] focuses on the lower tail:

$$L(\boldsymbol{y}) = -\frac{3n}{2} + 2\sum_{i=1}^{n} \Delta_{i:n} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log(\Delta_{i:n}).$$

Minimum distance estimation methods based on spacings

A class of estimators minimizes different measures of discrepancy between the spacings $\boldsymbol{\Phi}_i(\boldsymbol{y})$ and their expected value $(n+1)^{-1}$. The MSAD method [49] minimizes absolute deviations:

$$\zeta(\boldsymbol{y}) = \sum_{i=1}^{n+1} \left| \boldsymbol{\Phi}_i(\boldsymbol{y}) - \frac{1}{n+1} \right|.$$

The MSALD method [49] applies a log transformation to these deviations:

$$\Upsilon(\boldsymbol{y}) = \sum_{i=1}^{n+1} \left| \log \boldsymbol{\Phi}_i(\boldsymbol{y}) - \log\left(\frac{1}{n+1}\right) \right|.$$

Similarly, MSSD and MSSLDE estimation methods are minimized:

$$\phi(\boldsymbol{y}) = \sum_{i=1}^{n+1} \left(\boldsymbol{\Phi}_i(\boldsymbol{y}) - \frac{1}{n+1} \right)^2,$$

$$\Psi(\boldsymbol{y}) = \sum_{i=1}^{n+1} \left(\log \boldsymbol{\Phi}_i(\boldsymbol{y}) - \log\left(\frac{1}{n+1}\right) \right)^2.$$

The MSLD estimation employs an asymmetric function:

$$\Lambda(\boldsymbol{y}) = \sum_{i=1}^{n+1} \left[\exp\left(\boldsymbol{\Phi}_i(\boldsymbol{y}) - \frac{1}{n+1} \right) - \left(\boldsymbol{\Phi}_i(\boldsymbol{y}) - \frac{1}{n+1} \right) - 1 \right].$$

Second-order left-tail Anderson–Darling and Kolmogorov methods

The ADSO estimation [50] refines left-tail analysis by incorporating a second-order term:

$$LTS = 2\sum_{i=1}^{n} \log \Delta_{i:n} + \frac{1}{n}\sum_{i=1}^{n} \frac{(2i-1)}{\Delta_{i:n}}.$$

Finally, the Kolmogorov estimation method [50] minimizes the supremum norm between the empirical and theoretical CDFs, providing a straightforward measure of the largest vertical discrepancy:

$$KM = \max_{1\le i\le n} \left\{ \max\left[\frac{i}{n} - \Delta_{i:n}, \Delta_{i:n} - \frac{i-1}{n} \right] \right\}.$$

Numerical simulation

This section presents a detailed evaluation of the parameter estimation strategies developed earlier in Section "Estimation Methods". To ground this investigation in empirical evidence, we generated datasets consistent with the underlying theoretical model, thereby enabling controlled experimentation with known parameter values. Our assessment relies on a Monte Carlo simulation framework using R software, which produces a thousand replicate datasets, each adhering to the prescribed distributional and structural properties of the model. The subsequent comparison of the estimation techniques is based on a multifaceted process assessed against five specific performance metrics. These criteria were selected to collectively provide a comprehensive profile of estimator quality, capturing essential properties such as

- Average bias (BIAS), $|Bias(\hat{\boldsymbol{J}})| = \frac{1}{L}\sum_{i=1}^{L}|\hat{\boldsymbol{J}} - \boldsymbol{J}|$, $\boldsymbol{J} = (b,\ c,\ \delta)$.
- Mean squared errors (MSE), $MSE(\hat{\boldsymbol{J}}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{\boldsymbol{J}} - \boldsymbol{J})^2$.
- Mean relative errors (MRE), $MRE(\hat{\boldsymbol{J}}) = \frac{1}{L}\sum_{i=1}^{L}|\hat{\boldsymbol{J}} - \boldsymbol{J}|/\boldsymbol{J}$.
- Average absolute difference (D_{abs}), $D_{abs} = \frac{1}{nH}\sum_{i=1}^{H}\sum_{j=1}^{n}|F(x_{ij};\boldsymbol{J}) - F(x_{ij};\hat{\boldsymbol{J}})|$.
- Maximum absolute difference (D_{max}), $D_{max} = \frac{1}{H}\sum_{i=1}^{H}\max_{j=1,\ldots,n}|F(x_{ij};\boldsymbol{J}) - F(x_{ij};\hat{\boldsymbol{J}})|$,
- Average squared absolute error, $ASAE = \frac{1}{n}\sum_{i=1}^{n}\frac{|x_i - \hat{x}_i|}{x_n - x_1}$, where x_i are the ascendingly ordered observations and $\hat{x}_i$ are obtained by $Q(i/n+1, \hat{\boldsymbol{J}})$.

This study employs Monte Carlo simulations to rigorously evaluate and identify the most effective parameter estimation method for the proposed theoretical model. The core of this approach involves repeatedly generating random samples under the model's assumptions, applying each estimation technique to these samples, and collecting the resulting estimates. A substantial number of these simulation replications were conducted to guarantee the statistical reliability and robustness of our conclusions regarding estimator performance. The outcomes of these experiments are systematically presented in Tables 5 to 9. To synthesize the multifaceted results into a clear hierarchy of methods, a formal ranking procedure was implemented. These results are consolidated in Table 2, which provides both rankings based on individual performance metrics and an overall ranking that integrates all evaluated criteria. For each measure and sample size, the estimators were ranked from 1 (best) to 16 (worst); the superscripts in the tables denote these ranks. The final overall performance for each scenario is summarized by the sum of these ranks ($\sum Ranks$), where a lower total indicates a superior overall estimator. A consistent pattern across all five simulation scenarios (Tables 5–9) is the expected improvement in performance for all estimators as the sample size n increases. The average measures generally decrease with larger n, confirming the asymptotic consistency of the methods.

Scenario 1 (Table 5: $b = 1.5, c = 0.5, \delta = 0.25$): The MSALDE performs exceptionally well for $n = 20$ ($\sum Ranks = 48^{\{1\}}$). As the sample size increases, the MLE, PCE, and MSADE become strong competitors. The KE estimator is the top performer for estimating parameter c across all sample sizes.

Scenario 2 (Table 6: $b = 0.75, c = 1.25, \delta = 0.9$): The MSADE and MSALDE are dominant for smaller samples ($n = 20$), with sums of ranks of $34^{\{1\}}$ and $60^{\{2\}}$, respectively. For larger samples ($n \geq 140$), the ADSOE and KE take the lead, with the ADSOE achieving the lowest overall sum of ranks for $n = 200$ ($\sum Ranks = 44.5^{\{2\}}$) and $n = 300$ ($\sum Ranks = 37^{\{1\}}$).

Scenario 3 (Table 7: $b = 0.4, c = 2.0, \delta = 1.5$): This scenario strongly favors the KE and ADSOE. The KE is ranked first overall for $n = 70$ ($\sum Ranks = 40^{\{1\}}$) and $n = 200$ ($\sum Ranks = 38^{\{1\}}$), while the ADSOE ranks first for $n = 140$ ($\sum Ranks = 29.5^{\{1\}}$). Their performance in estimating the relatively large c and δ parameters is outstanding.

Scenario 4 (Table 8: $b = 2.0, c = 1.5, \delta = 2.5$): The PCE shows the best performance for $n = 20$ ($\sum Ranks = 58^{\{1\}}$). For moderate to large samples ($n \geq 70$), the ADSOE and KE again demonstrate superior performance, with the ADSOE being the top performer for $n = 140$ ($\sum Ranks = 24^{\{1\}}$) and $n = 200$ ($\sum Ranks = 28^{\{1\}}$).

Scenario 5 (Table 9: $b = 2.5, c = 0.6, \delta = 2.0$): The ADSOE is consistently the best or among the best for all sample sizes, achieving the top rank for $n = 70, 140, 200$ ($\sum Ranks = 23^{\{1\}}, 56^{\{2\}}, 35^{\{1\}}$). The MSALDE also performs very well for larger n ($n = 140$: $\sum Ranks = 53^{\{1\}}$).

The final overall performance, summarized in Table 2, is determined by the sum of the ranks ($\sum$ Ranks) achieved in each individual scenario, with a lower sum indicating superior performance. The performance of traditional methods is mixed. The OLSE ($\sum$ Ranks = 171.5) and MPSE ($\sum$ Ranks = 178.0) demonstrate respectable, mid-tier performance. The often-used benchmark MLE ranks 11th overall ($\sum$ Ranks = 238.5), being consistently outperformed by several more modern techniques. Conversely, the MSSDE ($\sum$ Ranks = 367.0), MSLNDE ($\sum$ Ranks = 355.5), and PCE ($\sum$ Ranks = 258.0) methods consistently ranked among the least effective estimators in this study.

Based on the sum of ranks, the KE and ADSOE frequently emerge as the top overall performers, especially for smaller sample sizes. Their superiority is most pronounced in scenarios with smaller values of c and δ (Tables 5 and 7). For instance, in Table 7 ($b = 0.4, c = 2.0, \delta = 1.5$), the ADSOE achieves a remarkable total rank of $\sum Ranks = 29.5^{\{1\}}$ for $n = 140$, significantly lower than its competitors. The ADSOE and KE demonstrate remarkable consistency and efficiency across a wide range of scenarios. They consistently provide estimates with low bias and MSE, particularly for the scale parameter c, and often achieve the best overall ranking based on the sum of all performance criteria.

Applications

In this section, we show the effectiveness of the STL-FRD in a data-fitting scenario using two real-world datasets. Several competing models are compared to the STL-FRD, including cotangent-FRD (Cot-FRD) [51], odd exponential-logarithmic-FRD (OEL-FRD) [52], Topp-Leone-FRD (TL-FRD) [53], odd-generalized exponential-IWD (OGE-IWD) [54], generalized-IWD (G-IWD) [55], arctan-IWD (Art-IWD) [56], new arctan-FRD (NArt-FRD) [57], Weibull (WD) [58], modified Weibull (MWD) [59], Modified Topp-Leone (MTLD) [60], Sine-FRD (S-FRD) [39], Weibull-FRD (W-FRD) [61] and FRD.

The log-likelihood ($logLik$), parameter (Parm), Kolmogorov–Smirnov statistic (KS_stat) and its p-value (KS_p), AD statistic (AD_stat) and p-value (AD_p), CVM statistic (CVM_stat) and p-value (CVM_p), together with the Akaike information criterion (AIC), Bayesian information criterion (BIC), Consistent AIC ($CAIC$), and the Hannan–Quinn criterion ($HQIC$), constitute ten well-established measures of goodness-of-fit used to compare the candidate models. In this framework, the preferred model is the

Table 2
Partial and overall ranks for all estimation methods of our proposed model.

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
$b = 1.5,\ c = 0.5,\ \delta = 0.25$	20	5.5	9.5	9.5	5.5	12.0	4.0	7.0	13.5	8.0	3.0	1.0	15.0	2.0	16.0	11.0	13.5
	70	2.0	7.0	13.5	6.0	13.5	1.0	12.0	8.0	10.5	3.0	4.0	10.5	5.0	16.0	9.0	15.0
	140	1.0	5.0	13.0	7.0	15.0	6.0	12.0	10.0	3.0	8.0	4.0	11.0	2.0	16.0	9.0	14.0
	200	1.0	4.0	14.0	5.5	12.0	9.0	13.0	5.5	7.0	8.0	3.0	10.0	2.0	16.0	11.0	15.0
	300	2.0	3.5	15.0	5.0	12.0	11.0	14.0	9.0	6.0	7.0	3.5	10.0	1.0	16.0	8.0	13.0
$b = 0.75,\ c = 1.5,\ \delta = 0.9$	20	14.0	4.0	12.0	10.0	9.0	16.0	15.0	6.5	2.0	1.0	5.0	8.0	6.5	13.0	3.0	11.0
	70	8.5	5.0	13.0	7.0	12.0	16.0	14.0	6.0	2.0	4.0	1.0	10.0	3.0	11.0	8.5	15.0
	140	12.0	7.0	9.0	5.0	6.0	13.5	16.0	11.0	2.0	3.0	4.0	10.0	1.0	15.0	8.0	13.5
	200	12.0	8.0	6.0	5.0	7.0	13.5	16.0	10.0	1.0	3.0	4.0	11.0	2.0	13.5	9.0	15.0
	300	12.0	9.0	10.0	4.0	6.0	15.0	16.0	7.0	2.0	3.0	5.0	11.0	1.0	14.0	8.0	13.0
$b = 0.4,\ c = 2.0,\ \delta = 1.5$	20	11.0	12.0	10.0	9.0	7.0	14.0	6.0	4.0	13.0	2.0	1.0	5.0	3.0	16.0	8.0	15.0
	70	6.0	15.0	10.0	5.0	7.0	14.0	4.0	8.0	11.0	3.0	2.0	9.0	1.0	13.0	12.0	16.0
	140	9.0	10.0	7.0	5.0	4.0	14.0	6.0	11.0	13.0	2.0	3.0	12.0	1.0	15.0	8.0	16.0
	200	11.0	12.0	5.0	7.0	4.0	14.0	2.0	8.0	9.5	3.0	6.0	9.5	1.0	15.0	13.0	16.0
	300	8.5	12.0	4.0	8.5	2.0	15.0	3.0	5.0	10.0	6.0	7.0	11.0	1.0	14.0	13.0	16.0
$b = 2.0,\ c = 1.5,\ \delta = 2.5$	20	7.0	5.0	8.5	6.0	8.5	1.0	12.0	10.0	14.0	2.0	3.5	15.5	3.5	13.0	11.0	15.5
	70	11.0	8.0	5.0	9.0	3.0	6.0	10.0	4.0	13.0	7.0	2.0	16.0	1.0	14.5	12.0	14.5
	140	11.0	4.0	5.0	9.0	7.0	10.0	3.0	6.0	12.0	8.0	2.0	15.0	1.0	14.0	16.0	13.0
	200	10.0	4.0	2.0	12.0	3.0	8.0	9.0	6.0	7.0	5.0	11.0	16.0	1.0	15.0	13.0	14.0
	300	12.0	11.0	2.0	9.5	1.0	6.0	5.0	4.0	7.0	9.5	8.0	16.0	3.0	15.0	14.0	13.0
$b = 2.5,\ c = 0.6,\ \delta = 2.0$	20	16.0	7.0	15.0	5.0	9.5	14.0	11.0	8.0	6.0	3.0	2.0	4.0	1.0	12.0	13.0	9.5
	70	15.0	9.0	11.0	8.0	4.0	10.0	7.0	6.0	5.0	3.0	2.0	12.0	1.0	16.0	13.0	14.0
	140	13.5	10.0	8.0	7.0	4.0	9.0	3.0	6.0	13.5	5.0	1.0	11.0	2.0	16.0	12.0	15.0
	200	14.0	8.0	3.0	9.0	2.0	10.0	6.0	7.0	11.0	4.5	4.5	12.0	1.0	16.0	13.0	15.0
	300	13.5	11.0	4.0	9.0	1.0	8.0	3.0	5.0	10.0	7.0	6.0	12.0	2.0	16.0	13.5	15.0
$\sum$ Ranks		238.5	200.0	214.5	178.0	171.5	258.0	225.0	184.5	198.5	113.0	95.5	282.5	49.0	367.0	269.0	355.5
Overall Rank		11	8	9	5	4	12	10	6	7	3	2	14	1	16	13	15

one that yields higher p-values for KS, AD, and CVM (KS_p, AD_p, CVM_p), while simultaneously attaining lower values for KS_{stat}, AD_{stat}, CVM_{stat}, AIC, $CAIC$, BIC, and $HQIC$.

First Data Set: The dataset consists of the survival times (in days) of 73 patients diagnosed with acute bone cancer. This dataset has been recently analyzed by [62] to fit a new extended Weibull distribution. The recorded survival times range from 0.09 days to 86.01 days, with key descriptive statistics: first quartile 0.92, median 1.57, mean 3.75, and third quartile 2.75. The full set of observed survival times is:

0.09, 0.76, 1.81, 1.10, 3.72, 0.72, 2.49, 1.00, 0.53, 0.66, 31.61, 0.60, 0.20, 1.61, 1.88, 0.70,
1.36, 0.43, 3.16, 1.57, 4.93, 11.07, 1.63, 1.39, 4.54, 3.12, 86.01, 1.92, 0.92, 4.04, 1.16, 2.26,
0.20, 0.94, 1.82, 3.99, 1.46, 2.75, 1.38, 2.76, 1.86, 2.68, 1.76, 0.67, 1.29, 1.56, 2.83, 0.71,
1.48, 2.41, 0.66, 0.65, 2.36, 1.29, 13.75, 0.67, 3.70, 0.76, 3.63, 0.68, 2.65, 0.95, 2.30, 2.57,
0.61, 3.93, 1.56, 1.29, 9.94, 1.67, 1.42, 4.18, 1.30.

Second Data Set: The real dataset reported by [63] originates from endurance tests on deep-groove ball bearings. It records the number of million revolutions before failure for each of 23 bearings in the life test. The observed lifetimes range from 17.88 to 173.40 million revolutions. Descriptive statistics are as follows: first quartile 43.86, median 55.56, mean 67.87, and third quartile 88.62. The complete data are:

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56,
67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92,
128.04, 173.40.

Table 3 report the MLEs with standard errors (SEs) and goodness-of-fit statistics — including $logLik$, AIC, BIC, $CAIC$, $HQIC$, KS, CVM_stat, and AD_stat tests with their respective *p-values* for the STL-FRD and alternative distributions.

As summarized in Table 4, sixteen different estimation techniques were applied to the two real datasets in order to evaluate the empirical performance of the proposed STL-FRD model. The considered estimators include MLE, MPSE, OLSE, WLSE, CVME, ADE, RTADE, PCE, LTADE, MSADE, MSALDE, ADSOE, KE, MSSDE, MSSLDE, and MSLNDE. For each method, the model parameters were estimated and the corresponding Kolmogorov–Smirnov statistics, p-values, and overall ranks were computed.

The results for *Dataset (1)* reveal that the CVME method provides the best fit to the data, achieving the smallest KS statistic and the highest associated p-value among all competing estimators. In contrast, for *Dataset (2)*, the MLE method ranked first and exhibited superior goodness-of-fit performance relative to the remaining techniques.

These findings clearly indicate that the most suitable estimation method is not universal; rather, it depends strongly on the underlying characteristics of the dataset. Consequently, selecting the optimal estimator requires empirical evaluation, since the method that performs best in simulation may not necessarily be the most effective when applied to real-world data.

Visual comparisons are provided in Figs. 4–9. Figs. 4 and 7 show the estimated PDFs, and Figs. 5 and 8 show the estimated CDFs of the competing models for both datasets. Figs. 6 and 9 also show the probability-probability (PP) plots of the competing models for both datasets, showing the goodness of fit for our distribution to the observed data.

Table 3
MLEs with SEs, and goodness-of-fit statistics — including $logLik$, AIC, BIC, $CAIC$, $HQIC$, KS, CVM_stat, and AD_stat tests with their respective p-$values$ for the various fitted distributions.

Data	Name	logLik	AIC	BIC	CAIC	HQIC	KS_stat	KS_p	CVM_stat	CVM_p	AD_stat	AD_p	Parm	MLEs (SEs)
Data (1)	**STL-FRD**	**−142.3400**	**290.6800**	**297.5513**	**291.0278**	**293.4183**	**0.0915**	**0.5438**	**0.1189**	**0.5008**	**0.8796**	**0.4653**	$\hat{c}$ $\hat{b}$ $\hat{\delta}$	**1.2600 (1.5287)** **0.5290 (0.1094)** **2.1816 (3.8468)**
	Cot-FRD	−148.9896	301.9792	306.5601	302.1506	303.8048	0.1305	0.1523	0.3585	0.0930	2.2921	0.0297	$\hat{c}$ $\hat{b}$	1.0072 (0.1300) 0.9926 (0.0785)
	OEL-FRD	−144.8559	297.7119	306.8737	298.3001	301.3630	0.1206	0.2206	0.2253	0.2236	1.4585	0.1485	$\hat{c}$ $\hat{b}$ $\hat{\theta}$ $\hat{\lambda}$	0.3762 (0.0678) 45.9280 (53.3845) 0.1348 (0.1753) 9.6700 (6.9280)
	TL-FRD	−144.0110	294.0220	300.8933	294.3698	296.7603	0.1078	0.3401	0.1817	0.3064	1.2417	0.1881	$\hat{c}$ $\hat{b}$ $\hat{\delta}$	1.7361 (1.2453) 0.7457 (0.1011) 0.9534 (0.9436)
	FRD	−148.0624	300.1249	304.7058	300.2963	301.9504	0.1280	0.1677	0.3218	0.1173	2.0919	0.1188	$\hat{c}$ $\hat{b}$	1.0222 (0.1262) 0.9750 (0.0776)
	OGE-IWD	−146.0813	298.1626	305.0339	298.5104	300.9009	0.1118	0.2980	0.2396	0.2024	1.5722	0.1980	$\hat{\alpha}$ $\hat{\beta}$ $\hat{\lambda}$	0.3630 (0.0833) 3.2773 (0.8217) 9.4724 (7.8847)
	G-IWD	−148.0624	302.1249	308.9962	302.4727	304.8632	0.1280	0.1677	0.3218	0.1173	2.0919	0.0990	$\hat{\beta}$ $\hat{\lambda}$ $\hat{\alpha}$	0.9750 (0.0776) 1.0381 (65.8967) 0.9843 (64.0854)
	Art-IWD	−147.0758	298.1516	302.7325	298.3230	299.9771	0.1275	0.1706	0.2748	0.1595	1.8507	0.1386	$\hat{c}$ $\hat{b}$	1.3125 (0.1707) 0.8859 (0.0727)
	NArt-FRD	−146.6744	297.3488	301.9297	297.5202	299.1744	0.1214	0.2138	0.2728	0.1616	1.8114	0.1089	$\hat{c}$ $\hat{b}$	1.0221 (0.1192) 0.9554 (0.0765)
	S-FRD	−148.5337	301.0674	305.6483	301.2389	302.8930	0.1674	0.0295	0.3884	0.0773	2.1246	0.0693	$\hat{c}$ $\hat{b}$	1.3802 (0.1325) 0.8657 (0.0682)
	WD	−161.3673	326.7347	331.3156	326.9061	328.5602	0.1888	0.0094	0.7755	0.0082	4.6328	0.0297	$\hat{\theta}$ $\hat{\lambda}$	0.7654 (0.0567) 2.9246 (0.4760)
	W-FRD	−141.4312	290.8624	300.0242	291.4506	294.5136	0.0928	0.5263	0.1893	0.4121	0.9836	0.2376	$\hat{\theta}$ $\hat{\lambda}$ $\hat{c}$ $\hat{b}$	459.1972 (73.2813) 0.4817 (0.0764) 8.7128 (3.0177) 0.1637 (0.0934)
	MWD	−185.3947	376.7893	383.6607	377.1372	379.5277	0.3798	0.0000	2.9387	0.0000	13.9568	0.0099	$\hat{\theta}$ $\hat{\lambda}$ $\hat{\eta}$	0.6128 (0.1095) 0.8847 (0.2050) 0.0434 (0.0091)
	MTLD	−147.1911	296.3822	298.6727	296.4385	297.2950	0.0955	0.4887	0.1715	0.4387	0.9879	0.2277	$\hat{\beta}$	4.0523 (0.4743)
Data (2)	**STL-FRD**	**−112.0554**	**230.1109**	**233.5173**	**231.3740**	**230.9676**	**0.0905**	**0.9828**	**0.0298**	**0.9792**	**0.1991**	**0.9802**	$\hat{c}$ $\hat{b}$ $\hat{\delta}$	292.6774 (14.6529) 1.1630 (0.0985) 0.5655 (0.2779)
	Cot-FRD	−114.0265	232.0530	234.3240	232.6530	232.6241	0.1397	0.7091	0.0813	0.6893	0.5344	0.6931	$\hat{c}$ $\hat{b}$	1560.4700 (101.0283) 1.9310 (0.0618)
	OEL-FRD	−113.9803	235.9605	240.5025	238.1827	237.1028	0.1464	0.6545	0.0752	0.7248	0.5169	0.7228	$\hat{c}$ $\hat{b}$ $\hat{\theta}$ $\hat{\lambda}$	0.8756 (0.1838) 0.0402 (0.0751) 28.9897 (31.9328) 0.0036 (0.0016)
	TL-FRD	−112.7100	231.4200	234.8265	232.6831	232.2767	0.1091	0.9197	0.0456	0.9068	0.3052	0.9703	$\hat{c}$ $\hat{b}$ $\hat{\delta}$	1168.6753 (2.8826) 1.5462 (0.1184) 0.4213 (0.2326)
	FRD	−113.9065	231.8129	234.0839	232.4129	232.3841	0.1308	0.7796	0.0725	0.7414	0.4937	0.7129	$\hat{c}$ $\hat{b}$	1335.0081 (77.7892) 1.8850 (0.0595)
	OGE-IWD	−112.1218	230.2436	233.6501	231.5067	231.1003	0.1025	0.9490	0.0308	0.9684	0.2173	0.9797	$\hat{\alpha}$ $\hat{\beta}$ $\hat{\lambda}$	1.4671 (0.2694) 285.6063 (97.7382) 0.6978 (0.9158)
	G-IWD	−113.9065	233.8129	237.2194	235.0761	234.6697	0.1308	0.7796	0.0725	0.7414	0.4937	0.7327	$\hat{\beta}$ $\hat{\lambda}$ $\hat{\alpha}$	1.8850 (0.2817) 23.1277 (90.9714) 8.5977 (18.0653)
	Art-IWD	−113.9371	231.8742	234.1452	232.4742	232.4454	0.1181	0.8688	0.0637	0.7958	0.4653	0.7723	$\hat{c}$ $\hat{b}$	51.8579 (6.4272) 1.6864 (0.2607)
	NArt-FRD	−113.6769	231.3539	233.6248	231.9539	231.9250	0.1182	0.8679	0.0604	0.8163	0.4331	0.8020	$\hat{c}$ $\hat{b}$	1090.8459 (66.1029) 1.8294 (0.0572)
	S-FRD	−114.3040	232.6081	234.8791	233.2081	233.1792	0.1060	0.9343	0.0433	0.9198	0.3286	0.9208	$\hat{\theta}$ $\hat{b}$	371.0467 (151.9176) 1.4178 (0.1105)
	WD	−113.6920	231.3839	233.6549	231.9839	231.9551	0.1510	0.6170	0.0579	0.8319	0.3285	0.9208	$\hat{\alpha}$ $\hat{\beta}$	2.1018 (0.3285) 81.8746 (8.5629)
	W-FRD	−112.9256	233.8512	238.3932	236.0734	234.9935	0.1137	0.8949	0.0331	0.9676	0.2955	0.9527	$\hat{\theta}$ $\hat{\lambda}$ $\hat{c}$ $\hat{b}$	65.8753 (86.2346) 0.9700 (0.9393) 1.4940 (4.5159) 1.3913 (1.0228)
	MWD	−114.9676	235.9352	239.3417	237.1983	236.7919	0.1811	0.3909	0.0991	0.5932	0.5543	0.6832	$\hat{\alpha}$ $\hat{\beta}$ $\hat{\theta}$	0.0055 (0.2541) 10.1619 (2.0489) 2.0791 (95.2470)
	MTLD	−174.6975	351.3950	352.5305	351.5855	351.6806	0.8815	0.0000	6.7263	0.0000	58.7111	0.0099	$\hat{\alpha}$	69.7596 (12.6429)

Table 4
Parameter estimates and KS statistics for the STL-FRD model under different estimation methods for both datasets.

Data	Name	$\hat{c}$	$\hat{b}$	$\hat{\delta}$	KS_stat	KS_p	Rank
Data (1)	MLE	1.2600	0.5290	2.1816	0.0915	0.5438	11
	MPSE	1.0583	0.5756	3.1771	0.0788	0.7546	7
	OLSE	1.9764	0.6668	1.2438	0.0728	0.8344	2
	WLSE	1.8446	0.6621	1.3665	0.0732	0.8289	3
	CVME	2.3355	0.6973	1.0041	0.0706	0.8600	1
	ADE	1.3910	0.6048	1.9882	0.0787	0.7571	6
	RTADE	0.5616	0.5557	8.2078	0.0774	0.7737	5
	PCE	0.4790	0.2137	3.8085	0.6262	0.0000	15
	LTADE	2.1115	0.6321	1.1195	0.0736	0.8243	4
	MSADE	0.8432	0.6324	5.4043	0.1483	0.0808	13
	MSALDE	1.4449	0.6300	2.2070	0.1139	0.2999	12
	ADSOE	2.0940	0.4672	0.6692	0.3317	0.0000	14
	KE	0.0444	0.8315	2.5710	0.9817	0.0000	16
	MSSDE	2.4470	0.7563	0.9574	0.0807	0.7288	9
	MSSLDE	1.3356	0.5742	2.2334	0.0842	0.6792	10
	MSLNDE	2.5247	0.7599	0.9179	0.0800	0.7380	8
Data (2)	MLE	292.6774	1.1630	0.5655	0.0905	0.9828	1
	MPSE	225.0515	0.8269	8.7976	0.9840	0.0000	16
	OLSE	243.5065	1.1062	0.6054	0.0995	0.9751	10
	WLSE	228.7665	1.1056	0.6528	0.0993	0.9780	8
	CVME	229.8999	1.1390	0.7801	0.0970	0.9815	3
	ADE	220.0815	1.1237	0.7552	0.0985	0.9803	5
	RTADE	219.5731	1.1488	0.8789	0.0910	0.9822	2
	PCE	283.0147	1.5304	0.3712	0.9814	0.0000	15
	LTADE	218.9734	1.0956	0.6611	0.0986	0.9789	6
	MSADE	195.8633	1.3903	0.4009	0.1027	0.9749	11
	MSALDE	235.6220	1.1204	0.6183	0.0982	0.9809	4
	ADSOE	35.0867	0.7914	0.9351	0.3474	0.0078	13
	KE	196.8872	1.7491	0.2868	0.9565	0.0000	14
	MSSDE	219.3823	1.2017	0.6613	0.0988	0.9782	7
	MSSLDE	220.3871	1.1242	0.4623	0.0994	0.9773	9
	MSLNDE	223.9173	1.0812	0.7039	0.1162	0.9744	12

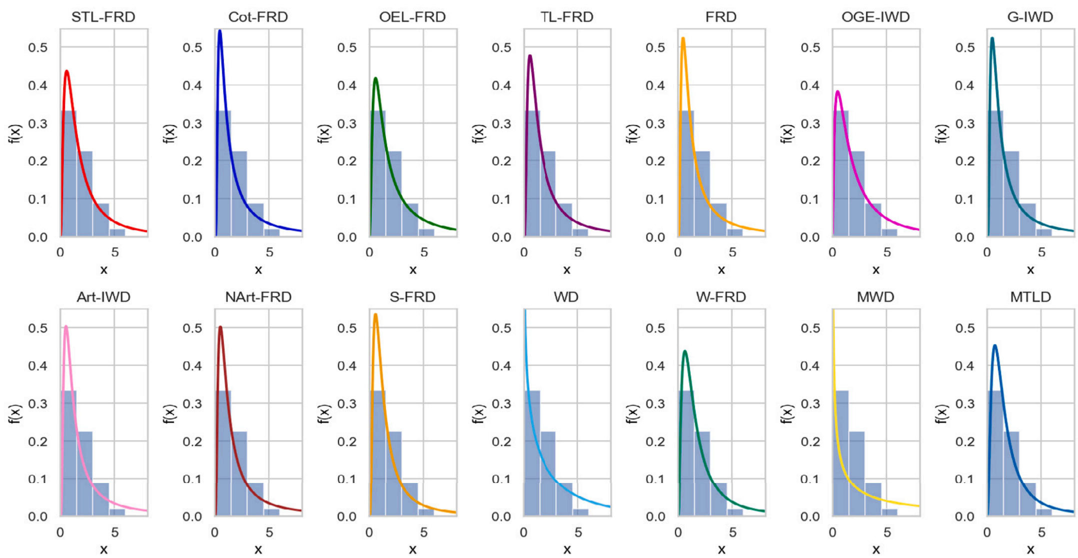

Fig. 4. Estimated PDFs for the competing models — First dataset.

The estimated parameters of the STL-FRD model provide insightful interpretations for the two distinct datasets under consideration.

Bone Cancer Survival Data For the bone cancer survival dataset, the estimated shape parameter $b = 0.529$ indicates a decreasing hazard rate over time. This aligns with clinical observations, where patients experience the highest mortality risk shortly after diagnosis, followed by a gradual reduction in hazard for those who survive the critical initial period. The scale parameter $c = 1.260$

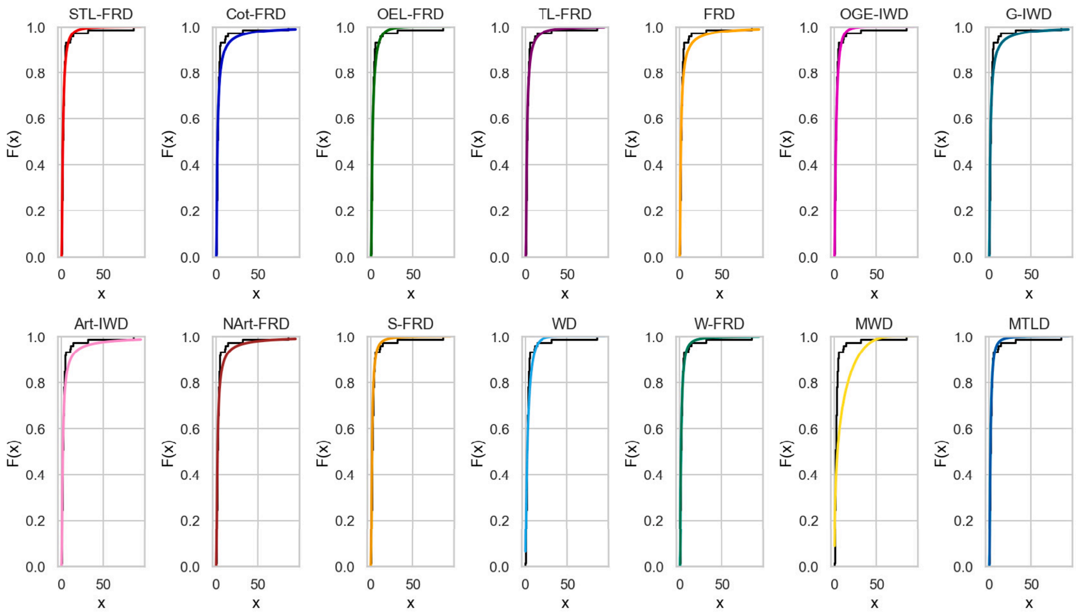

Fig. 5. Estimated CDFs for the competing models — First dataset.

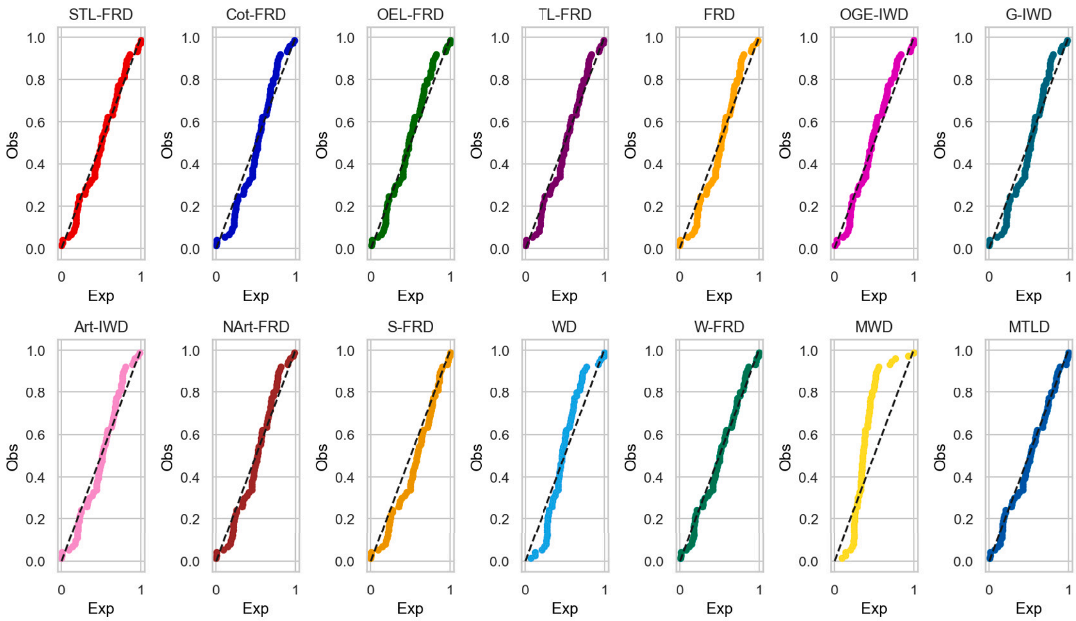

Fig. 6. PP plots or the competing models — First dataset.

reflects the moderate time scale of the survival distribution, while the flexibility parameter $\delta = 2.182$ allows the model to capture heavy-tailed behavior effectively, accounting for both early fatalities and long-term survivors. Collectively, these parameter estimates accurately describe the typical survival pattern in medical contexts, transitioning from high initial risk to stabilized long-term survival prospects.

Bearing Endurance Data In contrast, the bearing endurance dataset exhibits a classical mechanical wear-out failure mechanism. The shape parameter $b = 1.163$ signifies an increasing hazard rate over time, consistent with the gradual deterioration and accumulated damage experienced by mechanical components during extended operation. The large scale parameter $c = 292.677$

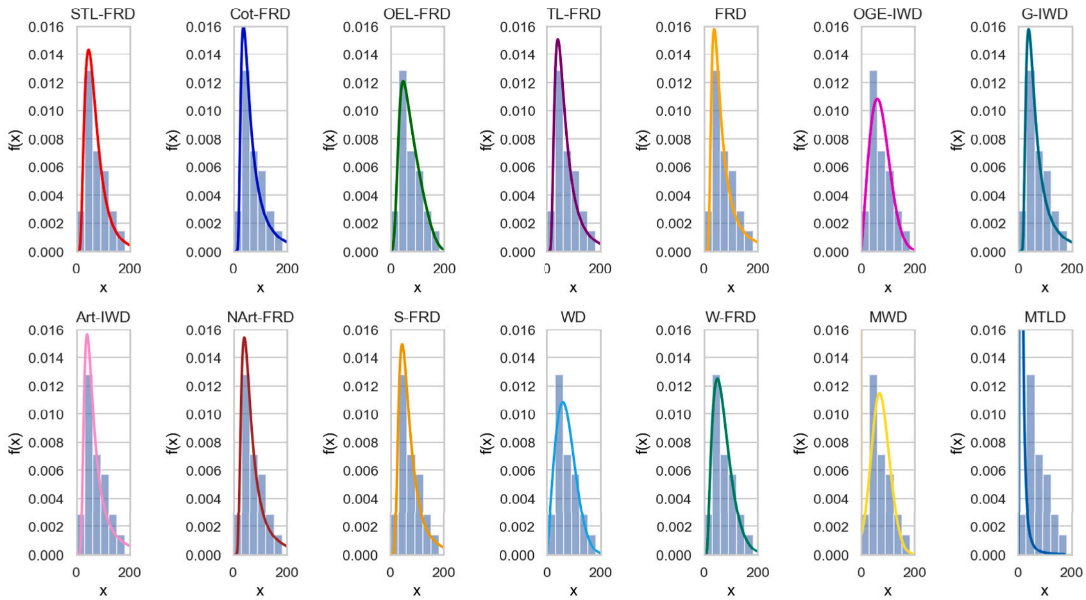

Fig. 7. Estimated PDFs for the competing models — Second dataset.

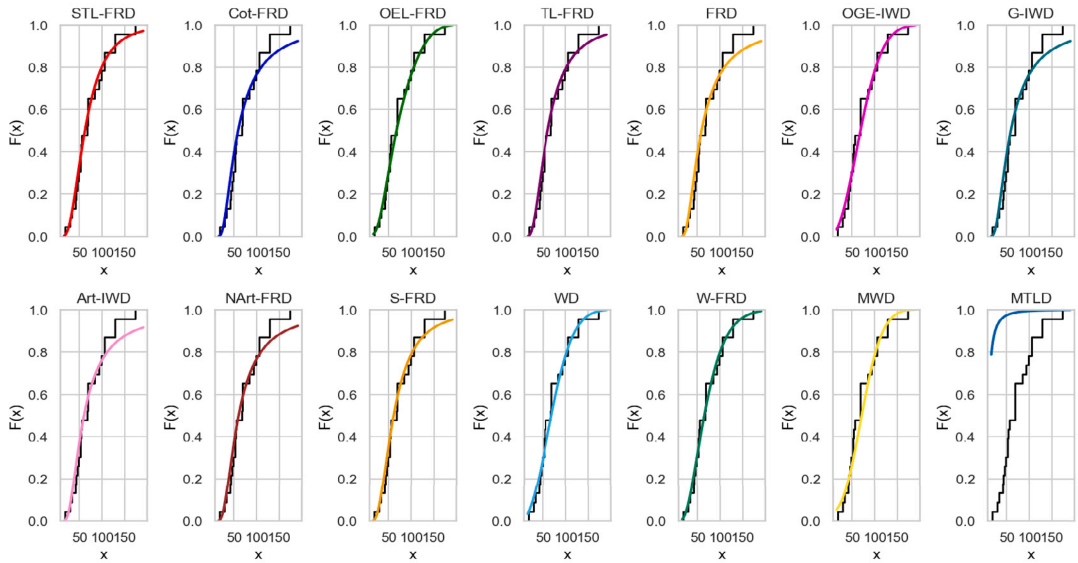

Fig. 8. Estimated CDFs for the competing models — Second dataset.

reflects the high-cycle fatigue nature of bearing failures, measured in millions of revolutions, while the flexibility parameter $\delta = 0.565$ enhances the model's accuracy in representing the early failure-free period and initial performance characteristics. This combination of parameters effectively captures the reliability behavior of bearings, showing consistent initial performance followed by progressive wear-out failure.

Overall Interpretation The contrasting parameter configurations between the two datasets highlight the STL-FRD model's adaptability to fundamentally different failure processes. It effectively distinguishes between the decreasing hazard pattern typical of biological systems and the increasing hazard pattern characteristic of mechanical systems. Furthermore, the flexibility of the model accommodates the unique characteristics of each application domain, providing both accurate statistical modeling and practical

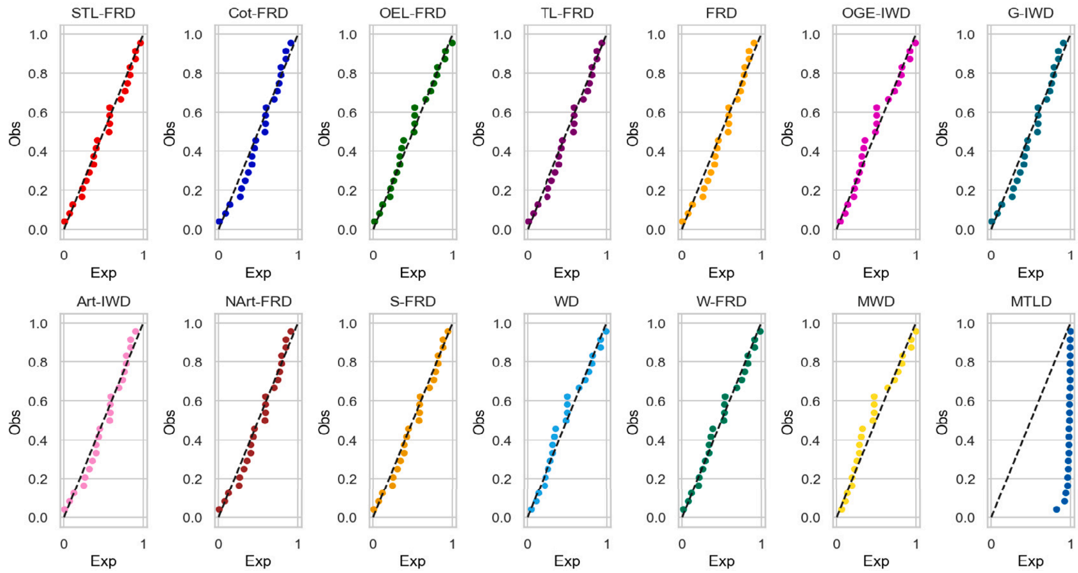

Fig. 9. PP plots or the competing models — Second dataset.

interpretability. These interpretive insights, supported by superior goodness-of-fit measures, establish the STL-FRD as a powerful tool for reliability analysis and survival modeling across diverse real-world datasets.

Conclusion

This paper introduces the STL-FRD, a new extension of the Fréchet distribution based on the STL-G family. The model can faithfully represent a broad range of hazard rate shapes, including increasing, decreasing, reversed J-shaped, and upside-down bathtub. Its probability density function may also be right-skewed, unimodal, or reversed J-shaped. For the proposed distribution, we provide several statistical properties, including QF, moments, MGF, expansion for PDF, S-SR, and INMs. This work presents a numerical and graphical analysis of several statistical features, including moments and quantiles. The use of both two- and three-dimensional plots enables a comprehensive visualization of these key statistical properties. We examined several distributional features of the proposed model and estimated its parameters using sixteen different estimation techniques. We use a Monte Carlo simulation to compare the effectiveness of different parameter estimates and assess the validity of the proposed model. In a range of scenarios, numerical analysis indicates that the KE and ADSOE demonstrate superior efficiency and consistency, especially for smaller sample sizes. They frequently receive the highest overall ranking based on the sum of all performance criteria and consistently produce estimates with low bias and MSE, especially for the scale parameter. The asymptotic consistency of the methods is confirmed by the average measures, which typically decrease as the sample size increases. Using two real-world datasets, we assessed the flexibility of the suggested distribution and demonstrated its exceptional ability to match the data perfectly. Our results show that, compared to other existing distributions, the proposed model has a higher quality of fit. The current study is limited to frequentist methods. Future research should explore Bayesian estimation of the STL-FRD parameters, particularly under different prior assumptions, and compare their performance with the classical estimates using Markov Chain Monte Carlo techniques. Investigate the parameter estimation for the STL-FRD under various censoring schemes common in reliability and survival analysis [64,65]. Apply the STL-FRD to model data in other scientific domains beyond the medical and engineering data showcased [66]. Construct a regression model based on the STL-FRD.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

See Tables 5–9.

Table 5
Numerical values of simulation measures for $b = 1.5,\ c = 0.5,\ \delta = 0.25$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{b}$)	0.28634 [9]	0.27072 [7]	0.29032 [10]	0.25418 [4]	0.34216 [14]	0.23543 [1]	0.24626 [3]	0.32885 [13]	0.3142 [11]	0.26726 [6]	0.2457 [2]	0.40988 [16]	0.28041 [8]	0.35285 [15]	0.26257 [5]	0.32233 [12]
20	BIAS($\hat{c}$)	0.17965 [6]	0.18816 [8]	0.17646 [3]	0.20392 [12]	0.22311 [16]	0.20516 [13]	0.19226 [10]	0.21635 [15]	0.179 [5]	0.17671 [4]	0.17367 [2]	0.19201 [9]	0.157 [1]	0.18621 [7]	0.2053 [14]	0.19684 [11]
20	BIAS($\hat{\delta}$)	0.11393 [4]	0.13366 [15]	0.12992 [12]	0.1238 [7]	0.12784 [8]	0.12145 [6]	0.13042 [13]	0.12986 [11]	0.12791 [9]	0.08336 [2]	0.10122 [3]	0.13395 [16]	0.0561 [1]	0.13311 [14]	0.12105 [5]	0.12808 [10]
20	MSE($\hat{b}$)	0.12853 [8]	0.12207 [7]	0.14001 [9]	0.09595 [3]	0.17444 [12]	0.08705 [1]	0.11746 [5]	0.19102 [14]	0.16343 [11]	0.11912 [6]	0.0879 [2]	0.26838 [16]	0.14003 [10]	0.19674 [15]	0.10794 [4]	0.17482 [13]
20	MSE($\hat{c}$)	0.04576 [4]	0.05002 [7]	0.04864 [6]	0.06283 [14]	0.07085 [16]	0.05618 [11]	0.05261 [8]	0.06469 [15]	0.0457 [3]	0.04545 [2]	0.04843 [5]	0.05726 [12]	0.03648 [1]	0.05469 [9]	0.0615 [13]	0.05582 [10]
20	MSE($\hat{\delta}$)	0.01627 [4]	0.02142 [15]	0.01998 [12]	0.01878 [8]	0.01934 [10]	0.01898 [9]	0.02039 [14]	0.0196 [11]	0.01874 [7]	0.0098 [2]	0.01413 [3]	0.01857 [6]	0.00471 [1]	0.02159 [16]	0.01778 [5]	0.02022 [13]
20	MRE($\hat{b}$)	0.19089 [9]	0.18048 [7]	0.19354 [10]	0.16945 [4]	0.2281 [14]	0.15695 [1]	0.16417 [3]	0.21923 [13]	0.20946 [11]	0.17818 [6]	0.1638 [2]	0.27325 [16]	0.18694 [8]	0.23524 [15]	0.17505 [5]	0.21489 [12]
20	MRE($\hat{c}$)	0.35929 [6]	0.37633 [8]	0.35292 [3]	0.40784 [12]	0.44622 [16]	0.41032 [13]	0.38452 [10]	0.4327 [15]	0.35799 [5]	0.35342 [4]	0.34734 [2]	0.38403 [9]	0.31401 [1]	0.37242 [7]	0.41061 [14]	0.39367 [11]
20	MRE($\hat{\delta}$)	0.45572 [4]	0.53466 [15]	0.51966 [12]	0.4952 [7]	0.51138 [8]	0.48578 [6]	0.52167 [13]	0.51945 [11]	0.51163 [9]	0.33345 [2]	0.40488 [3]	0.53581 [16]	0.22441 [1]	0.53244 [14]	0.48419 [5]	0.51233 [10]
20	D_{abs}	0.06236 [8]	0.05813 [5]	0.0627 [9]	0.05801 [4]	0.0607 [6]	0.05747 [3]	0.05285 [2]	0.06351 [12]	0.05265 [1]	0.06342 [11]	0.06325 [10]	0.06476 [13]	0.06229 [7]	0.07803 [15]	0.06649 [14]	0.08305 [16]
20	D_{max}	0.10195 [10]	0.09272 [4]	0.10163 [9]	0.09075 [3]	0.10024 [8]	0.09733 [5]	0.08655 [1]	0.1025 [12]	0.08657 [2]	0.09867 [7]	0.09755 [6]	0.11192 [14]	0.10229 [11]	0.12435 [15]	0.10565 [13]	0.12792 [16]
20	ASAE	0.04702 [9]	0.04064 [1]	0.04426 [4]	0.04205 [3]	0.04511 [5]	0.04552 [6]	0.0408 [2]	0.04593 [7]	0.04811 [11]	0.0547 [13]	0.04685 [8]	0.06287 [14]	0.04892 [12]	0.06811 [16]	0.04785 [10]	0.06419 [15]
20	$\sum Ranks$	81 [5.5]	99 [9.5]	99 [9.5]	81 [5.5]	133 [12]	75 [4]	84 [7]	149 [13.5]	85 [8]	65 [3]	48 [1]	157 [15]	62 [2]	158 [16]	107 [11]	149 [13.5]
70	BIAS($\hat{b}$)	0.13704 [7]	0.11888 [3]	0.17965 [15]	0.10724 [1]	0.1656 [13]	0.11361 [2]	0.14475 [8]	0.14594 [9]	0.1539 [10]	0.1258 [4]	0.12691 [5]	0.21849 [16]	0.15821 [12]	0.15446 [11]	0.12956 [6]	0.17732 [14]
70	BIAS($\hat{c}$)	0.14807 [3]	0.18387 [12]	0.16542 [7]	0.20082 [16]	0.17387 [8]	0.15641 [5]	0.1908 [14]	0.17413 [9]	0.17553 [11]	0.13625 [2]	0.16355 [6]	0.15217 [4]	0.10618 [1]	0.18534 [13]	0.1962 [15]	0.17493 [10]
70	BIAS($\hat{\delta}$)	0.09342 [5]	0.10705 [12]	0.11224 [14]	0.10564 [10]	0.11604 [15]	0.07681 [3]	0.12348 [16]	0.10289 [7]	0.10666 [11]	0.07526 [2]	0.07883 [4]	0.09631 [6]	0.03896 [1]	0.11087 [13]	0.10546 [9]	0.10531 [8]
70	MSE($\hat{b}$)	0.03066 [7]	0.02231 [3]	0.05064 [15]	0.01856 [2]	0.04295 [13]	0.01752 [1]	0.03095 [8]	0.033 [9]	0.03591 [10]	0.02591 [5]	0.0246 [4]	0.08344 [16]	0.0409 [12]	0.03894 [11]	0.02837 [6]	0.04808 [14]
70	MSE($\hat{c}$)	0.03608 [5]	0.04992 [12]	0.04315 [7]	0.06097 [16]	0.04805 [10]	0.03493 [3]	0.05203 [13]	0.04667 [9]	0.04298 [6]	0.02942 [2]	0.046 [8]	0.03553 [4]	0.01646 [1]	0.05267 [14]	0.05405 [15]	0.04894 [11]
70	MSE($\hat{\delta}$)	0.01066 [6]	0.01438 [12]	0.01587 [14]	0.01314 [8]	0.01608 [15]	0.0091 [4]	0.018 [16]	0.0128 [7]	0.01377 [10]	0.00807 [2]	0.00827 [3]	0.01012 [5]	0.00225 [1]	0.0155 [13]	0.01368 [9]	0.01423 [11]
70	MRE($\hat{b}$)	0.09136 [7]	0.07926 [3]	0.11977 [15]	0.07149 [1]	0.1104 [13]	0.07574 [2]	0.0965 [8]	0.09729 [9]	0.1026 [10]	0.08387 [4]	0.08461 [5]	0.14566 [16]	0.10547 [12]	0.10297 [11]	0.08638 [6]	0.11822 [14]
70	MRE($\hat{c}$)	0.29614 [3]	0.36775 [12]	0.33084 [7]	0.40614 [16]	0.34773 [8]	0.31282 [5]	0.38161 [14]	0.34826 [9]	0.35106 [11]	0.2725 [2]	0.3271 [6]	0.30435 [4]	0.21236 [1]	0.37011 [13]	0.3924 [15]	0.34985 [10]
70	MRE($\hat{\delta}$)	0.37369 [5]	0.42818 [12]	0.44898 [14]	0.42256 [10]	0.46414 [15]	0.30724 [3]	0.49393 [16]	0.41155 [7]	0.42665 [11]	0.30103 [2]	0.31532 [4]	0.38524 [6]	0.15582 [1]	0.44349 [13]	0.42183 [9]	0.42123 [8]
70	D_{abs}	0.02882 [2]	0.03077 [5]	0.03165 [8]	0.02844 [1]	0.03217 [10]	0.03088 [6]	0.02959 [3]	0.03368 [12]	0.03235 [11]	0.03727 [14]	0.0296 [4]	0.03632 [13]	0.0318 [9]	0.03958 [15]	0.03106 [7]	0.04357 [16]
70	D_{max}	0.04655 [2]	0.04927 [5]	0.05263 [10]	0.04458 [1]	0.05225 [9]	0.05218 [8]	0.04836 [4]	0.05384 [12]	0.05187 [7]	0.05864 [13]	0.04752 [3]	0.06133 [14]	0.05341 [11]	0.06268 [15]	0.05007 [6]	0.06949 [16]
70	ASAE	0.0181 [2]	0.01854 [4]	0.02077 [9]	0.01859 [5]	0.01961 [6]	0.02076 [8]	0.0171 [1]	0.01844 [3]	0.02094 [10]	0.02318 [12]	0.02358 [13]	0.02768 [14]	0.02049 [7]	0.02879 [15]	0.02147 [11]	0.02996 [16]
70	$\sum Ranks$	54 [2]	95 [7]	135 [13.5]	87 [6]	135 [13.5]	50 [1]	121 [12]	102 [8]	118 [10.5]	64 [3]	65 [4]	118 [10.5]	69 [5]	157 [16]	114 [9]	148 [15]
140	BIAS($\hat{b}$)	0.09468 [2]	0.09823 [7]	0.12015 [14]	0.09433 [1]	0.11731 [13]	0.10358 [9]	0.09609 [3]	0.10363 [10]	0.09615 [4]	0.10812 [11]	0.09622 [5]	0.15622 [16]	0.11002 [12]	0.12073 [15]	0.10182 [8]	0.09666 [6]
140	BIAS($\hat{c}$)	0.12411 [3]	0.15028 [9]	0.15497 [10]	0.1687 [12]	0.17165 [14]	0.13946 [5]	0.18969 [16]	0.15812 [11]	0.14884 [8]	0.13106 [4]	0.14005 [6]	0.11481 [2]	0.07762 [1]	0.16948 [13]	0.1435 [7]	0.17371 [15]
140	BIAS($\hat{\delta}$)	0.08958 [9]	0.08884 [6]	0.10199 [14]	0.1006 [13]	0.10842 [15]	0.06291 [2]	0.11148 [16]	0.09487 [12]	0.08935 [7]	0.07012 [3]	0.07071 [4]	0.0797 [5]	0.03663 [1]	0.09083 [11]	0.08937 [8]	0.08967 [10]
140	MSE($\hat{b}$)	0.01384 [3]	0.0143 [4]	0.02015 [13]	0.01474 [6]	0.02205 [14]	0.01558 [8]	0.01372 [2]	0.01676 [10]	0.01434 [5]	0.01668 [9]	0.01338 [1]	0.04029 [16]	0.01889 [12]	0.02295 [15]	0.01684 [11]	0.01515 [7]
140	MSE($\hat{c}$)	0.02244 [3]	0.03438 [9]	0.03851 [11]	0.03887 [12]	0.04565 [13]	0.02923 [5]	0.04991 [16]	0.03848 [10]	0.03337 [8]	0.02721 [4]	0.03301 [7]	0.0198 [2]	0.00881 [1]	0.04608 [14]	0.03212 [6]	0.04681 [15]
140	MSE($\hat{\delta}$)	0.01035 [8]	0.011 [11]	0.01306 [14]	0.01261 [13]	0.01416 [15]	0.00684 [2]	0.01472 [16]	0.01105 [12]	0.01027 [7]	0.00803 [5]	0.00689 [3]	0.00741 [4]	0.00189 [1]	0.01088 [10]	0.01038 [9]	0.0102 [6]
140	MRE($\hat{b}$)	0.06312 [2]	0.06549 [7]	0.0801 [14]	0.06288 [1]	0.07821 [13]	0.06905 [9]	0.06406 [3]	0.06909 [10]	0.0641 [4]	0.07208 [11]	0.06415 [5]	0.10415 [16]	0.07334 [12]	0.08048 [15]	0.06788 [8]	0.06444 [6]
140	MRE($\hat{c}$)	0.24822 [3]	0.30055 [9]	0.30995 [10]	0.3374 [12]	0.34675 [14]	0.27891 [5]	0.37877 [16]	0.31623 [11]	0.29768 [8]	0.26213 [4]	0.2801 [6]	0.22961 [2]	0.15524 [1]	0.33896 [13]	0.287 [7]	0.3493 [15]
140	MRE($\hat{\delta}$)	0.35834 [9]	0.35537 [6]	0.40796 [14]	0.40241 [13]	0.4337 [15]	0.25165 [2]	0.44592 [16]	0.37949 [12]	0.3574 [7]	0.28049 [3]	0.28283 [4]	0.31878 [5]	0.14654 [1]	0.36333 [11]	0.35747 [8]	0.35869 [10]
140	D_{abs}	0.02338 [6]	0.02323 [4]	0.02327 [5]	0.02203 [2]	0.02456 [9]	0.02577 [12]	0.02422 [7]	0.02168 [1]	0.02215 [3]	0.02664 [14]	0.02552 [11]	0.02605 [13]	0.02451 [8]	0.02865 [16]	0.02546 [10]	0.02832 [15]
140	D_{max}	0.03746 [4]	0.03765 [5]	0.03783 [6]	0.03524 [1]	0.04069 [10]	0.04307 [13]	0.04044 [9]	0.03578 [3]	0.03548 [2]	0.04306 [12]	0.04105 [11]	0.04349 [14]	0.03994 [7]	0.04658 [16]	0.04022 [8]	0.04482 [15]
140	ASAE	0.01237 [6]	0.01106 [2]	0.01267 [8]	0.0123 [5]	0.0117 [4]	0.01336 [10]	0.01067 [1]	0.01153 [3]	0.01269 [9]	0.01421 [13]	0.0139 [12]	0.01754 [16]	0.01265 [7]	0.01752 [15]	0.01389 [11]	0.01699 [14]
140	$\sum Ranks$	58 [1]	79 [5]	133 [13]	91 [7]	149 [15]	82 [6]	121 [12]	105 [10]	72 [3]	93 [8]	75 [4]	111 [11]	64 [2]	164 [16]	101 [9]	134 [14]
200	BIAS($\hat{b}$)	0.074 [2]	0.08758 [8]	0.10601 [16]	0.0706 [1]	0.09603 [14]	0.0943 [11]	0.09361 [10]	0.07905 [4]	0.08272 [5]	0.08919 [9]	0.07449 [3]	0.10139 [15]	0.08582 [7]	0.09487 [13]	0.08556 [6]	0.09432 [12]
200	BIAS($\hat{c}$)	0.10008 [2]	0.13477 [7]	0.15457 [10]	0.16466 [13]	0.16897 [15]	0.13449 [6]	0.16622 [14]	0.15714 [12]	0.13919 [8]	0.12283 [4]	0.12446 [5]	0.10642 [3]	0.05904 [1]	0.15559 [11]	0.13993 [9]	0.16951 [16]
200	BIAS($\hat{\delta}$)	0.06972 [5]	0.08703 [10]	0.10072 [15]	0.08842 [12]	0.09547 [14]	0.05774 [2]	0.11067 [16]	0.085 [9]	0.08216 [7]	0.06874 [3]	0.07035 [6]	0.06929 [4]	0.02814 [1]	0.09042 [13]	0.08497 [8]	0.0873 [11]
200	MSE($\hat{b}$)	0.00846 [2]	0.01096 [6]	0.01731 [15]	0.00777 [1]	0.01417 [13]	0.01345 [11]	0.01264 [10]	0.00901 [4]	0.0115 [8]	0.01244 [9]	0.00861 [3]	0.01736 [16]	0.01074 [5]	0.01382 [12]	0.01111 [7]	0.01464 [14]
200	MSE($\hat{c}$)	0.01612 [2]	0.02746 [6]	0.03684 [11]	0.03714 [12]	0.04517 [16]	0.0278 [7]	0.04017 [15]	0.03461 [10]	0.03037 [8]	0.02457 [4]	0.02479 [5]	0.01796 [3]	0.00594 [1]	0.04002 [14]	0.03086 [9]	0.03994 [13]
200	MSE($\hat{\delta}$)	0.00671 [5]	0.00871 [8]	0.01235 [16]	0.01004 [12]	0.0111 [14]	0.00534 [2]	0.01231 [15]	0.00863 [7]	0.00931 [10]	0.00698 [6]	0.00622 [4]	0.00579 [3]	0.00126 [1]	0.01051 [13]	0.00929 [9]	0.00992 [11]
200	MRE($\hat{b}$)	0.04933 [2]	0.05839 [8]	0.07067 [16]	0.04707 [1]	0.06402 [14]	0.06287 [11]	0.06241 [10]	0.0527 [4]	0.05514 [5]	0.05946 [9]	0.04966 [3]	0.06759 [15]	0.05722 [7]	0.06324 [13]	0.05704 [6]	0.06288 [12]
200	MRE($\hat{c}$)	0.20016 [2]	0.26954 [7]	0.30738 [10]	0.32933 [13]	0.3462 [16]	0.26897 [6]	0.33244 [14]	0.31428 [12]	0.27837 [8]	0.24566 [4]	0.24891 [5]	0.21284 [3]	0.11807 [1]	0.31118 [11]	0.27987 [9]	0.33902 [15]
200	MRE($\hat{\delta}$)	0.27889 [5]	0.35268 [11]	0.40536 [15]	0.35367 [12]	0.38187 [14]	0.23096 [2]	0.44563 [16]	0.34001 [9]	0.32865 [7]	0.27498 [3]	0.28064 [6]	0.27715 [4]	0.11257 [1]	0.36143 [13]	0.33987 [8]	0.34919 [10]
200	D_{abs}	0.01782 [5]	0.01757 [4]	0.01799 [6]	0.01728 [2]	0.01678 [1]	0.02085 [12]	0.01923 [8]	0.0173 [3]	0.01931 [9]	0.02326 [14]	0.02015 [11]	0.01998 [10]	0.01862 [7]	0.02563 [16]	0.02125 [13]	0.02478 [15]
200	D_{max}	0.0284 [5]	0.02816 [4]	0.03062 [7]	0.02791 [2]	0.02769 [1]	0.03561 [13]	0.03263 [11]	0.02797 [3]	0.03166 [8]	0.03665 [14]	0.03216 [9]	0.03257 [10]	0.0303 [6]	0.04108 [16]	0.0341 [12]	0.03975 [15]
200	ASAE	0.00955 [5.5]	0.00875 [1]	0.00955 [5.5]	0.00902 [3]	0.00947 [4]	0.01148 [11]	0.009 [2]	0.00966 [7]	0.0098 [8]	0.01275 [13]	0.01159 [12]	0.01323 [14]	0.01006 [9]	0.01352 [16]	0.01066 [10]	0.01329 [15]
200	$\sum Ranks$	42.5 [1]	80 [4]	142.5 [14]	84 [5.5]	136 [12]	94 [9]	141 [13]	84 [5.5]	91 [7]	92 [8]	72 [3]	100 [10]	47 [2]	161 [16]	106 [11]	159 [15]
300	BIAS($\hat{b}$)	0.06497 [2]	0.07343 [7]	0.0947 [15]	0.06513 [3]	0.08639 [12]	0.09248 [14]	0.09033 [13]	0.07697 [10]	0.07607 [9]	0.07532 [8]	0.06935 [6]	0.09988 [16]	0.06006 [1]	0.08155 [11]	0.0661 [4]	0.06797 [5]
300	BIAS($\hat{c}$)	0.09706 [3]	0.09866 [4]	0.15242 [14]	0.1285 [11]	0.16824 [16]	0.11776 [8]	0.14963 [12]	0.12552 [10]	0.10383 [6]	0.10156 [5]	0.11148 [7]	0.07129 [2]	0.05797 [1]	0.15397 [15]	0.12303 [9]	0.1523 [13]
300	BIAS($\hat{\delta}$)	0.06823 [5]	0.08121 [10]	0.10009 [15]	0.07753 [8]	0.08713 [13]	0.05739 [2]	0.10022 [16]	0.08283 [11]	0.0731 [7]	0.06061 [3]	0.07004 [6]	0.06446 [4]	0.02689 [1]	0.08989 [14]	0.07825 [9]	0.08465 [12]
300	MSE($\hat{b}$)	0.00683 [3]	0.00772 [8]	0.01262 [15]	0.00682 [2]	0.01165 [13]	0.01149 [12]	0.01185 [14]	0.00767 [7]	0.00899 [10]	0.00893 [9]	0.00699 [5]	0.01575 [16]	0.00584 [1]	0.01068 [11]	0.00686 [4]	0.0076 [6]
300	MSE($\hat{c}$)	0.01442 [3]	0.01493 [4]	0.03265 [13]	0.02313 [10]	0.04499 [16]	0.02467 [11]	0.03254 [12]	0.02174 [8]	0.01654 [6]	0.01571 [5]	0.01887 [7]	0.00805 [2]	0.00498 [1]	0.03607 [15]	0.02235 [9]	0.03489 [14]
300	MSE($\hat{\delta}$)	0.00667 [7]	0.00815 [10]	0.01188 [16]	0.00807 [9]	0.00912 [12]	0.00459 [4]	0.0115 [15]	0.00618 [6]	0.00802 [8]	0.00585 [5]	0.00426 [2]	0.00439 [3]	0.00113 [1]	0.00943 [13]	0.00861 [11]	0.00984 [14]
300	MRE($\hat{b}$)	0.04331 [2]	0.04895 [7]	0.06313 [15]	0.04342 [3]	0.0576 [12]	0.06188 [14]	0.06022 [13]	0.05255 [10]	0.05071 [9]	0.05021 [8]	0.04624 [6]	0.06658 [16]	0.04004 [1]	0.05437 [11]	0.04407 [4]	0.04531 [5]
300	MRE($\hat{c}$)	0.19412 [3]	0.19732 [4]	0.30415 [13]	0.257 [11]	0.34609 [16]	0.23551 [8]	0.29926 [12]	0.25104 [10]	0.20766 [6]	0.20313 [5]	0.22297 [7]	0.14258 [2]	0.11594 [1]	0.31013 [15]	0.24605 [9]	0.3046 [14]
300	MRE($\hat{\delta}$)	0.27292 [5]	0.32482 [10]	0.40036 [15]	0.31011 [8]	0.34853 [13]	0.22953 [2]	0.40089 [16]	0.33946 [12]	0.2924 [7]	0.24244 [3]	0.27762 [6]	0.25783 [4]	0.10755 [1]	0.35956 [14]	0.31301 [9]	0.33862 [11]
300	D_{abs}	0.01447 [2]	0.01532 [4]	0.01734 [11]	0.01366 [1]	0.01562 [6]	0.0185 [13]	0.01599 [7]	0.016 [8]	0.01526 [3]	0.01952 [15]	0.01561 [5]	0.01744 [12]	0.01658 [10]	0.0186 [14]	0.01636 [9]	0.02108 [16]
300	D_{max}	0.02274 [2]	0.02452 [3]	0.0282 [11]	0.02266 [1]	0.02654 [7]	0.03318 [15]	0.0269 [10]	0.02569 [6]	0.02483 [4]	0.03113 [14]	0.02509 [5]	0.02889 [12]	0.02656 [8]	0.0304 [13]	0.02666 [9]	0.03371 [16]
300	ASAE	0.00713 [6]	0.00652 [1]	0.00668 [2]	0.00733 [8]	0.00672 [3]	0.0083 [12]	0.00674 [4]	0.00696 [5]	0.00748 [9]	0.00941 [13]	0.00783 [10]	0.01001 [15]	0.00722 [7]	0.01043 [16]	0.00792 [11]	0.00993 [14]
300	$\sum Ranks$	43 [2]	72 [3.5]	155 [15]	75 [5]	139 [12]	115 [11]	144 [14]	103 [9]	84 [6]	93 [7]	72 [3.5]	104 [10]	34 [1]	162 [16]	97 [8]	140 [13]

Table 6
Numerical values of simulation measures for $b = 0.75,\ c = 1.25,\ \delta = 0.9$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{b}$)	0.15945 {13}	0.10914 {3}	0.16889 {14}	0.1154 {4}	0.12462 {7}	0.31037 {16}	0.18269 {15}	0.1259 {9}	0.097 {1}	0.10483 {2}	0.12123 {6}	0.13457 {10}	0.14204 {12}	0.13724 {11}	0.12007 {5}	0.12561 {8}
20	BIAS($\hat{c}$)	0.56905 {14}	0.44659 {10}	0.48184 {12}	0.46123 {11}	0.42155 {8}	0.65853 {16}	0.62237 {15}	0.38232 {7}	0.33393 {5}	0.12395 {1}	0.18155 {3}	0.25716 {4}	0.13862 {2}	0.44655 {9}	0.35539 {6}	0.50495 {13}
20	BIAS($\hat{\delta}$)	0.37837 {14}	0.27747 {9}	0.30867 {12}	0.2888 {11}	0.271 {8}	0.43496 {16}	0.40202 {15}	0.26519 {7}	0.21565 {5}	0.12168 {2}	0.18691 {4}	0.16211 {3}	0.1062 {1}	0.33411 {13}	0.25754 {6}	0.28469 {10}
20	MSE($\hat{b}$)	0.04672 {14}	0.02026 {3}	0.04592 {13}	0.02395 {7}	0.0266 {8}	0.12588 {16}	0.05616 {15}	0.02873 {9}	0.01489 {1}	0.019 {2}	0.02131 {4}	0.03001 {11}	0.03902 {12}	0.0291 {10}	0.02281 {6}	0.02157 {5}
20	MSE($\hat{c}$)	0.4539 {14}	0.29974 {10}	0.32732 {12}	0.32429 {11}	0.27393 {8}	0.56758 {16}	0.50329 {15}	0.23339 {7}	0.18479 {5}	0.0269 {1}	0.05527 {3}	0.11004 {4}	0.03034 {2}	0.28163 {9}	0.2039 {6}	0.37381 {13}
20	MSE($\hat{\delta}$)	0.17153 {14}	0.10283 {9}	0.11043 {11}	0.11474 {12}	0.08758 {6}	0.21421 {16}	0.18058 {15}	0.09387 {8}	0.06775 {5}	0.02533 {2}	0.05338 {4}	0.04566 {3}	0.02136 {1}	0.15657 {13}	0.0937 {7}	0.10461 {10}
20	MRE($\hat{b}$)	0.21261 {13}	0.14551 {3}	0.22518 {14}	0.15387 {4}	0.16616 {7}	0.41383 {16}	0.24359 {15}	0.16787 {9}	0.12934 {1}	0.13977 {2}	0.16165 {6}	0.17943 {10}	0.18939 {12}	0.18299 {11}	0.1601 {5}	0.16748 {8}
20	MRE($\hat{c}$)	0.45524 {14}	0.35727 {10}	0.38547 {12}	0.36898 {11}	0.33724 {8}	0.52683 {16}	0.4979 {15}	0.30586 {7}	0.26714 {5}	0.09916 {1}	0.14524 {3}	0.20573 {4}	0.11089 {2}	0.35724 {9}	0.28431 {6}	0.40396 {13}
20	MRE($\hat{\delta}$)	0.42041 {14}	0.3083 {9}	0.34297 {12}	0.32088 {11}	0.30112 {8}	0.48328 {16}	0.44668 {15}	0.29465 {7}	0.23961 {5}	0.1352 {2}	0.20768 {4}	0.18012 {3}	0.118 {1}	0.37123 {13}	0.28616 {6}	0.31632 {10}
20	D_{abs}	0.0614 {7}	0.05643 {2}	0.06291 {10}	0.05943 {5}	0.06207 {8}	0.06703 {13}	0.05245 {1}	0.05783 {3}	0.0659 {12}	0.05968 {6}	0.07153 {16}	0.0626 {9}	0.0649 {11}	0.06925 {14}	0.05824 {4}	0.07063 {15}
20	D_{max}	0.0989 {7}	0.08968 {1}	0.10679 {11}	0.09381 {4}	0.09921 {9}	0.12446 {16}	0.09048 {2}	0.09395 {5}	0.10128 {10}	0.09421 {6}	0.11312 {15}	0.09919 {8}	0.10701 {12}	0.11189 {13}	0.09097 {3}	0.11208 {14}
20	ASAE	0.04368 {3}	0.04564 {6}	0.04497 {5}	0.0508 {10}	0.04921 {9}	0.04298 {1}	0.04403 {4}	0.04364 {2}	0.05162 {11}	0.04744 {7}	0.04821 {8}	0.08222 {16}	0.05285 {12}	0.07299 {14}	0.05305 {13}	0.08109 {15}
20	$\sum Ranks$	141 {14}	75 {4}	138 {12}	101 {10}	94 {9}	174 {16}	142 {15}	80 {6.5}	66 {2}	34 {1}	76 {5}	85 {8}	80 {6.5}	139 {13}	73 {3}	134 {11}
70	BIAS($\hat{b}$)	0.06869 {5}	0.0621 {2}	0.0847 {13}	0.07062 {8}	0.08492 {14}	0.20708 {16}	0.12705 {15}	0.06917 {7}	0.05329 {1}	0.06913 {6}	0.06726 {4}	0.0811 {12}	0.07303 {9}	0.06706 {3}	0.07322 {10}	0.07593 {11}
70	BIAS($\hat{c}$)	0.3854 {10}	0.39057 {11}	0.47043 {14}	0.31176 {6}	0.39923 {12}	0.62213 {16}	0.61567 {15}	0.38025 {9}	0.22911 {5}	0.11909 {2}	0.18141 {3}	0.19683 {4}	0.08755 {1}	0.36962 {8}	0.32258 {7}	0.40143 {13}
70	BIAS($\hat{\delta}$)	0.27871 {11}	0.27574 {9}	0.3041 {14}	0.28087 {13}	0.26735 {8}	0.39395 {16}	0.37754 {15}	0.26259 {7}	0.16616 {4}	0.11969 {2}	0.181 {5}	0.15869 {3}	0.05683 {1}	0.27993 {12}	0.25587 {6}	0.27705 {10}
70	MSE($\hat{b}$)	0.00786 {8}	0.00636 {2}	0.01135 {13}	0.00774 {5}	0.0115 {14}	0.06661 {16}	0.02237 {15}	0.00785 {7}	0.0054 {1}	0.00777 {6}	0.00672 {3}	0.0095 {11}	0.00875 {9}	0.00714 {4}	0.00886 {10}	0.00957 {12}
70	MSE($\hat{c}$)	0.2473 {12}	0.22929 {9}	0.28231 {14}	0.17582 {6}	0.22445 {8}	0.55407 {16}	0.48756 {15}	0.23016 {10}	0.09158 {5}	0.02651 {2}	0.05392 {3}	0.06042 {4}	0.01163 {1}	0.24059 {11}	0.18806 {7}	0.2476 {13}
70	MSE($\hat{\delta}$)	0.10442 {11}	0.09459 {9}	0.10506 {12}	0.11166 {13}	0.08467 {6}	0.19295 {16}	0.16112 {15}	0.0915 {7}	0.04374 {3}	0.02473 {2}	0.05129 {5}	0.04551 {4}	0.00538 {1}	0.11252 {14}	0.0934 {8}	0.10329 {10}
70	MRE($\hat{b}$)	0.09158 {5}	0.0828 {2}	0.11294 {13}	0.09416 {8}	0.11323 {14}	0.27611 {16}	0.1694 {15}	0.09222 {7}	0.07105 {1}	0.09217 {6}	0.08968 {4}	0.10813 {12}	0.09738 {9}	0.08942 {3}	0.09763 {10}	0.10124 {11}
70	MRE($\hat{c}$)	0.30832 {10}	0.31246 {11}	0.37635 {14}	0.24941 {6}	0.31939 {12}	0.4977 {16}	0.49254 {15}	0.30508 {9}	0.18329 {5}	0.09887 {2}	0.14325 {3}	0.15746 {4}	0.07004 {1}	0.29569 {8}	0.25807 {7}	0.32114 {13}
70	MRE($\hat{\delta}$)	0.30968 {11}	0.30638 {9}	0.33789 {14}	0.31208 {13}	0.29706 {8}	0.43773 {16}	0.41949 {15}	0.29419 {7}	0.18462 {4}	0.13394 {2}	0.20111 {5}	0.17983 {3}	0.06315 {1}	0.31103 {12}	0.28538 {6}	0.30784 {10}
70	D_{abs}	0.03357 {8}	0.03192 {4}	0.03148 {3}	0.03004 {2}	0.03389 {10}	0.05035 {16}	0.02963 {1}	0.03289 {6}	0.03356 {7}	0.03809 {13}	0.03215 {5}	0.04033 {15}	0.03394 {11}	0.03722 {12}	0.03368 {9}	0.04027 {14}
70	D_{max}	0.05267 {5}	0.0512 {2}	0.05283 {6}	0.04929 {1}	0.05607 {10}	0.0921 {16}	0.05196 {4}	0.05354 {8}	0.0529 {7}	0.05998 {13}	0.05176 {3}	0.0641 {14}	0.05625 {11}	0.05931 {12}	0.05411 {9}	0.06554 {15}
70	ASAE	0.01744 {3}	0.01819 {4}	0.01943 {8}	0.01822 {5}	0.01838 {6}	0.01893 {7}	0.01717 {2}	0.01689 {1}	0.0221 {12}	0.02271 {13}	0.02206 {11}	0.03526 {16}	0.01977 {9}	0.02752 {14}	0.02116 {10}	0.02917 {15}
70	$\sum Ranks$	99 {8.5}	74 {5}	138 {13}	86 {7}	122 {12}	183 {16}	142 {14}	85 {6}	55 {2}	69 {4}	54 {1}	102 {10}	64 {3}	113 {11}	99 {8.5}	147 {15}
140	BIAS($\hat{b}$)	0.05443 {8}	0.04977 {6}	0.06924 {13}	0.04704 {5}	0.05233 {7}	0.08648 {16}	0.08331 {15}	0.0552 {9}	0.03786 {1}	0.04214 {3}	0.04601 {4}	0.07919 {14}	0.04146 {2}	0.06169 {12}	0.05984 {11}	0.0578 {10}
140	BIAS($\hat{c}$)	0.38434 {13}	0.388 {15}	0.35588 {9}	0.29979 {6}	0.37169 {11}	0.32372 {8}	0.58046 {16}	0.37526 {12}	0.22841 {5}	0.11903 {2}	0.18053 {3}	0.19579 {4}	0.06019 {1}	0.36567 {10}	0.31779 {7}	0.38517 {14}
140	BIAS($\hat{\delta}$)	0.27829 {12}	0.2676 {10}	0.25835 {7}	0.27843 {13}	0.25985 {8}	0.29473 {15}	0.36954 {16}	0.26223 {9}	0.16352 {4}	0.11852 {2}	0.18017 {5}	0.15525 {3}	0.04787 {1}	0.27844 {14}	0.25438 {6}	0.27704 {11}
140	MSE($\hat{b}$)	0.00489 {8}	0.00381 {6}	0.00724 {13}	0.00342 {5}	0.00457 {7}	0.01679 {16}	0.01036 {15}	0.00538 {9}	0.0023 {1}	0.00292 {3}	0.00321 {4}	0.00937 {14}	0.0026 {2}	0.00568 {10}	0.00569 {11}	0.00634 {12}
140	MSE($\hat{c}$)	0.24303 {15}	0.22065 {12}	0.18697 {9}	0.15155 {6}	0.19423 {10}	0.17792 {8}	0.44697 {16}	0.21048 {11}	0.08381 {5}	0.02334 {2}	0.05359 {3}	0.05947 {4}	0.0055 {1}	0.22488 {13}	0.17697 {7}	0.24251 {14}
140	MSE($\hat{\delta}$)	0.10153 {11}	0.0926 {9}	0.08205 {6}	0.10979 {13}	0.08284 {7}	0.12481 {15}	0.15577 {16}	0.0854 {8}	0.04008 {3}	0.02302 {2}	0.04945 {5}	0.04513 {4}	0.0036 {1}	0.11035 {14}	0.09273 {10}	0.10187 {12}
140	MRE($\hat{b}$)	0.07258 {8}	0.06637 {6}	0.09232 {13}	0.06272 {5}	0.06977 {7}	0.1153 {16}	0.11108 {15}	0.0736 {9}	0.05048 {1}	0.05618 {3}	0.06134 {4}	0.10785 {14}	0.05529 {2}	0.08225 {12}	0.07979 {11}	0.07706 {10}
140	MRE($\hat{c}$)	0.30809 {13}	0.3104 {15}	0.2847 {9}	0.23983 {6}	0.29735 {11}	0.25898 {8}	0.46437 {16}	0.30021 {12}	0.183 {5}	0.09824 {2}	0.14225 {3}	0.15543 {4}	0.04816 {1}	0.29253 {10}	0.25423 {7}	0.30814 {14}
140	MRE($\hat{\delta}$)	0.30694 {12}	0.29733 {10}	0.28705 {7}	0.30978 {13}	0.28872 {8}	0.32748 {15}	0.4106 {16}	0.29179 {9}	0.1815 {4}	0.13292 {2}	0.19714 {5}	0.17722 {3}	0.05319 {1}	0.30984 {14}	0.28297 {6}	0.30564 {11}
140	D_{abs}	0.02238 {5}	0.02108 {2}	0.02355 {7}	0.01962 {1}	0.02114 {3}	0.03336 {16}	0.02449 {9}	0.02436 {8}	0.02191 {4}	0.02556 {11}	0.02601 {12}	0.02867 {13}	0.02251 {6}	0.03002 {15}	0.02493 {10}	0.02939 {14}
140	D_{max}	0.0357 {5}	0.03438 {3}	0.03856 {7}	0.03109 {1}	0.03435 {2}	0.05856 {16}	0.04043 {10}	0.03916 {8}	0.03566 {4}	0.04052 {11}	0.04099 {12}	0.04766 {13}	0.03643 {6}	0.04831 {15}	0.04027 {9}	0.04825 {14}
140	ASAE	0.01145 {7}	0.01063 {3}	0.0108 {5}	0.0111 {6}	0.01198 {10}	0.00893 {1}	0.00987 {2}	0.0107 {4}	0.01315 {11}	0.01398 {12}	0.01463 {13}	0.02135 {16}	0.01149 {8}	0.01668 {15}	0.01157 {9}	0.01539 {14}
140	$\sum Ranks$	117 {12}	97 {7}	105 {9}	80 {5}	91 {6}	150 {13.5}	162 {16}	108 {11}	48 {2}	55 {3}	73 {4}	106 {10}	32 {1}	154 {15}	104 {8}	150 {13.5}
200	BIAS($\hat{b}$)	0.05317 {11}	0.04452 {6}	0.04807 {7}	0.03879 {3}	0.04996 {8}	0.08555 {16}	0.0759 {14}	0.05389 {12}	0.03506 {1}	0.03822 {2}	0.04195 {5}	0.07622 {15}	0.03948 {4}	0.05132 {9}	0.05486 {13}	0.05186 {10}
200	BIAS($\hat{c}$)	0.38382 {14}	0.3879 {15}	0.35278 {10}	0.29966 {6}	0.35667 {11}	0.32182 {8}	0.47246 {16}	0.37392 {13}	0.18817 {4}	0.1176 {2}	0.17375 {3}	0.19554 {5}	0.05638 {1}	0.326 {9}	0.31612 {7}	0.36528 {12}
200	BIAS($\hat{\delta}$)	0.27742 {14}	0.26544 {10}	0.25533 {8}	0.26776 {11}	0.2498 {6}	0.27386 {12}	0.31148 {16}	0.26151 {9}	0.15925 {4}	0.11699 {2}	0.1801 {5}	0.15508 {3}	0.04706 {1}	0.27806 {15}	0.25416 {7}	0.27703 {13}
200	MSE($\hat{b}$)	0.00371 {8}	0.00306 {6}	0.00347 {7}	0.00238 {2}	0.0041 {11}	0.01229 {16}	0.00892 {14}	0.00404 {9}	0.00184 {1}	0.00241 {3}	0.00257 {5}	0.00919 {15}	0.00252 {4}	0.00405 {10}	0.00452 {13}	0.0043 {12}
200	MSE($\hat{c}$)	0.24212 {15}	0.22028 {14}	0.18471 {9}	0.14991 {6}	0.18601 {10}	0.17765 {8}	0.35436 {16}	0.20987 {12}	0.05722 {4}	0.02271 {2}	0.05188 {3}	0.05845 {5}	0.00483 {1}	0.18606 {11}	0.17567 {7}	0.21671 {13}
200	MSE($\hat{\delta}$)	0.10059 {12}	0.09254 {10}	0.08107 {7}	0.10539 {13}	0.07935 {6}	0.11573 {15}	0.12518 {16}	0.08462 {8}	0.03968 {3}	0.02203 {2}	0.04891 {5}	0.04375 {4}	0.00331 {1}	0.10991 {14}	0.09178 {9}	0.09893 {11}
200	MRE($\hat{b}$)	0.07136 {11}	0.05935 {6}	0.06409 {7}	0.05171 {3}	0.06661 {8}	0.11305 {16}	0.1012 {14}	0.07186 {12}	0.04675 {1}	0.05097 {2}	0.05593 {5}	0.10693 {15}	0.05265 {4}	0.06843 {9}	0.07315 {13}	0.06915 {10}
200	MRE($\hat{c}$)	0.30647 {14}	0.30896 {15}	0.28217 {10}	0.23941 {6}	0.28534 {11}	0.25746 {8}	0.37797 {16}	0.29738 {13}	0.15053 {4}	0.0955 {2}	0.139 {3}	0.1541 {5}	0.0451 {1}	0.2608 {9}	0.25279 {7}	0.29222 {12}
200	MRE($\hat{\delta}$)	0.3059 {14}	0.29682 {10}	0.28501 {8}	0.29751 {11}	0.27755 {6}	0.30428 {12}	0.34609 {16}	0.29104 {9}	0.17695 {4}	0.13114 {2}	0.19468 {5}	0.17625 {3}	0.05097 {1}	0.30857 {15}	0.28133 {7}	0.30434 {13}
200	D_{abs}	0.01659 {3}	0.01654 {2}	0.01886 {5}	0.01644 {1}	0.01984 {7}	0.03315 {16}	0.02148 {11}	0.01925 {6}	0.01816 {4}	0.02359 {13}	0.02067 {10}	0.02512 {14}	0.02018 {8}	0.02513 {15}	0.02034 {9}	0.02217 {12}
200	D_{max}	0.02676 {3}	0.02666 {2}	0.03071 {5}	0.02622 {1}	0.0324 {7}	0.05672 {16}	0.03561 {11}	0.03133 {6}	0.02958 {4}	0.03674 {13}	0.03309 {10}	0.04215 {15}	0.03302 {9}	0.0403 {14}	0.03274 {8}	0.03596 {12}
200	ASAE	0.00802 {3}	0.00829 {4}	0.00899 {8}	0.01004 {13}	0.00874 {6}	0.00731 {1}	0.00743 {2}	0.0083 {5}	0.00891 {7}	0.00932 {9.5}	0.00972 {11}	0.01832 {16}	0.00932 {9.5}	0.0121 {14}	0.00982 {12}	0.01221 {15}
200	$\sum Ranks$	122 {12}	100 {8}	91 {6}	76 {5}	97 {7}	144 {13.5}	162 {16}	114 {10}	41 {1}	54.5 {3}	70 {4}	115 {11}	44.5 {2}	144 {13.5}	112 {9}	145 {15}
300	BIAS($\hat{b}$)	0.04582 {11}	0.04417 {10}	0.04753 {13}	0.03742 {3}	0.04036 {6}	0.08226 {16}	0.06757 {14}	0.03992 {5}	0.02507 {1}	0.03779 {4}	0.04172 {8}	0.07279 {15}	0.03737 {2}	0.04634 {12}	0.04071 {7}	0.04261 {9}
300	BIAS($\hat{c}$)	0.38221 {15}	0.36164 {12}	0.34715 {11}	0.29892 {6}	0.34535 {10}	0.3217 {8}	0.42007 {16}	0.37113 {14}	0.18696 {4}	0.11472 {2}	0.17346 {3}	0.19536 {5}	0.05576 {1}	0.32475 {9}	0.31595 {7}	0.36483 {13}
300	BIAS($\hat{\delta}$)	0.27685 {15}	0.26297 {10}	0.24466 {6}	0.26718 {11}	0.24768 {7}	0.2737 {12}	0.29566 {16}	0.25994 {9}	0.15822 {4}	0.11589 {2}	0.17664 {5}	0.15343 {3}	0.04591 {1}	0.27403 {13}	0.25228 {8}	0.27559 {14}
300	MSE($\hat{b}$)	0.00338 {12}	0.00288 {10}	0.00293 {11}	0.00223 {5}	0.00244 {6}	0.01082 {16}	0.00755 {15}	0.00256 {7}	0.00097 {2}	0.0022 {4}	3e-05 {1}	0.00727 {14}	0.00214 {3}	0.00348 {13}	0.00287 {9}	0.0027 {8}
300	MSE($\hat{c}$)	0.24079 {15}	0.18169 {12}	0.17542 {9}	0.14924 {6}	0.17555 {10}	0.17567 {11}	0.27482 {16}	0.20919 {13}	0.05532 {4}	0.02238 {2}	0.0498 {3}	0.05739 {5}	0.00423 {1}	0.16814 {7}	0.17528 {8}	0.21119 {14}
300	MSE($\hat{\delta}$)	0.09636 {11}	0.08514 {9}	0.07411 {6}	0.10362 {13}	0.07779 {7}	0.11257 {16}	0.1125 {15}	0.08444 {8}	0.03752 {3}	0.01954 {2}	0.0479 {5}	0.03997 {4}	0.00261 {1}	0.10828 {14}	0.08978 {10}	0.09857 {12}
300	MRE($\hat{b}$)	0.06109 {11}	0.0589 {10}	0.06298 {13}	0.05135 {4}	0.05381 {7}	0.11267 {16}	0.09009 {14}	0.05323 {6}	0.03343 {1}	0.05039 {3}	0.05164 {5}	0.09705 {15}	0.04983 {2}	0.06179 {12}	0.05428 {8}	0.05682 {9}
300	MRE($\hat{c}$)	0.30547 {15}	0.28931 {12}	0.27772 {11}	0.23913 {6}	0.27628 {10}	0.25641 {8}	0.33605 {16}	0.29497 {14}	0.14759 {4}	0.09446 {2}	0.13641 {3}	0.15293 {5}	0.04461 {1}	0.25915 {9}	0.25199 {7}	0.29186 {13}
300	MRE($\hat{\delta}$)	0.30545 {14}	0.29218 {10}	0.27185 {6}	0.29509 {11}	0.2752 {7}	0.30224 {12}	0.32851 {16}	0.28882 {9}	0.17591 {3}	0.13095 {2}	0.19375 {5}	0.17618 {4}	0.05032 {1}	0.30741 {15}	0.28017 {8}	0.3029 {13}
300	D_{abs}	0.01422 {1}	0.01571 {6}	0.01699 {11}	0.01454 {2}	0.01506 {5}	0.03214 {16}	0.01656 {9}	0.0148 {4}	0.01473 {3}	0.01911 {13}	0.01691 {10}	0.0232 {15}	0.01581 {7}	0.02004 {14}	0.01638 {8}	0.01809 {12}
300	D_{max}	0.02309 {1}	0.02555 {6}	0.02821 {11}	0.02327 {2}	0.02493 {5}	0.05398 {16}	0.02792 {10}	0.02444 {4}	0.02371 {3}	0.03062 {13}	0.02687 {9}	0.0385 {15}	0.02628 {8}	0.03272 {14}	0.02625 {7}	0.02966 {12}
300	ASAE	0.00584 {4}	0.00615 {6}	0.00639 {7}	0.00607 {5}	0.00658 {9}	0.00518 {1}	0.00554 {2}	0.00569 {3}	0.00703 {11}	0.00737 {13}	0.0065 {8}	0.01433 {16}	0.00686 {10}	0.00895 {15}	0.0072 {12}	0.00862 {14}
300	$\sum Ranks$	124 {12}	112 {9}	114 {10}	73 {4}	88 {6}	147 {15}	158 {16}	95 {7}	42 {2}	61 {3}	80 {5}	115 {11}	37 {1}	146 {14}	98 {8}	142 {13}

Table 7
Numerical values of simulation measures for $b = 0.4,\ c = 2.0,\ \delta = 1.5$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{b}$)	0.06417 [8]	0.06067 [4]	0.08032 [15]	0.05428 [3]	0.06801 [10]	0.22052 [16]	0.0704 [12]	0.05278 [2]	0.06779 [9]	0.06249 [6.5]	0.06086 [5]	0.06249 [6.5]	0.06841 [11]	0.07743 [13]	0.04881 [1]	0.07813 [14]
20	BIAS($\hat{c}$)	1.0271 [14]	1.03901 [16]	0.723 [7]	1.00189 [13]	0.88821 [9]	0.60506 [5]	0.51793 [4]	0.98117 [12]	1.03826 [15]	0.24208 [2]	0.35552 [3]	0.65637 [6]	0.23844 [1]	0.88563 [8]	0.97041 [11]	0.94895 [10]
20	BIAS($\hat{\delta}$)	0.62547 [12]	0.62599 [13]	0.46259 [6]	0.63151 [14]	0.50825 [7]	0.5554 [9]	0.3542 [4]	0.5984 [10]	0.54346 [8]	0.19719 [2]	0.29881 [3]	0.45959 [5]	0.19694 [1]	0.65454 [15]	0.65632 [16]	0.59869 [11]
20	MSE($\hat{b}$)	0.00643 [5]	0.00693 [7]	0.01098 [15]	0.00519 [3]	0.00867 [11]	0.05681 [16]	0.00924 [12]	0.00468 [2]	0.00776 [9.5]	0.00699 [8]	0.00576 [4]	0.0066 [6]	0.00776 [9.5]	0.0098 [14]	0.00366 [1]	0.00928 [13]
20	MSE($\hat{c}$)	1.45581 [16]	1.43518 [15]	0.9247 [7]	1.37256 [13]	1.2128 [9]	0.55625 [5]	0.54546 [4]	1.32683 [12]	1.37336 [14]	0.10253 [2]	0.28496 [3]	0.60337 [6]	0.10248 [1]	1.06919 [8]	1.22136 [10]	1.22923 [11]
20	MSE($\hat{\delta}$)	0.4721 [13]	0.45698 [12]	0.33237 [6]	0.48082 [14]	0.35256 [8]	0.44419 [10]	0.21642 [4]	0.43822 [9]	0.35209 [7]	0.08191 [2]	0.15544 [3]	0.24384 [5]	0.07894 [1]	0.50818 [16]	0.48525 [15]	0.44597 [11]
20	MRE($\hat{b}$)	0.16043 [8]	0.15166 [4]	0.20081 [15]	0.1357 [3]	0.17001 [10]	0.55131 [16]	0.176 [12]	0.13194 [2]	0.16946 [9]	0.15621 [6]	0.15214 [5]	0.15622 [7]	0.17103 [11]	0.19356 [13]	0.12202 [1]	0.19532 [14]
20	MRE($\hat{c}$)	0.51355 [14]	0.5195 [16]	0.3615 [7]	0.50095 [13]	0.44411 [9]	0.30253 [5]	0.25897 [4]	0.49059 [12]	0.51913 [15]	0.12104 [2]	0.17776 [3]	0.32818 [6]	0.11922 [1]	0.44282 [8]	0.4852 [11]	0.47447 [10]
20	MRE($\hat{\delta}$)	0.41698 [12]	0.41732 [13]	0.30839 [6]	0.42101 [14]	0.33883 [7]	0.37026 [9]	0.23614 [4]	0.39893 [10]	0.36231 [8]	0.13146 [2]	0.19921 [3]	0.3064 [5]	0.1313 [1]	0.43636 [15]	0.43755 [16]	0.39912 [11]
20	D_{abs}	0.0552 [4]	0.05509 [3]	0.06326 [7]	0.05584 [5]	0.05412 [1]	0.22947 [16]	0.06487 [12]	0.05485 [2]	0.06365 [9]	0.06947 [13]	0.06456 [10]	0.06345 [8]	0.06478 [11]	0.07321 [15]	0.0568 [6]	0.07035 [14]
20	D_{max}	0.0865 [2]	0.08727 [3]	0.10574 [10]	0.08638 [1]	0.08906 [6]	0.44959 [16]	0.1066 [12]	0.08819 [5]	0.10061 [7]	0.11027 [13]	0.10314 [9]	0.10214 [8]	0.10586 [11]	0.11772 [15]	0.08788 [4]	0.11488 [14]
20	ASAE	0.048 [4]	0.05227 [11]	0.0503 [6]	0.05156 [10]	0.04975 [5]	0.01381 [1]	0.03727 [2]	0.04219 [3]	0.05802 [12]	0.0504 [7]	0.0511 [9]	0.10266 [16]	0.05101 [8]	0.07617 [14]	0.06369 [13]	0.08873 [15]
20	$\sum Ranks$	112 [11]	117 [12]	107 [10]	106 [9]	92 [7]	124 [14]	86 [6]	81 [4]	122.5 [13]	65.5 [2]	60 [1]	84.5 [5]	67.5 [3]	154 [16]	105 [8]	148 [15]
70	BIAS($\hat{b}$)	0.03469 [5]	0.04212 [14]	0.04998 [15]	0.03188 [2]	0.03937 [12]	0.2119 [16]	0.03746 [7]	0.0362 [6]	0.03764 [8]	0.0384 [10]	0.02978 [1]	0.03803 [9]	0.03277 [3]	0.03997 [13]	0.0336 [4]	0.03904 [11]
70	BIAS($\hat{c}$)	0.79609 [12]	1.01035 [16]	0.52693 [7]	0.72535 [10]	0.40807 [5]	0.60426 [8]	0.40179 [4]	0.85034 [14]	0.85225 [15]	0.24047 [2]	0.35478 [3]	0.46093 [6]	0.13217 [1]	0.7249 [9]	0.83447 [13]	0.79324 [11]
70	BIAS($\hat{\delta}$)	0.50181 [9]	0.57886 [16]	0.32594 [5]	0.52796 [12]	0.2866 [3]	0.55514 [13]	0.3433 [6]	0.52555 [11]	0.49824 [8]	0.19656 [2]	0.29739 [4]	0.34715 [7]	0.09684 [1]	0.52257 [10]	0.56411 [15]	0.5625 [14]
70	MSE($\hat{b}$)	0.00207 [7]	0.00263 [11.5]	0.00369 [15]	0.0016 [2]	0.00267 [13]	0.05424 [16]	0.00263 [11.5]	0.00198 [5]	0.00232 [9]	0.00203 [6]	0.00137 [1]	0.00219 [8]	0.00185 [4]	0.00268 [14]	0.00173 [3]	0.00244 [10]
70	MSE($\hat{c}$)	1.03785 [14]	1.41269 [16]	0.60945 [8]	0.8409 [10]	0.38687 [6]	0.55455 [7]	0.27038 [3]	1.04089 [15]	1.00166 [13]	0.09967 [2]	0.28454 [4]	0.34053 [5]	0.04671 [1]	0.83179 [9]	0.99402 [12]	0.90305 [11]
70	MSE($\hat{\delta}$)	0.35492 [10]	0.41381 [15]	0.18118 [7]	0.37905 [12]	0.13619 [3]	0.44327 [16]	0.17845 [6]	0.3311 [9]	0.30619 [8]	0.08007 [2]	0.15474 [5]	0.14864 [4]	0.01593 [1]	0.36157 [11]	0.40212 [14]	0.38593 [13]
70	MRE($\hat{b}$)	0.08672 [5]	0.1053 [14]	0.12494 [15]	0.07969 [2]	0.09842 [12]	0.52976 [16]	0.09364 [7]	0.09049 [6]	0.0941 [8]	0.09599 [10]	0.07446 [1]	0.09507 [9]	0.08193 [3]	0.09992 [13]	0.08401 [4]	0.0976 [11]
70	MRE($\hat{c}$)	0.39805 [12]	0.50518 [16]	0.26346 [7]	0.36267 [10]	0.20403 [5]	0.30228 [8]	0.2009 [4]	0.42517 [14]	0.42612 [15]	0.12083 [2]	0.17759 [3]	0.23046 [6]	0.06609 [1]	0.36245 [9]	0.41724 [13]	0.39662 [11]
70	MRE($\hat{\delta}$)	0.33454 [9]	0.38591 [16]	0.21729 [5]	0.35197 [12]	0.19106 [3]	0.36865 [13]	0.22887 [6]	0.35037 [11]	0.33216 [8]	0.13132 [2]	0.19889 [4]	0.23143 [7]	0.06456 [1]	0.34838 [10]	0.37607 [15]	0.375 [14]
70	D_{abs}	0.02567 [1]	0.03168 [5]	0.03275 [6]	0.03017 [4]	0.03773 [11]	0.22888 [16]	0.03672 [10]	0.02751 [2]	0.02891 [3]	0.03885 [13]	0.0328 [7]	0.03807 [12]	0.03283 [8]	0.03933 [14]	0.03581 [9]	0.03996 [15]
70	D_{max}	0.0409 [1]	0.05042 [5]	0.05566 [8]	0.04859 [4]	0.06183 [12]	0.44848 [16]	0.05964 [10]	0.04538 [2]	0.04769 [3]	0.06355 [13]	0.0522 [6]	0.06181 [11]	0.05376 [7]	0.06404 [14]	0.05676 [9]	0.06445 [15]
70	ASAE	0.01509 [3]	0.01681 [5]	0.01793 [7]	0.01768 [6]	0.02008 [12]	0.00727 [1]	0.01461 [2]	0.01554 [4]	0.02233 [13]	0.01935 [10]	0.01971 [11]	0.04314 [16]	0.0193 [9]	0.02564 [15]	0.01877 [8]	0.02486 [14]
70	$\sum Ranks$	88 [6]	149.5 [15]	105 [10]	86 [5]	97 [7]	146 [14]	76.5 [4]	99 [8]	111 [11]	74 [3]	50 [2]	100 [9]	40 [1]	141 [13]	119 [12]	150 [16]
140	BIAS($\hat{b}$)	0.02628 [8]	0.03085 [11]	0.02754 [9]	0.02398 [4]	0.02615 [7]	0.21185 [16]	0.0279 [10]	0.03202 [14]	0.02597 [6]	0.02315 [3]	0.02273 [2]	0.03205 [15]	0.02104 [1]	0.03162 [13]	0.02464 [5]	0.03136 [12]
140	BIAS($\hat{c}$)	0.73764 [14]	0.70955 [11]	0.30873 [4]	0.61823 [9]	0.27038 [3]	0.603 [8]	0.33982 [5]	0.73305 [13]	0.78724 [15]	0.2377 [2]	0.35236 [6]	0.36 [7]	0.0725 [1]	0.7209 [12]	0.69122 [10]	0.79268 [16]
140	BIAS($\hat{\delta}$)	0.49401 [12]	0.48195 [11]	0.25708 [4]	0.45941 [9]	0.21115 [3]	0.5541 [15]	0.30794 [7]	0.45904 [8]	0.4595 [10]	0.19519 [2]	0.29736 [6]	0.2849 [5]	0.07086 [1]	0.52135 [13]	0.52876 [14]	0.56076 [16]
140	MSE($\hat{b}$)	0.00106 [7]	0.00143 [11]	0.00123 [9]	0.00089 [4]	0.00129 [10]	0.05161 [16]	0.0012 [8]	0.00167 [15]	0.00105 [6]	0.00086 [3]	0.00083 [1.5]	0.00154 [13]	0.00083 [1.5]	0.00165 [14]	0.00099 [5]	0.00146 [12]
140	MSE($\hat{c}$)	0.90206 [14]	0.81491 [11]	0.23204 [6]	0.63725 [9]	0.1744 [3]	0.55387 [8]	0.20403 [5]	0.91236 [15]	0.9363 [16]	0.09675 [2]	0.28049 [7]	0.19271 [4]	0.00924 [1]	0.82902 [12]	0.73802 [10]	0.90056 [13]
140	MSE($\hat{\delta}$)	0.34038 [12]	0.32109 [11]	0.12062 [5]	0.30572 [10]	0.08081 [3]	0.44185 [16]	0.15251 [7]	0.28834 [9]	0.27075 [8]	0.07887 [2]	0.15239 [6]	0.1069 [4]	0.00795 [1]	0.35874 [13]	0.38711 [15]	0.38469 [14]
140	MRE($\hat{b}$)	0.06569 [8]	0.07713 [11]	0.06885 [9]	0.05995 [4]	0.06537 [7]	0.52964 [16]	0.06975 [10]	0.08005 [14]	0.06494 [6]	0.05789 [3]	0.05682 [2]	0.08012 [15]	0.0526 [1]	0.07906 [13]	0.06161 [5]	0.07839 [12]
140	MRE($\hat{c}$)	0.36882 [14]	0.35477 [11]	0.15436 [4]	0.30912 [9]	0.13519 [3]	0.29982 [8]	0.16991 [5]	0.36653 [13]	0.39362 [15]	0.11885 [2]	0.17718 [6]	0.18 [7]	0.03625 [1]	0.35784 [12]	0.34561 [10]	0.39471 [16]
140	MRE($\hat{\delta}$)	0.32934 [12]	0.3213 [11]	0.17139 [4]	0.30627 [9]	0.14077 [3]	0.36852 [15]	0.20529 [7]	0.30603 [8]	0.30633 [10]	0.12994 [2]	0.19435 [6]	0.18994 [5]	0.04724 [1]	0.3445 [13]	0.35251 [14]	0.37329 [16]
140	D_{abs}	0.02205 [3]	0.02191 [2]	0.02585 [11]	0.02217 [4]	0.0252 [10]	0.22867 [16]	0.02405 [8]	0.02027 [1]	0.0247 [9]	0.02665 [13]	0.02337 [6]	0.02794 [14]	0.02245 [5]	0.02594 [12]	0.02387 [7]	0.02817 [15]
140	D_{max}	0.0349 [2]	0.0355 [4]	0.04252 [12]	0.03503 [3]	0.0411 [10]	0.44528 [16]	0.03995 [9]	0.03377 [1]	0.03893 [8]	0.04257 [13]	0.03764 [6]	0.04726 [15]	0.03614 [5]	0.04185 [11]	0.03818 [7]	0.04655 [14]
140	ASAE	0.00826 [2]	0.00953 [6]	0.01105 [9]	0.00873 [4]	0.01011 [8]	0.00543 [1]	0.00847 [3]	0.00954 [7]	0.01202 [12]	0.01145 [11]	0.01401 [15]	0.02654 [16]	0.01135 [10]	0.01254 [14]	0.00943 [5]	0.01216 [13]
140	$\sum Ranks$	108 [9]	111 [10]	86 [7]	78 [5]	70 [4]	151 [14]	84 [6]	118 [11]	121 [13]	58 [2]	69.5 [3]	120 [12]	29.5 [1]	152 [15]	107 [8]	169 [16]
200	BIAS($\hat{b}$)	0.02611 [12]	0.02606 [11]	0.0238 [7]	0.02254 [5]	0.02378 [6]	0.20004 [16]	0.02212 [4]	0.02743 [13]	0.02586 [10]	0.02171 [2]	0.02203 [3]	0.024 [8]	0.01953 [1]	0.02817 [15]	0.02446 [9]	0.02772 [14]
200	BIAS($\hat{c}$)	0.73533 [15]	0.63826 [12]	0.22916 [2]	0.61802 [11]	0.23796 [4]	0.60286 [9]	0.25546 [5]	0.57262 [8]	0.60492 [10]	0.23687 [3]	0.35086 [6]	0.35649 [7]	0.07224 [1]	0.72015 [14]	0.68314 [13]	0.7924 [16]
200	BIAS($\hat{\delta}$)	0.49399 [12]	0.4268 [10]	0.20019 [3]	0.45668 [11]	0.20784 [4]	0.55225 [16]	0.24155 [5]	0.39756 [9]	0.3913 [8]	0.19443 [2]	0.29668 [7]	0.28353 [6]	0.07084 [1]	0.51801 [13]	0.52824 [14]	0.53832 [15]
200	MSE($\hat{b}$)	0.00048 [2]	0.00112 [12]	9e-04 [8]	0.00046 [1]	0.00093 [9]	0.05024 [16]	0.00087 [6]	0.00117 [13]	0.00101 [11]	0.00075 [5]	0.00072 [4]	0.00089 [7]	0.00062 [3]	0.00126 [15]	0.00095 [10]	0.00121 [14]
200	MSE($\hat{c}$)	0.89809 [15]	0.74086 [13]	0.1247 [3]	0.63502 [11]	0.13555 [4]	0.55277 [8]	0.14051 [5]	0.60288 [10]	0.59384 [9]	0.09605 [2]	0.27893 [7]	0.19151 [6]	0.00796 [1]	0.82655 [14]	0.73718 [12]	0.89823 [16]
200	MSE($\hat{\delta}$)	0.33939 [12]	0.26157 [10]	0.07614 [2]	0.30229 [11]	0.07989 [4]	0.4379 [16]	0.0912 [5]	0.2237 [9]	0.20719 [8]	0.07882 [3]	0.14906 [7]	0.10637 [6]	0.00751 [1]	0.35045 [13]	0.38657 [15]	0.38318 [14]
200	MRE($\hat{b}$)	0.06536 [12]	0.06514 [11]	0.05949 [6]	0.05982 [7]	0.05945 [5]	0.50009 [16]	0.05531 [4]	0.06858 [13]	0.06465 [10]	0.05428 [2]	0.05507 [3]	0.06 [8]	0.04881 [1]	0.07044 [15]	0.0616 [9]	0.06931 [14]
200	MRE($\hat{c}$)	0.36613 [15]	0.31913 [12]	0.11458 [2]	0.30834 [11]	0.11898 [4]	0.29758 [9]	0.12773 [5]	0.28631 [8]	0.30246 [10]	0.11668 [3]	0.17607 [6]	0.17812 [7]	0.03612 [1]	0.35739 [14]	0.34157 [13]	0.39307 [16]
200	MRE($\hat{\delta}$)	0.32902 [12]	0.28453 [10]	0.13346 [3]	0.30327 [11]	0.13852 [4]	0.36618 [16]	0.16104 [5]	0.26504 [9]	0.26087 [8]	0.12834 [2]	0.19331 [7]	0.18902 [6]	0.04607 [1]	0.34271 [13]	0.35084 [14]	0.35888 [15]
200	D_{abs}	0.01689 [1]	0.02018 [9]	0.02134 [10]	0.01862 [5]	0.01932 [7]	0.22713 [16]	0.01799 [3]	0.01768 [2]	0.0185 [4]	0.02244 [13]	0.02206 [12]	0.02522 [14]	0.01966 [8]	0.02152 [11]	0.01873 [6]	0.02633 [15]
200	D_{max}	0.02763 [1]	0.03246 [9]	0.03489 [10]	0.02997 [4]	0.03187 [7]	0.44365 [16]	0.02965 [3]	0.02911 [2]	0.03015 [5]	0.03621 [13]	0.03525 [11]	0.04162 [14]	0.03233 [8]	0.03552 [12]	0.03025 [6]	0.04212 [15]
200	ASAE	0.00626 [2]	0.00705 [6]	0.00765 [9]	0.00688 [5]	0.00727 [7]	0.00414 [1]	0.00676 [4]	0.00669 [3]	0.0087 [13]	0.00822 [12]	0.00755 [8]	0.01356 [16]	0.00793 [11]	0.00881 [14]	0.00789 [10]	0.00945 [15]
200	$\sum Ranks$	111 [11]	124 [12]	73 [5]	93 [7]	64 [4]	154 [14]	54 [2]	98 [8]	105 [9.5]	62 [3]	81 [6]	105 [9.5]	38 [1]	162 [15]	130 [13]	178 [16]
300	BIAS($\hat{b}$)	0.02097 [8]	0.02535 [14]	0.01714 [3]	0.0194 [5]	0.01703 [2]	0.19759 [16]	0.01912 [4]	0.02083 [6]	0.02423 [13]	0.02089 [7]	0.021 [9]	0.02211 [11]	0.01701 [1]	0.02171 [10]	0.02254 [12]	0.02701 [15]
300	BIAS($\hat{c}$)	0.62087 [13]	0.55845 [10]	0.15163 [2]	0.59182 [11]	0.19564 [3]	0.60237 [12]	0.2294 [4]	0.46106 [8]	0.54982 [9]	0.23653 [5]	0.35021 [7]	0.30186 [6]	0.05454 [1]	0.65227 [14]	0.66746 [15]	0.77607 [16]
300	BIAS($\hat{\delta}$)	0.47471 [12]	0.39066 [10]	0.16455 [2]	0.45626 [11]	0.1846 [3]	0.54934 [16]	0.21724 [5]	0.33274 [8]	0.37776 [9]	0.19371 [4]	0.29643 [7]	0.24094 [6]	0.06399 [1]	0.49442 [14]	0.48867 [13]	0.53743 [15]
300	MSE($\hat{b}$)	−0.00064 [2]	0.00097 [14]	5e-04 [5]	−0.0018 [1]	0.00049 [4]	0.04917 [16]	0.00061 [6]	0.00073 [10]	0.00083 [13]	0.00066 [7]	0.00072 [9]	0.00081 [12]	0.00045 [3]	0.00069 [8]	0.00075 [11]	0.00109 [15]
300	MSE($\hat{c}$)	0.65167 [13]	0.58062 [11]	0.04202 [2]	0.61681 [12]	0.05808 [3]	0.55267 [10]	0.08501 [4]	0.40987 [8]	0.46759 [9]	0.09427 [5]	0.277 [7]	0.17161 [6]	0.00649 [1]	0.69117 [14]	0.69508 [15]	0.89676 [16]
300	MSE($\hat{\delta}$)	0.33893 [14]	0.21653 [10]	0.05358 [2]	0.30207 [11]	0.05467 [3]	0.43665 [16]	0.07343 [4]	0.1631 [8]	0.18834 [9]	0.0781 [5]	0.14716 [7]	0.09448 [6]	0.00716 [1]	0.33571 [13]	0.31827 [12]	0.38054 [15]
300	MRE($\hat{b}$)	0.05244 [8]	0.06337 [14]	0.04286 [3]	0.04851 [5]	0.04259 [2]	0.49728 [16]	0.04779 [4]	0.05209 [6]	0.06057 [13]	0.05224 [7]	0.0536 [9]	0.05528 [11]	0.04251 [1]	0.05428 [10]	0.05635 [12]	0.06752 [15]
300	MRE($\hat{c}$)	0.31044 [13]	0.27923 [10]	0.07582 [2]	0.29591 [12]	0.09782 [3]	0.29346 [11]	0.1147 [4]	0.23053 [8]	0.27491 [9]	0.1157 [5]	0.17543 [7]	0.15093 [6]	0.02727 [1]	0.32613 [14]	0.33373 [15]	0.38803 [16]
300	MRE($\hat{\delta}$)	0.31647 [12]	0.26044 [10]	0.1097 [2]	0.30255 [11]	0.12307 [3]	0.36451 [16]	0.14482 [5]	0.22183 [8]	0.25184 [9]	0.12722 [4]	0.19216 [7]	0.16063 [6]	0.04266 [1]	0.32961 [14]	0.32578 [13]	0.35559 [15]
300	D_{abs}	0.01538 [4]	0.01564 [6]	0.01647 [9]	0.01658 [10]	0.01514 [3]	0.22554 [16]	0.01539 [5]	0.01503 [2]	0.01492 [1]	0.01917 [12]	0.01707 [11]	0.0208 [15]	0.01616 [8]	0.01952 [13]	0.01569 [7]	0.02023 [14]
300	D_{max}	0.02432 [1]	0.02547 [6]	0.0268 [9]	0.02617 [8]	0.02476 [2.5]	0.44337 [16]	0.02539 [5]	0.02476 [2.5]	0.02495 [4]	0.03071 [12]	0.02788 [11]	0.0343 [15]	0.02711 [10]	0.03115 [13]	0.02568 [7]	0.03291 [14]
300	ASAE	0.00454 [2]	0.0047 [5]	0.00578 [12]	0.00455 [3]	0.00571 [10]	0.00323 [1]	0.00458 [4]	0.00487 [6]	0.00609 [14]	0.00604 [13]	0.00558 [8]	0.01066 [16]	0.00572 [11]	0.00634 [15]	0.00516 [7]	0.00568 [9]
300	$\sum Ranks$	101 [8.5]	119 [12]	64 [4]	101 [8.5]	41.5 [2]	161 [15]	53 [3]	79.5 [5]	111 [10]	85 [6]	98 [7]	115 [11]	40 [1]	151 [14]	138 [13]	174 [16]

Table 8
Numerical values of simulation measures for $b = 2.0,\ c = 1.5,\ \delta = 2.5$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{b}$)	0.30847 {6}	0.306 {5}	0.39143 {15}	0.28221 {3}	0.34649 {12}	0.29475 {4}	0.40862 {16}	0.36371 {14}	0.3445 {10}	0.27211 {1}	0.35233 {13}	0.34588 {11}	0.34323 {9}	0.32559 {8}	0.27356 {2}	0.31681 {7}
20	BIAS($\hat{c}$)	0.6484 {12}	0.55014 {9}	0.29922 {5}	0.63262 {11}	0.28452 {4}	0.33931 {6}	0.34021 {7}	0.40265 {8}	0.82401 {16}	0.21901 {2}	0.2141 {1}	0.7869 {15}	0.22311 {3}	0.62803 {10}	0.77619 {14}	0.67635 {13}
20	BIAS($\hat{\delta}$)	0.95207 {10}	0.82322 {9}	0.39998 {4}	0.97497 {11}	0.44143 {5}	0.62465 {8}	0.49686 {6}	0.58362 {7}	1.03269 {13}	0.34155 {3}	0.33122 {2}	1.06826 {14}	0.2275 {1}	0.99345 {12}	1.12464 {15}	1.1324 {16}
20	MSE($\hat{b}$)	0.15478 {5}	0.15525 {6}	0.26921 {15}	0.1141 {1}	0.20689 {13}	0.14044 {3}	0.27529 {16}	0.20017 {12}	0.19268 {11}	0.11797 {2}	0.18793 {10}	0.17727 {9}	0.22026 {14}	0.17011 {8}	0.14762 {4}	0.16345 {7}
20	MSE($\hat{c}$)	0.62472 {11}	0.48901 {9}	0.18373 {5}	0.63501 {12}	0.16619 {4}	0.18445 {6}	0.22503 {7}	0.31452 {8}	0.8998 {16}	0.08481 {2}	0.08159 {1}	0.85924 {15}	0.09723 {3}	0.56805 {10}	0.83522 {14}	0.65801 {13}
20	MSE($\hat{\delta}$)	1.26755 {11}	0.96936 {9}	0.31334 {4}	1.29564 {12}	0.33949 {5}	0.55838 {8}	0.45032 {6}	0.53921 {7}	1.34677 {13}	0.24329 {3}	0.18741 {2}	1.43501 {14}	0.10816 {1}	1.25876 {10}	1.56231 {15}	1.6093 {16}
20	MRE($\hat{b}$)	0.15423 {6}	0.153 {5}	0.19572 {15}	0.1411 {3}	0.17324 {12}	0.14738 {4}	0.20431 {16}	0.18185 {14}	0.17225 {10}	0.13605 {1}	0.17616 {13}	0.17294 {11}	0.17161 {9}	0.1628 {8}	0.13678 {2}	0.1584 {7}
20	MRE($\hat{c}$)	0.43227 {12}	0.36676 {9}	0.19948 {5}	0.42175 {11}	0.18968 {4}	0.22621 {6}	0.2268 {7}	0.26843 {8}	0.54934 {16}	0.146 {2}	0.14274 {1}	0.5246 {15}	0.14874 {3}	0.41868 {10}	0.51746 {14}	0.4509 {13}
20	MRE($\hat{\delta}$)	0.38083 {10}	0.32929 {9}	0.15999 {4}	0.38999 {11}	0.17657 {5}	0.24986 {8}	0.19875 {6}	0.23345 {7}	0.41307 {13}	0.13662 {3}	0.13249 {2}	0.4273 {14}	0.091 {1}	0.39738 {12}	0.44985 {15}	0.45296 {16}
20	D_{abs}	0.05538 {1}	0.05921 {3}	0.06132 {6}	0.05964 {5}	0.0668 {12}	0.0585 {2}	0.07179 {14}	0.06469 {11}	0.05922 {4}	0.07308 {15}	0.06244 {9}	0.06141 {7}	0.06262 {10}	0.07366 {16}	0.06229 {8}	0.06729 {13}
20	D_{max}	0.0893 {1}	0.09309 {4}	0.10352 {10}	0.09295 {3}	0.10637 {13}	0.09188 {2}	0.11972 {16}	0.10424 {11}	0.09648 {5}	0.11143 {14}	0.10007 {8}	0.09991 {7}	0.10326 {9}	0.11282 {15}	0.09745 {6}	0.10444 {12}
20	ASAE	0.04087 {8}	0.03957 {7}	0.03873 {6}	0.0371 {3}	0.0385 {5}	0.03442 {1}	0.03741 {4}	0.03597 {2}	0.04373 {10}	0.05391 {13}	0.04711 {12}	0.06194 {16}	0.04687 {11}	0.05509 {14}	0.04359 {9}	0.05672 {15}
20	$\sum Ranks$	93 {7}	84 {5}	94 {8.5}	86 {6}	94 {8.5}	58 {1}	121 {12}	109 {10}	137 {14}	61 {2}	74 {3.5}	148 {15.5}	74 {3.5}	133 {13}	118 {11}	148 {15.5}
70	BIAS($\hat{b}$)	0.17035 {4}	0.18326 {6}	0.18914 {10}	0.16293 {3}	0.18712 {9}	0.18464 {7}	0.19897 {12}	0.18603 {8}	0.22983 {15}	0.17754 {5}	0.16183 {2}	0.1991 {13}	0.198 {11}	0.21217 {14}	0.15589 {1}	0.24565 {16}
70	BIAS($\hat{c}$)	0.57108 {12}	0.37716 {9}	0.2052 {3}	0.57303 {13}	0.1945 {2}	0.30189 {8}	0.25836 {6}	0.2658 {7}	0.41381 {10}	0.21068 {4}	0.21138 {5}	0.78548 {16}	0.118 {1}	0.62423 {15}	0.61195 {14}	0.56335 {11}
70	BIAS($\hat{\delta}$)	0.93859 {12}	0.67185 {10}	0.33376 {4}	0.90534 {11}	0.32775 {2}	0.62424 {9}	0.47342 {6}	0.54708 {7}	0.60271 {8}	0.34153 {5}	0.32893 {3}	1.06639 {15}	0.15293 {1}	0.97258 {13}	1.07145 {16}	1.03114 {14}
70	MSE($\hat{b}$)	0.0492 {5}	0.04959 {6}	0.05729 {9}	0.04042 {2}	0.05156 {7}	0.04682 {4}	0.07567 {14}	0.05857 {10}	0.09344 {16}	0.05258 {8}	0.04063 {3}	0.06456 {12}	0.06218 {11}	0.06714 {13}	0.03968 {1}	0.0909 {15}
70	MSE($\hat{c}$)	0.52248 {13}	0.26495 {9}	0.07316 {2}	0.50726 {12}	0.07605 {3}	0.14071 {8}	0.11202 {6}	0.1248 {7}	0.33764 {10}	0.07813 {4}	0.0782 {5}	0.85895 {16}	0.02635 {1}	0.56292 {15}	0.53832 {14}	0.46819 {11}
70	MSE($\hat{\delta}$)	1.24423 {12}	0.68666 {10}	0.20897 {4}	1.12284 {11}	0.20498 {3}	0.55764 {8}	0.35995 {6}	0.43235 {7}	0.56984 {9}	0.24167 {5}	0.18553 {2}	1.41435 {15}	0.05003 {1}	1.2508 {13}	1.43767 {16}	1.4101 {14}
70	MRE($\hat{b}$)	0.08518 {4}	0.09163 {6}	0.09457 {10}	0.08146 {3}	0.09356 {9}	0.09232 {7}	0.09948 {12}	0.09301 {8}	0.11491 {15}	0.08877 {5}	0.08091 {2}	0.09955 {13}	0.099 {11}	0.10608 {14}	0.07795 {1}	0.12282 {16}
70	MRE($\hat{c}$)	0.38072 {12}	0.25144 {9}	0.1368 {3}	0.38202 {13}	0.12967 {2}	0.20126 {8}	0.17224 {6}	0.1772 {7}	0.27588 {10}	0.14045 {4}	0.14212 {5}	0.52376 {16}	0.07867 {1}	0.41616 {15}	0.40797 {14}	0.37557 {11}
70	MRE($\hat{\delta}$)	0.37543 {12}	0.26874 {10}	0.1335 {4}	0.36214 {11}	0.1311 {3}	0.24904 {9}	0.18937 {6}	0.21883 {7}	0.24109 {8}	0.13415 {5}	0.13057 {2}	0.42426 {15}	0.06117 {1}	0.38903 {13}	0.42858 {16}	0.41246 {14}
70	D_{abs}	0.03158 {4}	0.03407 {7}	0.03718 {11}	0.03232 {6}	0.03725 {12}	0.03219 {5}	0.03411 {8}	0.02826 {1}	0.0343 {9}	0.03848 {13}	0.03572 {10}	0.03853 {14}	0.03144 {3}	0.03916 {15}	0.03114 {2}	0.04006 {16}
70	D_{max}	0.04975 {2}	0.05467 {7}	0.06065 {12}	0.0521 {4}	0.05998 {11}	0.05294 {6}	0.05644 {8}	0.04681 {1}	0.05668 {9}	0.06121 {13}	0.05758 {10}	0.06345 {14}	0.05281 {5}	0.06381 {15}	0.05062 {3}	0.06587 {16}
70	ASAE	0.01722 {7}	0.01684 {3}	0.01711 {5}	0.0172 {6}	0.01671 {2}	0.01572 {1}	0.01737 {8}	0.01709 {4}	0.01945 {9}	0.02446 {13}	0.02411 {12}	0.03623 {16}	0.02044 {11}	0.02623 {14}	0.02041 {10}	0.02702 {15}
70	$\sum Ranks$	99 {11}	92 {8}	77 {5}	95 {9}	65 {3}	80 {6}	98 {10}	74 {4}	128 {13}	84 {7}	61 {2}	175 {16}	58 {1}	169 {14.5}	108 {12}	169 {14.5}
140	BIAS($\hat{b}$)	0.13695 {8}	0.12801 {4}	0.13822 {9}	0.13318 {5}	0.14543 {12}	0.15228 {14}	0.13528 {6}	0.11916 {1}	0.15399 {16}	0.13652 {7}	0.12031 {3}	0.14417 {11}	0.11972 {2}	0.14836 {13}	0.1538 {15}	0.1401 {10}
140	BIAS($\hat{c}$)	0.57053 {12}	0.23781 {7}	0.16221 {3}	0.57281 {13}	0.15876 {2}	0.3017 {10}	0.1939 {5}	0.25178 {8}	0.28922 {9}	0.1896 {4}	0.21085 {6}	0.66373 {16}	0.06219 {1}	0.62326 {15}	0.60869 {14}	0.53519 {11}
140	BIAS($\hat{\delta}$)	0.93719 {13}	0.48148 {7}	0.3016 {2}	0.90433 {11}	0.30581 {3}	0.62218 {10}	0.45925 {6}	0.49591 {8}	0.50414 {9}	0.34016 {5}	0.32826 {4}	0.93847 {14}	0.11171 {1}	0.96815 {15}	1.05922 {16}	0.90814 {12}
140	MSE($\hat{b}$)	0.03158 {8}	0.0275 {5}	0.03327 {10}	0.02638 {4}	0.03576 {14}	0.03451 {12}	0.031 {7}	0.02485 {3}	0.04118 {16}	0.02956 {6}	0.02155 {1}	0.03726 {15}	0.02283 {2}	0.03312 {9}	0.03501 {13}	0.03331 {11}
140	MSE($\hat{c}$)	0.52144 {13}	0.10124 {7}	0.0568 {3}	0.50719 {12}	0.04151 {2}	0.13859 {9}	0.06845 {4}	0.11749 {8}	0.18503 {10}	0.07739 {6}	0.07702 {5}	0.56564 {16}	0.00756 {1}	0.56203 {15}	0.53701 {14}	0.45461 {11}
140	MSE($\hat{\delta}$)	1.24413 {14}	0.36133 {8}	0.16693 {3}	1.12216 {13}	0.16483 {2}	0.55445 {10}	0.34728 {6}	0.34965 {7}	0.41138 {9}	0.2409 {5}	0.18358 {4}	1.03988 {11}	0.02096 {1}	1.24626 {15}	1.36814 {16}	1.08404 {12}
140	MRE($\hat{b}$)	0.06847 {8}	0.064 {4}	0.06911 {9}	0.06659 {5}	0.07272 {12}	0.07614 {14}	0.06764 {6}	0.05958 {1}	0.077 {16}	0.06826 {7}	0.06016 {3}	0.07208 {11}	0.05986 {2}	0.07418 {13}	0.0769 {15}	0.07005 {10}
140	MRE($\hat{c}$)	0.38042 {13}	0.15854 {7}	0.10814 {3}	0.38026 {12}	0.10584 {2}	0.1979 {10}	0.12926 {5}	0.16785 {8}	0.19281 {9}	0.1264 {4}	0.14028 {6}	0.44249 {16}	0.04146 {1}	0.41588 {15}	0.40711 {14}	0.35679 {11}
140	MRE($\hat{\delta}$)	0.37427 {13}	0.19259 {7}	0.12064 {2}	0.36058 {11}	0.12233 {3}	0.24838 {10}	0.1837 {6}	0.19836 {8}	0.20166 {9}	0.13256 {5}	0.13017 {4}	0.37539 {14}	0.04468 {1}	0.38749 {15}	0.42369 {16}	0.36326 {12}
140	D_{abs}	0.02162 {3}	0.02298 {5}	0.02426 {9}	0.02216 {4}	0.02719 {14}	0.02091 {2}	0.02309 {6}	0.02458 {10}	0.02378 {8}	0.02655 {12}	0.02349 {7}	0.02983 {16}	0.02068 {1}	0.0271 {13}	0.02918 {15}	0.02646 {11}
140	D_{max}	0.03455 {3}	0.03739 {5}	0.03987 {10}	0.03534 {4}	0.04414 {13}	0.03417 {2}	0.03771 {6}	0.03902 {9}	0.03822 {8}	0.04276 {11}	0.03805 {7}	0.04995 {16}	0.03411 {1}	0.04448 {14}	0.04663 {15}	0.04383 {12}
140	ASAE	0.01075 {2}	0.01091 {3}	0.01132 {8}	0.01107 {7}	0.011 {4}	0.01104 {6}	0.01056 {1}	0.01102 {5}	0.01248 {9}	0.01597 {13}	0.01463 {12}	0.02067 {16}	0.01332 {10}	0.01648 {14}	0.01387 {11}	0.01691 {15}
140	$\sum Ranks$	110 {11}	69 {4}	71 {5}	101 {9}	83 {7}	109 {10}	64 {3}	76 {6}	128 {12}	85 {8}	62 {2}	172 {15}	24 {1}	166 {14}	174 {16}	138 {13}
200	BIAS($\hat{b}$)	0.11413 {10}	0.10112 {3}	0.09668 {2}	0.10995 {7}	0.11375 {9}	0.10464 {5}	0.12817 {14}	0.10811 {6}	0.11081 {8}	0.1044 {4}	0.11512 {11}	0.1267 {13}	0.09588 {1}	0.13197 {15}	0.11578 {12}	0.13883 {16}
200	BIAS($\hat{c}$)	0.47581 {11}	0.23692 {9}	0.13422 {2}	0.53581 {14}	0.13623 {3}	0.30007 {10}	0.19154 {5}	0.22203 {8}	0.20397 {6}	0.18527 {4}	0.20946 {7}	0.64669 {16}	0.05152 {1}	0.52304 {13}	0.58437 {15}	0.51884 {12}
200	BIAS($\hat{\delta}$)	0.8535 {11}	0.48119 {8}	0.27071 {2}	0.87808 {12}	0.30365 {3}	0.62065 {10}	0.4449 {7}	0.49136 {9}	0.38968 {6}	0.33924 {5}	0.32813 {4}	0.93444 {14}	0.09017 {1}	0.96623 {15}	0.98895 {16}	0.90745 {13}
200	MSE($\hat{b}$)	0.0206 {10}	0.01735 {4}	0.01524 {2}	0.01772 {5}	0.02041 {8}	0.01848 {6}	0.02533 {13}	0.01953 {7}	0.02087 {12}	0.01719 {3}	0.02075 {11}	0.02586 {14}	0.01356 {1}	0.02819 {15}	0.02056 {9}	0.02985 {16}
200	MSE($\hat{c}$)	0.37217 {11}	0.10076 {9}	0.0349 {3}	0.46574 {14}	0.03012 {2}	0.1376 {10}	0.06614 {4}	0.06676 {5}	0.08694 {8}	0.07492 {6}	0.07646 {7}	0.5132 {15}	0.00475 {1}	0.42605 {13}	0.52004 {16}	0.4168 {12}
200	MSE($\hat{\delta}$)	0.99381 {11}	0.36005 {9}	0.13543 {2}	1.0695 {13}	0.16409 {3}	0.55251 {10}	0.32657 {7}	0.34138 {8}	0.2781 {6}	0.23948 {5}	0.18342 {4}	0.99412 {12}	0.01423 {1}	1.24357 {15}	1.28905 {16}	1.08343 {14}
200	MRE($\hat{b}$)	0.05707 {10}	0.05056 {3}	0.04834 {2}	0.05498 {7}	0.05687 {9}	0.05232 {5}	0.06408 {14}	0.05405 {6}	0.05541 {8}	0.0522 {4}	0.05756 {11}	0.06335 {13}	0.04794 {1}	0.06599 {15}	0.05789 {12}	0.06656 {16}
200	MRE($\hat{c}$)	0.31721 {11}	0.15652 {9}	0.08948 {2}	0.35721 {14}	0.09082 {3}	0.19772 {10}	0.12862 {5}	0.14802 {8}	0.13598 {6}	0.12351 {4}	0.1382 {7}	0.43113 {16}	0.03434 {1}	0.34869 {13}	0.38958 {15}	0.34589 {12}
200	MRE($\hat{\delta}$)	0.3414 {11}	0.19185 {8}	0.10828 {2}	0.35123 {12}	0.12041 {3}	0.24602 {10}	0.17796 {7}	0.19654 {9}	0.15587 {6}	0.13018 {5}	0.1291 {4}	0.37378 {14}	0.03607 {1}	0.38547 {15}	0.39558 {16}	0.36303 {13}
200	D_{abs}	0.01645 {2}	0.01738 {3}	0.01888 {6}	0.01557 {1}	0.01881 {5}	0.01926 {8}	0.01972 {10}	0.01924 {7}	0.01933 {9}	0.02134 {12}	0.02231 {13}	0.02804 {16}	0.01856 {4}	0.02365 {14}	0.02034 {11}	0.02572 {15}
200	D_{max}	0.0271 {2}	0.02811 {3}	0.03077 {6}	0.02585 {1}	0.031 {7}	0.03026 {4}	0.03268 {11}	0.03139 {9}	0.03126 {8}	0.03458 {12}	0.03569 {13}	0.04656 {16}	0.03047 {5}	0.03887 {14}	0.03202 {10}	0.04242 {15}
200	ASAE	0.00836 {2}	0.00852 {4}	0.00879 {7}	0.00895 {8}	0.00862 {6}	0.00857 {5}	0.00789 {1}	0.00848 {3}	0.00942 {9}	0.01271 {13}	0.01157 {12}	0.01822 {16}	0.01039 {10}	0.0131 {14}	0.01041 {11}	0.01405 {15}
200	$\sum Ranks$	102 {10}	72 {4}	38 {2}	108 {12}	61 {3}	93 {8}	98 {9}	85 {6}	92 {7}	77 {5}	104 {11}	175 {16}	28 {1}	171 {15}	159 {13}	169 {14}
300	BIAS($\hat{b}$)	0.1118 {14}	0.09423 {10}	0.08056 {4}	0.09206 {7}	0.07396 {2}	0.08151 {5}	0.07935 {3}	0.07303 {1}	0.09335 {9}	0.09269 {8}	0.09779 {11}	0.10849 {13}	0.08245 {6}	0.11784 {16}	0.11415 {15}	0.10411 {12}
300	BIAS($\hat{c}$)	0.47447 {12}	0.21961 {9}	0.12621 {2}	0.4397 {11}	0.12642 {3}	0.29986 {10}	0.17364 {5}	0.17891 {7}	0.16395 {4}	0.17518 {6}	0.20603 {8}	0.64172 {16}	0.03998 {1}	0.50252 {13}	0.55262 {15}	0.5183 {14}
300	BIAS($\hat{\delta}$)	0.85315 {12}	0.47797 {9}	0.27011 {2}	0.84687 {11}	0.30245 {3}	0.61991 {10}	0.39736 {8}	0.39666 {7}	0.36172 {6}	0.33883 {5}	0.3251 {4}	0.92906 {15}	0.08199 {1}	0.92435 {14}	0.95524 {16}	0.90606 {13}
300	MSE($\hat{b}$)	0.01935 {13}	0.01527 {9}	0.01179 {6}	0.01357 {8}	0.00913 {2}	0.01034 {4}	0.0099 {3}	0.00815 {1}	0.01579 {12}	0.01344 {7}	0.01542 {10}	0.01966 {14}	0.01128 {5}	0.02127 {16}	0.02037 {15}	0.01548 {11}
300	MSE($\hat{c}$)	0.3707 {12}	0.08387 {9}	0.02761 {3}	0.34937 {11}	0.02478 {2}	0.13592 {10}	0.04959 {4}	0.05355 {5}	0.05835 {6}	0.06037 {7}	0.07632 {8}	0.49512 {16}	0.00276 {1}	0.37932 {13}	0.48859 {15}	0.41644 {14}
300	MSE($\hat{\delta}$)	0.99341 {12}	0.35901 {9}	0.135 {2}	1.06835 {13}	0.16374 {3}	0.55049 {10}	0.26884 {8}	0.2351 {6}	0.20766 {5}	0.23887 {7}	0.1831 {4}	0.95664 {11}	0.00998 {1}	1.14182 {15}	1.21611 {16}	1.08291 {14}
300	MRE($\hat{b}$)	0.0559 {14}	0.04711 {10}	0.04028 {4}	0.04603 {7}	0.03698 {2}	0.04075 {5}	0.03968 {3}	0.03652 {1}	0.04668 {9}	0.04635 {8}	0.04889 {11}	0.05424 {13}	0.04123 {6}	0.05892 {16}	0.05708 {15}	0.05205 {12}
300	MRE($\hat{c}$)	0.31575 {12}	0.14641 {9}	0.08414 {2}	0.29313 {11}	0.08428 {3}	0.19763 {10}	0.11576 {5}	0.11927 {7}	0.1093 {4}	0.11678 {6}	0.13639 {8}	0.42781 {16}	0.02666 {1}	0.33501 {13}	0.36841 {15}	0.34354 {14}
300	MRE($\hat{\delta}$)	0.34079 {12}	0.19059 {9}	0.10651 {2}	0.33875 {11}	0.11914 {3}	0.24316 {10}	0.15895 {8}	0.15866 {7}	0.14469 {6}	0.128 {5}	0.12613 {4}	0.37163 {15}	0.0328 {1}	0.36974 {14}	0.3821 {16}	0.36173 {13}
300	D_{abs}	0.01523 {7}	0.01638 {11}	0.01472 {4}	0.01467 {3}	0.01529 {8}	0.01341 {1}	0.01479 {5}	0.01432 {2}	0.01607 {10}	0.01869 {14}	0.01574 {9}	0.02244 {16}	0.01511 {6}	0.0208 {15}	0.01665 {12}	0.01701 {13}
300	D_{max}	0.02477 {7}	0.02657 {11}	0.02409 {5}	0.02304 {2}	0.02451 {6}	0.02155 {1}	0.02398 {4}	0.02331 {3}	0.02624 {10}	0.02975 {14}	0.02544 {9}	0.03764 {16}	0.02488 {8}	0.03368 {15}	0.02704 {12}	0.02746 {13}
300	ASAE	0.00659 {4}	0.00629 {3}	0.0069 {8}	0.00663 {5}	0.00668 {6}	0.00674 {7}	0.0061 {1}	0.00627 {2}	0.00759 {9}	0.0096 {13}	0.00863 {12}	0.01372 {16}	0.00794 {11}	0.01033 {14}	0.00786 {10}	0.01038 {15}
300	$\sum Ranks$	131 {12}	108 {11}	44 {2}	100 {9.5}	43 {1}	83 {6}	57 {5}	49 {4}	90 {7}	100 {9.5}	98 {8}	177 {16}	48 {3}	174 {15}	172 {14}	158 {13}

Table 9
Numerical values of simulation measures for $b = 2.5$, $c = 0.6$, $\delta = 2.0$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{b}$)	0.69682 {12}	0.55723 {9}	1.01861 {16}	0.40002 {1}	0.82416 {14}	0.62061 {11}	0.90251 {15}	0.76796 {13}	0.58589 {10}	0.45937 {3}	0.40918 {2}	0.52995 {5}	0.47789 {4}	0.53932 {8}	0.53546 {6}	0.53856 {7}
20	BIAS($\hat{c}$)	0.36451 {16}	0.28327 {11}	0.22633 {6}	0.32082 {15}	0.22015 {5}	0.28938 {12}	0.21202 {4}	0.25084 {7}	0.25211 {8}	0.09059 {2}	0.10197 {3}	0.27132 {10}	0.07046 {1}	0.29012 {13}	0.29882 {14}	0.25694 {9}
20	BIAS($\hat{\delta}$)	1.18096 {16}	0.97127 {12}	0.98074 {13}	0.94206 {11}	0.8318 {7}	1.01408 {14}	0.81788 {4}	0.92213 {9}	0.93136 {10}	0.36701 {3}	0.3636 {2}	0.82233 {6}	0.24547 {1}	0.87606 {8}	1.03844 {15}	0.81909 {5}
20	MSE($\hat{b}$)	0.74647 {12}	0.45669 {5}	1.42303 {16}	0.26071 {2}	1.05183 {14}	0.58886 {11}	1.32317 {15}	0.89069 {13}	0.53221 {10}	0.36063 {3}	0.24549 {1}	0.47955 {7}	0.43035 {4}	0.46566 {6}	0.50565 {9}	0.49632 {8}
20	MSE($\hat{c}$)	0.15493 {16}	0.10855 {11}	0.08002 {6}	0.14034 {15}	0.07591 {5}	0.12234 {13}	0.07003 {4}	0.09092 {8}	0.08664 {7}	0.01413 {2}	0.01948 {3}	0.10497 {10}	0.01008 {1}	0.11607 {12}	0.12441 {14}	0.10159 {9}
20	MSE($\hat{\delta}$)	1.50232 {16}	1.10132 {12}	1.18734 {13}	1.07856 {11}	0.92732 {6}	1.27369 {15}	0.96305 {8}	1.03363 {10}	0.95784 {7}	0.23948 {2}	0.27348 {3}	0.83238 {4}	0.11604 {1}	0.96478 {9}	1.21916 {14}	0.88204 {5}
20	MRE($\hat{b}$)	0.27873 {12}	0.22289 {9}	0.40744 {16}	0.16001 {1}	0.32966 {14}	0.24824 {11}	0.36101 {15}	0.30718 {13}	0.23436 {10}	0.18375 {3}	0.16367 {2}	0.21198 {5}	0.19115 {4}	0.21573 {8}	0.21418 {6}	0.21542 {7}
20	MRE($\hat{c}$)	0.60752 {16}	0.47212 {11}	0.37722 {6}	0.5347 {15}	0.36692 {5}	0.4823 {12}	0.35336 {4}	0.41807 {7}	0.42018 {8}	0.15098 {2}	0.16995 {3}	0.45219 {10}	0.11743 {1}	0.48353 {13}	0.49803 {14}	0.42824 {9}
20	MRE($\hat{\delta}$)	0.59048 {16}	0.48563 {12}	0.49037 {13}	0.47103 {11}	0.4159 {7}	0.50704 {14}	0.40894 {4}	0.46107 {9}	0.46568 {10}	0.18351 {3}	0.1818 {2}	0.41117 {6}	0.12274 {1}	0.43803 {8}	0.51922 {15}	0.40955 {5}
20	D_{abs}	0.06634 {8}	0.05592 {2}	0.07305 {15}	0.05814 {5}	0.06754 {9}	0.06928 {10}	0.07406 {16}	0.05849 {6}	0.05691 {4}	0.06987 {11}	0.07017 {12}	0.05612 {3}	0.05555 {1}	0.07179 {14}	0.06603 {7}	0.07115 {13}
20	D_{max}	0.10533 {9}	0.09046 {2}	0.12482 {16}	0.08922 {1}	0.11333 {14}	0.1036 {8}	0.12421 {15}	0.09642 {6}	0.09442 {4}	0.11096 {11}	0.11087 {10}	0.09386 {3}	0.09445 {5}	0.11266 {13}	0.10256 {7}	0.11258 {12}
20	ASAE	0.04287 {6}	0.0399 {3}	0.04464 {10}	0.04076 {4}	0.04202 {5}	0.03901 {1}	0.04389 {8}	0.0396 {2}	0.04394 {9}	0.05218 {13}	0.04847 {12}	0.05306 {14}	0.04785 {11}	0.05715 {15}	0.04347 {7}	0.05748 {16}
20	$\sum Ranks$	155 {16}	99 {7}	146 {15}	92 {5}	105 {9.5}	132 {14}	112 {11}	103 {8}	97 {6}	58 {3}	55 {2}	83 {4}	35 {1}	127 {12}	128 {13}	105 {9.5}
70	BIAS($\hat{b}$)	0.31531 {8}	0.349 {13}	0.39859 {16}	0.25933 {4}	0.37442 {15}	0.33529 {11}	0.37334 {14}	0.3407 {12}	0.32295 {9}	0.25001 {3}	0.23554 {1}	0.29138 {6}	0.23598 {2}	0.32831 {10}	0.27071 {5}	0.29237 {7}
70	BIAS($\hat{c}$)	0.27126 {16}	0.22942 {9}	0.12587 {5}	0.25205 {11}	0.11369 {4}	0.20476 {7}	0.13979 {6}	0.20667 {8}	0.22957 {10}	0.08978 {2}	0.09034 {3}	0.26704 {15}	0.0331 {1}	0.26512 {14}	0.25796 {13}	0.2538 {12}
70	BIAS($\hat{\delta}$)	0.92275 {16}	0.80166 {10}	0.55235 {5}	0.8311 {12}	0.52591 {4}	0.84811 {14}	0.59074 {6}	0.73988 {8}	0.75207 {9}	0.36379 {3}	0.3616 {2}	0.73419 {7}	0.17953 {1}	0.84859 {15}	0.83842 {13}	0.81524 {11}
70	MSE($\hat{b}$)	0.17353 {9}	0.18341 {10}	0.25838 {15}	0.10176 {3}	0.23666 {14}	0.22025 {13}	0.28362 {16}	0.20315 {12}	0.15918 {8}	0.1051 {4}	0.08823 {2}	0.13044 {6}	0.08488 {1}	0.18425 {11}	0.12123 {5}	0.14719 {7}
70	MSE($\hat{c}$)	0.10356 {15}	0.08458 {10}	0.03568 {5}	0.09199 {11}	0.02845 {4}	0.07159 {7}	0.04157 {6}	0.07754 {8}	0.081 {9}	0.01382 {2}	0.01683 {3}	0.09567 {12}	0.00191 {1}	0.10587 {16}	0.09938 {13}	0.10062 {14}
70	MSE($\hat{\delta}$)	1.04828 {16}	0.83481 {10}	0.45204 {5}	0.85621 {11}	0.42729 {4}	0.99428 {15}	0.54733 {6}	0.75204 {9}	0.70817 {8}	0.23542 {2}	0.27304 {3}	0.58817 {7}	0.05261 {1}	0.90127 {14}	0.86571 {12}	0.87987 {13}
70	MRE($\hat{b}$)	0.12612 {8}	0.1396 {13}	0.15944 {16}	0.10373 {4}	0.14977 {15}	0.13412 {11}	0.14934 {14}	0.13628 {12}	0.12918 {9}	0.1 {3}	0.09421 {1}	0.11655 {6}	0.09439 {2}	0.13132 {10}	0.10828 {5}	0.11695 {7}
70	MRE($\hat{c}$)	0.4521 {16}	0.38236 {9}	0.20978 {5}	0.42008 {11}	0.18948 {4}	0.34127 {7}	0.23298 {6}	0.34445 {8}	0.38262 {10}	0.14963 {2}	0.15057 {3}	0.44506 {15}	0.05516 {1}	0.44186 {14}	0.42994 {13}	0.42791 {12}
70	MRE($\hat{\delta}$)	0.46138 {16}	0.40083 {10}	0.27617 {5}	0.41555 {12}	0.26296 {4}	0.42406 {14}	0.29537 {6}	0.36994 {8}	0.37604 {9}	0.18223 {3}	0.17971 {2}	0.36709 {7}	0.08977 {1}	0.42429 {15}	0.41921 {13}	0.40763 {11}
70	D_{abs}	0.03543 {10}	0.03151 {3}	0.03724 {13}	0.03453 {9}	0.03433 {8}	0.03363 {6}	0.0335 {5}	0.03323 {4}	0.03083 {2}	0.03714 {12}	0.03404 {7}	0.03967 {15}	0.03002 {1}	0.03668 {11}	0.03946 {14}	0.0432 {16}
70	D_{max}	0.05556 {8}	0.05147 {3}	0.06328 {14}	0.05429 {6}	0.05859 {10}	0.05385 {4}	0.05745 {9}	0.05401 {5}	0.05123 {2}	0.0603 {12}	0.05477 {7}	0.06518 {15}	0.04926 {1}	0.05959 {11}	0.06158 {13}	0.06809 {16}
70	ASAE	0.01842 {5}	0.01755 {3}	0.01893 {7.5}	0.01868 {6}	0.01893 {7.5}	0.01729 {1}	0.01827 {4}	0.01739 {2}	0.01963 {9}	0.02648 {14}	0.02334 {12}	0.03116 {16}	0.02072 {10}	0.02784 {15}	0.02091 {11}	0.02513 {13}
70	$\sum Ranks$	143 {15}	103 {9}	111.5 {11}	100 {8}	93.5 {4}	110 {10}	98 {7}	96 {6}	94 {5}	62 {3}	46 {2}	127 {12}	23 {1}	156 {16}	130 {13}	139 {14}
140	BIAS($\hat{b}$)	0.2355 {14}	0.2432 {15}	0.20686 {9}	0.15143 {1}	0.20922 {10}	0.19234 {7}	0.178 {4}	0.21043 {11}	0.27154 {16}	0.19442 {8}	0.15673 {3}	0.15276 {2}	0.18529 {6}	0.21847 {12}	0.1811 {5}	0.23366 {13}
140	BIAS($\hat{c}$)	0.23945 {15}	0.15465 {8}	0.09006 {5}	0.20305 {11}	0.06863 {3}	0.1748 {9}	0.09173 {6}	0.1229 {7}	0.20837 {12}	0.05813 {2}	0.08813 {4}	0.22242 {13}	0.02867 {1}	0.25091 {16}	0.19949 {10}	0.23378 {14}
140	BIAS($\hat{\delta}$)	0.89749 {16}	0.64329 {9}	0.38364 {5}	0.72016 {12}	0.33913 {3}	0.68097 {11}	0.45467 {6}	0.51098 {7}	0.67973 {10}	0.32099 {2}	0.35853 {4}	0.63619 {8}	0.12563 {1}	0.84416 {15}	0.74439 {13}	0.81271 {14}
140	MSE($\hat{b}$)	0.09225 {14}	0.10046 {15}	0.09015 {13}	0.04164 {3}	0.07927 {9}	0.06196 {8}	0.05528 {6}	0.08355 {10}	0.10776 {16}	0.05958 {7}	0.03958 {2}	0.03646 {1}	0.05044 {5}	0.08556 {11}	0.04813 {4}	0.08886 {12}
140	MSE($\hat{c}$)	0.07991 {15}	0.04605 {8}	0.02055 {6}	0.06907 {12}	0.01237 {3}	0.05434 {9}	0.01315 {4}	0.03136 {7}	0.0717 {13}	0.0058 {2}	0.0163 {5}	0.06482 {11}	0.00131 {1}	0.09346 {16}	0.0602 {10}	0.0797 {14}
140	MSE($\hat{\delta}$)	0.95699 {16}	0.57475 {9}	0.29873 {5}	0.73579 {13}	0.19495 {2}	0.64181 {11}	0.32022 {6}	0.40005 {7}	0.61455 {10}	0.21165 {3}	0.27245 {4}	0.46274 {8}	0.02863 {1}	0.86506 {15}	0.70824 {12}	0.85755 {14}
140	MRE($\hat{b}$)	0.0942 {14}	0.09728 {15}	0.08274 {9}	0.06057 {1}	0.08369 {10}	0.07693 {7}	0.0712 {4}	0.08417 {11}	0.10861 {16}	0.07777 {8}	0.06269 {3}	0.0611 {2}	0.07412 {6}	0.08739 {12}	0.07244 {5}	0.09346 {13}
140	MRE($\hat{c}$)	0.39908 {15}	0.25775 {8}	0.1501 {5}	0.33841 {11}	0.11438 {3}	0.29134 {9}	0.15288 {6}	0.20483 {7}	0.34729 {12}	0.09688 {2}	0.14745 {4}	0.3707 {13}	0.04779 {1}	0.41818 {16}	0.33248 {10}	0.38963 {14}
140	MRE($\hat{\delta}$)	0.44875 {16}	0.32164 {9}	0.19182 {5}	0.36008 {12}	0.16957 {3}	0.34048 {11}	0.22733 {6}	0.25549 {7}	0.33987 {10}	0.16049 {2}	0.17768 {4}	0.31809 {8}	0.06281 {1}	0.42208 {15}	0.37219 {13}	0.40757 {14}
140	D_{abs}	0.02255 {3}	0.02304 {5}	0.02406 {10}	0.02098 {1}	0.02389 {8}	0.02327 {6}	0.0236 {7}	0.02242 {2}	0.02393 {9}	0.02853 {15}	0.02281 {4}	0.0284 {14}	0.02435 {11}	0.03055 {16}	0.02502 {12}	0.02707 {13}
140	D_{max}	0.03605 {2}	0.03786 {6}	0.03954 {9}	0.03293 {1}	0.0397 {10}	0.03722 {5}	0.03864 {7}	0.03685 {3}	0.03971 {11}	0.04639 {15}	0.037 {4}	0.04582 {14}	0.04009 {12}	0.04812 {16}	0.03939 {8}	0.04384 {13}
140	ASAE	0.01122 {4}	0.01082 {1}	0.01198 {8}	0.01175 {7}	0.01129 {6}	0.01106 {2}	0.01115 {3}	0.01126 {5}	0.01229 {9}	0.01693 {14}	0.01408 {12}	0.02111 {16}	0.01358 {10}	0.01723 {15}	0.01379 {11}	0.01684 {13}
140	$\sum Ranks$	144 {13.5}	108 {10}	89 {8}	85 {7}	70 {4}	95 {9}	65 {3}	84 {6}	144 {13.5}	80 {5}	53 {1}	110 {11}	56 {2}	175 {16}	113 {12}	161 {15}
200	BIAS($\hat{b}$)	0.22089 {16}	0.17138 {8}	0.14713 {4}	0.14987 {5}	0.15507 {7}	0.17374 {10}	0.17262 {9}	0.18409 {12}	0.21589 {14}	0.14529 {3}	0.12045 {1}	0.15202 {6}	0.13251 {2}	0.21824 {15}	0.17721 {11}	0.2072 {13}
200	BIAS($\hat{c}$)	0.2382 {15}	0.14852 {8}	0.05637 {4}	0.2017 {12}	0.05372 {2}	0.15397 {9}	0.07583 {5}	0.08474 {7}	0.18383 {10}	0.05573 {3}	0.08375 {6}	0.20506 {13}	0.02098 {1}	0.25024 {16}	0.19858 {11}	0.23273 {14}
200	BIAS($\hat{\delta}$)	0.89429 {16}	0.59963 {8}	0.30446 {3}	0.7197 {12}	0.27389 {2}	0.68037 {11}	0.38964 {6}	0.45956 {7}	0.64653 {10}	0.31958 {4}	0.35522 {5}	0.61462 {9}	0.11162 {1}	0.84288 {15}	0.74346 {13}	0.81161 {14}
200	MSE($\hat{b}$)	0.07256 {15}	0.05221 {11}	0.04243 {7}	0.03965 {5}	0.04067 {6}	0.05077 {10}	0.04895 {9}	0.05659 {12}	0.07038 {14}	0.03428 {3}	0.02318 {1}	0.03644 {4}	0.02682 {2}	0.0824 {16}	0.04663 {8}	0.06701 {13}
200	MSE($\hat{c}$)	0.07842 {15}	0.04489 {9}	0.00761 {4}	0.06813 {13}	0.00725 {3}	0.03997 {8}	0.00974 {5}	0.01276 {6}	0.06174 {12}	0.0053 {2}	0.01467 {7}	0.05362 {10}	0.00077 {1}	0.09072 {16}	0.05872 {11}	0.07784 {14}
200	MSE($\hat{\delta}$)	0.95372 {16}	0.5151 {9}	0.1676 {3}	0.73398 {13}	0.12078 {2}	0.6407 {11}	0.22838 {5}	0.34724 {7}	0.58671 {10}	0.20981 {4}	0.27222 {6}	0.4272 {8}	0.01998 {1}	0.86417 {15}	0.70468 {12}	0.85518 {14}
200	MRE($\hat{b}$)	0.08835 {16}	0.06855 {8}	0.05885 {5}	0.05744 {3}	0.06203 {7}	0.0695 {10}	0.06905 {9}	0.07363 {12}	0.08636 {14}	0.05811 {4}	0.04818 {1}	0.0608 {6}	0.05301 {2}	0.08675 {15}	0.07088 {11}	0.08288 {13}
200	MRE($\hat{c}$)	0.39716 {15}	0.24754 {8}	0.09395 {3}	0.33566 {12}	0.08953 {2}	0.25661 {9}	0.12638 {5}	0.14124 {6}	0.30638 {10}	0.0954 {4}	0.14605 {7}	0.34177 {13}	0.03497 {1}	0.41762 {16}	0.3318 {11}	0.38831 {14}
200	MRE($\hat{\delta}$)	0.44723 {16}	0.29982 {8}	0.15223 {3}	0.35576 {12}	0.13695 {2}	0.33884 {11}	0.19482 {6}	0.22978 {7}	0.32327 {10}	0.15599 {4}	0.17621 {5}	0.30731 {9}	0.05581 {1}	0.41958 {15}	0.37179 {13}	0.40605 {14}
200	D_{abs}	0.01641 {1}	0.01905 {6}	0.02092 {11}	0.01793 {3}	0.01894 {5}	0.02002 {8}	0.0209 {10}	0.01846 {4}	0.0173 {2}	0.02198 {12}	0.02046 {9}	0.02546 {15}	0.01944 {7}	0.02419 {14}	0.02256 {13}	0.02577 {16}
200	D_{max}	0.02601 {1}	0.03094 {4}	0.03391 {10}	0.02968 {3}	0.03156 {6}	0.03279 {9}	0.03499 {11}	0.03099 {5}	0.0286 {2}	0.03529 {12}	0.03245 {8}	0.0427 {16}	0.03191 {7}	0.03888 {14}	0.03614 {13}	0.04075 {15}
200	ASAE	0.00869 {3}	0.00883 {7}	0.00882 {6}	0.00938 {8}	0.00878 {5}	0.00873 {4}	0.00853 {2}	0.00851 {1}	0.01059 {10}	0.01262 {13}	0.01112 {12}	0.01758 {16}	0.0102 {9}	0.01385 {14}	0.01093 {11}	0.01457 {15}
200	$\sum Ranks$	145 {14}	94 {8}	63 {3}	101 {9}	49 {2}	110 {10}	82 {6}	86 {7}	118 {11}	68 {4.5}	68 {4.5}	125 {12}	35 {1}	181 {16}	138 {13}	169 {15}
300	BIAS($\hat{b}$)	0.17296 {14}	0.14454 {11}	0.13097 {8}	0.12625 {5}	0.11177 {2}	0.13202 {9}	0.10548 {1}	0.12735 {6}	0.16153 {12}	0.13418 {10}	0.11715 {3}	0.1301 {7}	0.11725 {4}	0.17393 {15}	0.17217 {13}	0.19295 {16}
300	BIAS($\hat{c}$)	0.20091 {14}	0.11046 {9}	0.04751 {2}	0.17158 {11}	0.04904 {3}	0.14025 {10}	0.07271 {5}	0.08335 {7}	0.10427 {8}	0.05567 {4}	0.08174 {6}	0.1975 {13}	0.02009 {1}	0.23428 {16}	0.19657 {12}	0.22884 {15}
300	BIAS($\hat{\delta}$)	0.82955 {15}	0.50956 {9}	0.28566 {3}	0.69532 {12}	0.24086 {2}	0.67915 {11}	0.38789 {6}	0.44009 {8}	0.43977 {7}	0.31895 {4}	0.3543 {5}	0.587 {10}	0.08705 {1}	0.83423 {16}	0.74322 {13}	0.81047 {14}
300	MSE($\hat{b}$)	0.04508 {13}	0.03815 {11}	0.03157 {10}	0.02439 {5}	0.01958 {2}	0.03027 {9}	0.01776 {1}	0.02821 {6}	0.04021 {12}	0.02907 {8}	0.02221 {4}	0.02834 {7}	0.02194 {3}	0.04614 {14}	0.04636 {15}	0.0588 {16}
300	MSE($\hat{c}$)	0.06395 {14}	0.02582 {9}	0.00328 {3}	0.05034 {11}	0.00406 {4}	0.03156 {10}	0.00746 {5}	0.0123 {6}	0.02487 {8}	0.00208 {2}	0.01257 {7}	0.05048 {12}	0.00067 {1}	0.08083 {16}	0.0578 {13}	0.07738 {15}
300	MSE($\hat{\delta}$)	0.92402 {16}	0.40411 {10}	0.13282 {3}	0.71507 {13}	0.10398 {2}	0.63966 {11}	0.22572 {5}	0.30241 {8}	0.30232 {7}	0.2077 {4}	0.27001 {6}	0.40138 {9}	0.01335 {1}	0.86388 {15}	0.70392 {12}	0.85453 {14}
300	MRE($\hat{b}$)	0.06919 {14}	0.05782 {11}	0.05239 {8}	0.0505 {5}	0.04471 {2}	0.05281 {9}	0.04219 {1}	0.05094 {6}	0.06461 {12}	0.05367 {10}	0.04727 {4}	0.05204 {7}	0.0469 {3}	0.06957 {15}	0.06887 {13}	0.07718 {16}
300	MRE($\hat{c}$)	0.33485 {14}	0.18409 {9}	0.07919 {2}	0.28597 {11}	0.08174 {3}	0.23374 {10}	0.12118 {5}	0.13902 {6}	0.17378 {8}	0.09394 {4}	0.14524 {7}	0.32917 {12}	0.03348 {1}	0.39046 {16}	0.33161 {13}	0.3814 {15}
300	MRE($\hat{\delta}$)	0.41478 {15}	0.25478 {9}	0.14283 {3}	0.34766 {12}	0.12043 {2}	0.33855 {11}	0.19462 {6}	0.22005 {8}	0.21988 {7}	0.15574 {4}	0.17465 {5}	0.2935 {10}	0.04352 {1}	0.41712 {16}	0.37103 {13}	0.40419 {14}
300	D_{abs}	0.01455 {3}	0.01559 {7}	0.01651 {9}	0.01452 {2}	0.01624 {8}	0.01321 {1}	0.01692 {11}	0.01547 {6}	0.01505 {4}	0.02008 {14}	0.01727 {12}	0.02142 {16}	0.01674 {10}	0.02045 {15}	0.01522 {5}	0.01906 {13}
300	D_{max}	0.02338 {2}	0.02552 {7}	0.02731 {10}	0.02379 {3}	0.02633 {8}	0.02146 {1}	0.02725 {9}	0.025 {6}	0.02471 {5}	0.03209 {14}	0.02749 {11}	0.03538 {16}	0.02789 {12}	0.03292 {15}	0.02447 {4}	0.03106 {13}
300	ASAE	0.00663 {3}	0.00666 {4.5}	0.00685 {7}	0.00698 {8}	0.00669 {6}	0.00666 {4.5}	0.00652 {2}	0.00647 {1}	0.00759 {9}	0.00998 {13}	0.00858 {12}	0.01351 {16}	0.00797 {10}	0.01056 {14}	0.00842 {11}	0.01079 {15}
300	$\sum Ranks$	137 {13.5}	106.5 {11}	68 {4}	98 {9}	44 {1}	96.5 {8}	57 {3}	74 {5}	99 {10}	91 {7}	82 {6}	135 {12}	48 {2}	183 {16}	137 {13.5}	176 {15}

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