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Point and Interval Estimation of the Stress-Strength Reliability for discrete model based on different sampling plans

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Abstract

In recent years, ranked set sampling, a cheap and effective sampling method, has been used for statistical inferences as an alternative to traditional simple random sampling. Although this trendy topic has been frequently studied in continuous models, it has only been studied in parameter estimation for the discrete Weibull distribution in discrete models. Additionally, it is seen in the literature that stress-strength reliability has not been studied under ranked set sampling for discrete models. This paper discusses statistical inference for the stress-strength reliability when stress and strength are independent discrete Poisson-Ailamujia random variables under ranked set sampling. Moreover, stress-strength reliability estimators are obtained using point and bootstrap confidence interval estimation methods in simple random and ranked set sampling methods. The performances of the proposed estimators are compared using a Monte Carlo simulation. The simulation results and three real data applications show that the estimators obtained by ranked set sampling are preferable to the traditional simple random sampling in terms of efficiency.

Keywords: Bootstrap confidence intervals, Discrete distribution, Estimation, Ranked set sampling, Stress-strength reliability, Monte Carlo simulation.

1 Introduction

There are many fields such as engineering, medicine, statistics, physics, psychology, biostatistics, and economics of study in the statistical literature that investigates how to estimate system stress-strength reliability ($R = P(X<Y)$). The random variables stress-strength reliability (SSR) X and Y are assumed to be independent, and Y represents the strength of the system and X represents the stress. Obviously, the system will fail if the applied stress X is greater than the strength Y (X>Y), otherwise, the system keeps running if (X <Y), so R = P(X <Y) is a measure of the system performance.

Most of the work done with the SSR is related to continuous probability distributions. But in some real life, stress or strength can have a discrete random variable. For example, if X represents the number of products that customers want to buy, Y represents the number of products that the business produces, R represents the probability of meeting demand, in medical research, X represents the control without treatment, and Y represents the treatment results, then R represents the effectiveness of the treatment. Another example is given in engineering studies. A multitude of engineering problems can be solved using stress strength models, such as determining whether the power of a rocket engine can exceed the operating pressure, determining whether a building's strength can withstand an earthquake designed for it, and comparing the two processes. Nowadays, SSR models are widely used in these experiments when X and Y represent the lifetime times of two devices in life tests, and one wants to predict the probability that it will fail before the other. The estimation of SSR has been discussed by many authors, starting with the pioneering work of Birnbaum [1]. Maiti [2] and Ahmad et al. [3] investigated the Bayesian estimation of $R = P(Y > X)$ for two independent, non-identically distributed geometric random variables using Lindley's approximation. Maiti and Murmu [6] obtained the Bayesian estimator of R when X and Y have independent two-parameter geometric distributions. The negative binomial distribution is evaluated by Ivshin and Lumelskii [4] and Sathe and Dixit [5]. In the context of Poisson processes, Belyaev and Lumelskii [7] developed the uniformly minimum variance unbiased estimator for the SSR. In recent years, Tarvirdizade and Ahmadpour [8], Babayi and Khorram [9], Hassan et al. [10] and Akgül and Şenoğlu [11] conducted studies on SSR.

The variables of stress and strength involved in studies are difficult to quantify and often involve complex procedures that are time-consuming and/or expensive. However, access to some variables can be obtained from sampling units that can be easily measured at a low cost. Therefore, the use of cost-effective sampling methods to obtain the measurement of the features of interest is more preferable than traditional sampling methods. The ranked set sampling (RSS) method, proposed by McIntyre [13] as a cost-effective and efficient alternative to a simple random sample (SRS), is widely used in the estimation of SSR, [see [14]-[19]]. The RSS is a useful method in cases where actual measurements of sampling units are difficult or expensive to obtain, but it is still easy and more convenient to assign sequence numbers using visual comparisons or rough methods according to the variable of interest by Al-Mutairi et al. [12]. Inference procedures based on RSS are often superior to their SRS counterparts if they have the same sample size by Mahdizadeh [20].

Some of the studies on the RSS-based SSR are as follows: Safariyan et al. [21] examined improved point and interval estimation of SSR based on RSS exponential distribution. Al-Omari et al. [22] worked on the estimation of SSR for the exponential Pareto distribution using median rank set sampling and RSS methods. Akgül and Şenoğlu [23] examined the estimation of R=P(X <Y) for the Burr Type X distribution based on RSS. As it can be understood from these studies, RSS-based SSR, are carried out using continuous distributions. In the literature on SSR, no studies based on the RSS method have been encountered in discrete distributions. Therefore, this article is the first study on the estimation of R=P(X <Y) based on RSS for discrete distributions. The first purpose of this paper is to estimate SSR using different estimators based on SRS and RSS methods in situations where both stress and strength are independent of the Poisson-Ailamujia (PoA) distribution. The PoA distribution is proposed by Hassan et al. [24]. The cumulative distribution and probability mass functions (pmf) are as follows, respectively,

$$F_X(x) = 1 - \frac{(4\alpha + 2\alpha x + 1)}{(1+2\alpha)^{x+2}}, \tag{1}$$

$$f(x) = \frac{4\alpha^2(1+x)}{(1+2\alpha)^{x+2}}, \tag{2}$$

where $x = 0, 1, ...$ and $\alpha > 0$. X and Y are two independent random variables with the same distribution, $X \sim PoA(\alpha_1)$ represents the stress and $Y \sim PoA(\alpha_2)$ are represents strength. Then SSR for X and Y is obtained by

$$\begin{aligned} R = P(X < Y) &= \sum_{x=0}^{\infty} \frac{4\alpha_1^2(1+x)}{(1+2\alpha_1)^{(x+2)}} \sum_{y=x+1}^{\infty} \frac{4\alpha_2^2(1+y)}{(1+2\alpha_2)^{(y+2)}}, \\ &= \frac{\alpha_1^2\left(8\alpha_1\alpha_2^2 + 4\alpha_2^2 + 3\alpha_2 + 6\alpha_1\alpha_2 + \alpha_1\right)}{(\alpha_2 + \alpha_1 + 2\alpha_1\alpha_2)^3}. \end{aligned}$$

In this study, we assume that X and Y are follows PoA distributions with pmf $f(x, \alpha_1)$ and $f(y, \alpha_2)$, respectively. We then derive the maximum likelihood (ML), least squares (LS), weighted least squares (WLS), and Cramer-von Mises (CVM) estimators for the system reliability based on SRS and RSS. Also to evaluate the performance of these estimators under changing situations. The second purpose bootstrap confidence intervals (BCIs) are obtained using bootstrap methods that include student, standard, percentile, normal, and bias-corrected percentile bootstrap methods for the SSR distribution based on different estimation methods.

The remainder of the article is structured as follows. In Section 2, the estimate of SSR was obtained using the methods of ML, LS, WLS, and CVM for both SRS and RSS. In Section 3, the BCIs are discussed based on different estimation methods using SRS and RSS sampling methods for SSR. In Section 4, a Monte Carlo simulation study was conducted to evaluate the performance of BCI between different estimation and sampling methods. Three real data applications are presented in Section 5.

2 Estimation stress-strength reliability for Poisson-Ailamujia distribution

2.1 Estimation of stress-strength reliability based on ranked set sample

In this section, the RSS method, first proposed by McIntyre [13], is mentioned. To select a sample of size $n = mr$ from a population using the RSS method, the implementation steps are as follows.

Step 1: A sample of m^2 size is randomly selected from the population. The selected sample is randomly allocated to m clusters of m size.

Step 2: Units in each cluster are ordered from smallest to largest. This ranking can be done without precise visual measurement or by utilizing an auxiliary variable that is easy to measure and highly correlated with the variable of interest.

Step 3: Of the units listed; the unit in the first row from the first cluster, the unit in the second row from the second cluster, and so on, the unit in the m row from the m cluster is selected and the selected units are measured in terms of the variable of interest.

Step 4: Steps 1 to 3 are repeated r times until the sample size is $n = mr$.
Here, m and r correspond to the set size or the number of cycles. It should be noted that the setting size m plays an important role in the RSS method. Let $X_{(i)ik}$, $(i = 1, ..., m_x, k = 1, ..., r_x)$ and $Y_{(j)jl}$, $(j = 1, ..., m_y, l = 1, ..., r_y)$ indicate the RSS from $PoA\,(\alpha_1)$ and $PoA\,(\alpha_2)$ with sample size $n_1 = m_x r_x$ and $n_2 = m_y r_y$. For the sake of brevity, we'll use the notation X_{ik} and Y_{jl} instead of $X_{(i)ik}$ and $Y_{(j)jl}$, respectively. When the judgment ranking is perfect, the pmfs of X_{ik} and Y_{jl} are indicated by

$$f\,(x) = \sum_{s=r}^{n_1} \binom{m_x}{s} \left\{ [F\,(x)]^s\,[1 - F\,(x)]^{m_x - s} - [F\,(x-1)]^s\,[1 - F\,(x-1)]^{m_x - s} \right\}, \tag{3}$$

$$f\,(y) = \sum_{w=r}^{n_2} \binom{m_y}{w} \left\{ [F\,(y)]^w\,[1 - F\,(y)]^{m_y - w} - [F\,(y-1)]^w\,[1 - F\,(y-1)]^{m_y - w} \right\}. \tag{4}$$

To obtain the ML estimator of the SSR using Equations (3)-(4), the RSS-based likelihood assuming a perfect ranking is written as follows,

$$L(\alpha_1, \alpha_2) = \prod_{k=1}^{r_x} \prod_{i=1}^{m_x} f\,(x_{ik}) \prod_{l=1}^{r_y} \prod_{j=1}^{m_y} f\,(y_{jl})\,, \tag{5}$$

when Equations (3)-(4) are replaced in Equation (5), the log-likelihood equation is obtained as follows.

$$\begin{aligned} \ell(\alpha_1, \alpha_2) &= \sum_{k=1}^{r_x} \sum_{i=1}^{m_x} \log f\,(x_{ik}) \sum_{l=1}^{r_y} \sum_{j=1}^{m_y} \log f\,(y_{jl}) \\ &= \sum_{k=1}^{r_x} \sum_{i=1}^{m_x} \log \left\{ \sum_{s=r}^{n_1} \binom{n_1}{s} (-1)^{2s} \left[\frac{\left[4\left(\frac{\alpha_1+1}{2}\right)^2 (1+2\alpha_1)^{x_{ik}} - \alpha_1\,(2x_{ik}+4) - 1 \right]^s}{(1+2\alpha_1)^{n_1 (x_{ik}+2)}\,[1+\alpha_1\,(2x_{ik}+4)]^{s-n_1}} \right] \right\} \end{aligned}$$

$$
-\sum_{k=1}^{r_x}\sum_{i=1}^{m_x}\log\left\{\sum_{s=r}^{n_1}\binom{n_1}{s}(-1)^{2s}\left[\frac{\left[(1+2\alpha_1)^{x_{ik}}+2\alpha_1(1+2\alpha_1)^{x_{ik}}-2\alpha_1(x_{ik}+1)-1\right]^s}{(1+2\alpha_1)^{n_1(x_{ik}+1)}\left[1+\alpha_1(2x_{ik}+2)\right]^{s-n_1}}\right]\right\}
$$

$$
+\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\log\left\{\sum_{w=r}^{n_2}\binom{n_2}{w}(-1)^{2w}\left[\frac{\left[4\left(\frac{\alpha_2+1}{2}\right)^2(1+2\alpha_2)^{y_{jl}}-\alpha_2\left(2^{y_{jl}}+4\right)-1\right]^w}{(1+2\alpha_2)^{n_2\left(y_{jl}+2\right)}\left[1+\alpha_2\left(2^{y_{jl}}+4\right)\right]^{w-n_2}}\right]\right\}
$$

$$
-\sum_{l=1}^{r_y}\sum_{j=1}^{m_y}\log\left\{\sum_{w=r}^{n_2}\binom{n_2}{w}(-1)^{2w}\left[\frac{\left[(1+2\alpha_2)^{y_{jl}}+2\alpha_2(1+2\alpha_2)^{y_{jl}}-2\alpha_2(y_{jl}+1)-1\right]^w}{(1+2\alpha_2)^{n_2\left(y_{jl}+1\right)}\left[1+\alpha_2(2y_{jl}+2)\right]^{w-n_2}}\right]\right\},
$$

where the RSS-based ML estimators of α_1 and α_2, (say $\tilde{\alpha_1}_{ML}$ and $\tilde{\alpha_2}_{ML}$), are solutions to the following linear equations: $\partial\ell(\alpha_1,\alpha_2)/\partial\alpha_1 = 0$ and $\partial\ell(\alpha_1,\alpha_2)/\partial\alpha_2 = 0$. The ML estimators of α_1 and α_2 cannot be determined explicitly. Therefore, they can be determined using numerical methods. The optim function in R is used for this purpose. After that, the ML estimate of SSR based on RSS, namely, $\tilde{R}_{ML-RSS}$ is obtained as follows by using the invariance property of the ML estimators.

$$
\tilde{R}_{ML-RSS}=\frac{\tilde{\alpha}_{1ML}^2\left(8\tilde{\alpha}_{1ML}\tilde{\alpha}_{2ML}^2+4\tilde{\alpha}_{2ML}^2+3\tilde{\alpha}_{2ML}+6\tilde{\alpha}_{1ML}\tilde{\alpha}_{2ML}+\tilde{\alpha}_{1ML}\right)}{\left(\tilde{\alpha}_{2ML}+\tilde{\alpha}_{1ML}+2\tilde{\alpha}_{1ML}\tilde{\alpha}_{2ML}\right)^3}.
$$

2.2 Estimation of stress-strength reliability based on simple random sample

Suppose that X and Y are random variables in the SSR that are independently distributed as $X \sim PoA(\alpha_1)$ and $Y \sim PoA(\alpha_2)$ respectively. Let X_i, $(i = 1, ..., n_1)$ and Y_j, $(j = 1, ..., n_2)$ be two independent SRS from $PoA(\alpha_1)$ and $PoA(\alpha_2)$, respectively. Then the likelihood function is given by

$$
\begin{aligned}
L(\alpha_1,\alpha_2) &= \prod_{i=1}^{n_1} f(x_i)\prod_{j=1}^{n_2} f(y_j), \\
&= 4^{n_1+n_2}\alpha_1^{2n_1}\alpha_2^{2n_2}\prod_{i=1}^{n_1}\frac{1+x_i}{(1+2\alpha_1)^{(x_i+2)}}\prod_{j=1}^{n_2}\frac{1+y_j}{(1+2\alpha_2)^{(y_j+2)}}. \qquad (6)
\end{aligned}
$$

To obtain the ML estimators of α_1 and α_2, we first derive the log-likelihood function by taking the logarithm of Equation 6.

$$
\begin{aligned}
\ell(\alpha_1,\alpha_2) &= (n_1+n_2)\ln(4)+2n_1\ln(\alpha_1)+2n_2\ln(\alpha_2)+\ln\sum_{i=1}^{n_1}(1+x_i) \\
&-\sum_{i=1}^{n_1}(x_i+2)\ln(1+2\alpha_1)+\ln\sum_{j=1}^{n_2}\left(1+y_j\right)-\sum_{j=1}^{n_2}\left(y_j+2\right)\ln(1+2\alpha_2). \qquad (7)
\end{aligned}
$$

We then take the derivatives of the log-likelihood function according to the parameters of interest and obtain the likelihood Equations (8)-(9).

$$
\frac{\partial\ell(\alpha_1,\alpha_2)}{\partial\alpha_1}=\frac{2n_1}{\alpha_1}-\frac{2\sum_{i=1}^{n_1}(x_i+2)}{1+2\alpha_1}=0, \qquad (8)
$$

$$\frac{\partial \ell(\alpha_1,\alpha_2)}{\partial \alpha_2} = \frac{2n_2}{\alpha_2} - \frac{2\sum_{j=1}^{n_2}(y_j+2)}{1+2\alpha_2} = 0. \tag{9}$$

Solutions to likelihood Equations (8)-(9) can be obtained in closed form, as well as estimates of the unknown parameters, can be obtained $\widehat{\alpha}_{1ML} = \frac{1}{\overline{x}}$ and $\widehat{\alpha}_{2ML} = \frac{1}{\overline{y}}$ using numerical methods. After that, the ML estimation of system reliability based on SRS, namely, $\widehat{R}_{ML-SRS}$ is obtained as follows:

$$\widehat{R}_{ML-SRS} = \frac{\overline{x}^{-2}\left(8\overline{x}^{-1}\overline{y}^{-2} + 4\overline{y}^{-2} + 3\overline{y}^{-1} + 6\overline{x}^{-1}\overline{y}^{-1} + \overline{x}^{-1}\right)}{\left(\overline{y}^{-1} + \overline{x}^{-1} + 2\overline{x}^{-1}\right)^3} \tag{10}$$

by using the invariance property of the ML estimators.

2.3 Least square estimation, weight least square estimation and Cramer-von Mises estimate

The LS, WLS, and CVM method estimation yield similar directions for parameter estimation when using SRS and RSS. However, the differences in the parameter estimations obtained by these three methods are largely due to the sampling selection method employed. In this context, the process of obtaining parameter estimates using these methods is explained as follows.

The LS and WLS estimators are obtained by minimizing the sum of the squares of the residuals. Let $X_1, X_2, ..., X_{n_1}$ be a random sample from the distribution with pmf is iid sample from $PoA(\alpha_1)$ and $Y_1, Y_2, ..., Y_{n_2}$ is iid sample from $PoA(\alpha_2)$. It assumed that X_i and Y_j $(i \neq j)$ are independent. $X_{(1)}, X_{(2)}, ..., X_{(n_1)}$ and $Y_{(1)}, Y_{(2)}, ..., Y_{(n_2)}$ denote the rank statistics of the random samples. The mean rank is used to estimate the values of the cumulative distribution function $F(x)$,

$$\widehat{F}\left(x_{(i)}\right) = \frac{i}{n_1+1}; \widehat{F}\left(y_{(j)}\right) = \frac{j}{n_2+1}$$

$$\widehat{F}\left(x_{(i)}, \alpha_1\right) = \sum_{i=1}^{n_1}\left(F\left(x_{i:n_1}|\alpha_1\right) - \widehat{F}\left(x_{(i)}\right)\right)^2$$

$$\widehat{F}\left(y_{(j)}, \alpha_2\right) = \sum_{j=1}^{n_2}\left(F\left(y_{j:n_2}|\alpha_2\right) - \widehat{F}\left(y_{(j)}\right)\right)^2$$

The parameter vector that minimizes the quadratic expression of the equation is obtained. With respect α_1 and α_2, where $F(.)$ is the cdf in Equation 1. Equivalently they can be obtained by solving:

$$\sum_{i=1}^{n_1}\left[F\left(x_{i:n_1}|\alpha_1\right) - \frac{i}{n_1+1}\right]\eta_1\left(x_{i:n_1}|\alpha_1\right) = 0,$$

$$\sum_{i=1}^{n_2}\left[F\left(y_{j:n_2}|\alpha_2\right) - \frac{j}{n_2+1}\right]\eta_2\left(y_{j:n_2}|\alpha_2\right) = 0,$$

where

$$\eta_1 = \frac{dF(x_{i:n_1})}{d\alpha_1} = 4(x+2)\alpha_1(1+x)(1+2\alpha_1)^{-x-3} \tag{11}$$

$$\eta_2 = \frac{dF(y_{j:n_2})}{d\alpha_2} = 4(y+2)\alpha_2(1+y)(1+2\alpha_2)^{-y-3} \tag{12}$$

The solutions to Equations (11)-(12) can be obtained in closed form, and the estimates of the unknown parameters, obtained using the SRS method as $\widehat{\alpha}_{1LS}$ and $\widehat{\alpha}_{2LS}$, and using the RSS method as $\tilde{\alpha}_{1LS}$ and $\tilde{\alpha}_{2LS}$, can be determined using numerical methods. After that, the LS estimations of SSR denoted as $\widehat{R}_{LS-SRS}$ and $\tilde{R}_{LS-RSS}$, are obtained as follows:

$$\widehat{R}_{LS-SRS} = \frac{\widehat{\alpha}_{1LS}^2\left(8\widehat{\alpha}_{1LS}\widehat{\alpha}_{2LS}^2 + 4\widehat{\alpha}_{2LS}^2 + 3\widehat{\alpha}_{2LS} + 6\widehat{\alpha}_{1LS}\widehat{\alpha}_{2LS} + \widehat{\alpha}_{1LS}\right)}{\left(\widehat{\alpha}_{2LS} + \widehat{\alpha}_{1LS} + 2\widehat{\alpha}_{1LS}\widehat{\alpha}_{2LS}\right)^3}$$

and

$$\tilde{R}_{\text{LS-RSS}} = \frac{\tilde{\alpha}_{1\text{LS}}^2\left(8\tilde{\alpha}_{1\text{Ls}}\tilde{\alpha}_{2\text{LS}}^2 + 4\tilde{\alpha}_{2\text{LS}}^2 + 3\tilde{\alpha}_{2\text{LS}} + 6\tilde{\alpha}_{1\text{LS}}\tilde{\alpha}_{2\text{LS}} + \tilde{\alpha}_{1\text{LS}}\right)}{\left(\tilde{\alpha}_{2\text{Ls}} + \tilde{\alpha}_{1\text{LS}} + 2\tilde{\alpha}_{1\text{LS}}\tilde{\alpha}_{2\text{LS}}\right)^3}$$

using the LS estimators. Then the WLS estimators of the parameter of $PoA(\alpha_1)$ and $PoA(\alpha_2)$ distributions are obtained by minimizing

$$W(\alpha_1) = \sum_{i=1}^{n_1} \frac{(n_1+1)^2(n_1+2)}{i(n_1-i+1)}\left[F(x_{i:n_1}|\alpha_1) - \frac{i}{n_1+1}\right]^2$$

$$W(\alpha_2) = \sum_{j=1}^{n2} \frac{(n2+1)^2(n_2+2)}{j(n_2-j+1)}\left[F(y_{j:n_2}|\alpha_2) - \frac{j}{n_2+1}\right]^2$$

The parameter estimates are obtained by solving the equations:

$$\sum_{i=1}^{n_1} \frac{(n_1+1)^2(n_1+2)}{i(n_1-i+1)}\left[F(x_{i:n_1}|\alpha_1) - \frac{i}{n_1+1}\right]\eta_1(x_{i:n_1}|\alpha_1) = 0 \tag{13}$$

$$\sum_{j=1}^{n2} \frac{(n2+1)^2(n_2+2)}{j(n_2-j+1)}\left[F(y_{j:n_2}|\alpha_2) - \frac{j}{n_2+1}\right]\eta_2(y_{j:n_2}|\alpha_2) = 0 \tag{14}$$

Solutions to Equations (13)-(14) can be obtained in closed form, as well as estimates of the unknown parameters, obtained using the SRS method as $\widehat{\alpha}_{1WLS}$ and $\widehat{\alpha}_{2WLS}$, and using the RSS method as $\tilde{\alpha}_{1WLS}$ and $\tilde{\alpha}_{2WLS}$, can be determined using numerical methods. After that, the WLS estimations of SSR denoted as $\widehat{R}_{WLS-SRS}$ and $\tilde{R}_{WLS-RSS}$, are obtained as follows:

$$\widehat{R}_{WLS-SRS} = \frac{\widehat{\alpha}_{1WLS}^2\left(8\widehat{\alpha}_{1WLS}\widehat{\alpha}_{2WLS}^2 + 4\widehat{\alpha}_{2WLS}^2 + 3\widehat{\alpha}_{2WLS} + 6\widehat{\alpha}_{1WLS}\widehat{\alpha}_{2WLS} + \widehat{\alpha}_{1WLS}\right)}{\left(\widehat{\alpha}_{2WLS} + \widehat{\alpha}_{1WLS} + 2\widehat{\alpha}_{1WLS}\widehat{\alpha}_{2WLS}\right)^3}$$

and

$$\tilde{R}_{\text{WLS-RSS}}=\frac{\tilde{\alpha}_{1\text{WLS}}^{2}\left(8\tilde{\alpha}_{1\text{WLS}}\tilde{\alpha}_{2\text{WLS}}^{2}+4\tilde{\alpha}_{2\text{WLS}}^{2}+3\tilde{\alpha}_{2\text{WLS}}+6\tilde{\alpha}_{1\text{WLS}}\tilde{\alpha}_{2\text{WLS}}+\tilde{\alpha}_{1\text{WLS}}\right)}{\left(\tilde{\alpha}_{2\text{WLS}}+\tilde{\alpha}_{1\text{WLS}}+2\tilde{\alpha}_{1\text{WLS}}\tilde{\alpha}_{2\text{WLS}}\right)^{3}}$$

using the WLS estimators.

The CVM estimators can be determined depending on the difference between both the estimated and exact distributions. Let $X_1, X_2, ..., X_{n_1}$ be a random sample from the distribution with pmf is iid sample from $PoA\left(\alpha_1\right)$ and $Y_1, Y_2, ..., Y_{n_2}$ is iid sample from $PoA\left(\alpha_2\right)$. The CVM estimators

$$CVM\left(\alpha_1\right)=\frac{1}{12n_1}+\sum_{i=1}^{n_1}\left[F\left(x_{i:n_1}|\alpha_1\right)-\frac{2i-1}{2n_1}\right]^2, \tag{15}$$

$$CVM\left(\alpha_2\right)=\frac{1}{12n_2}+\sum_{j=1}^{n_2}\left[F\left(y_{j:n_2}|\alpha_2\right)-\frac{2j-1}{2n_2}\right]^2. \tag{16}$$

Solutions to Equations (15)-(16) can be obtained in closed form, as well as estimates of the unknown parameters, obtained using the SRS method as $\widehat{\alpha}_{1CVM}$ and $\widehat{\alpha}_{2CVM}$ and using the RSS method as $\tilde{\alpha}_{1CVM}$ and $\tilde{\alpha}_{2CVM}$, can be determined using numerical methods. After that, the CVM estimations of SSR denoted as $\widehat{R}_{CVM-SRS}$ and $\tilde{R}_{CVM-RSS}$, are obtained as follows:

$$\widehat{R}_{CVM-SRS}=\frac{\widehat{\alpha}_{1CVM}^{2}\left(8\widehat{\alpha}_{1CVM}\widehat{\alpha}_{2CVM}^{2}+4\widehat{\alpha}_{2CVM}^{2}+3\widehat{\alpha}_{2CVM}+6\widehat{\alpha}_{1CVM}\widehat{\alpha}_{2CVM}+\widehat{\alpha}_{1CVM}\right)}{\left(\widehat{\alpha}_{2CVM}+\widehat{\alpha}_{1CVM}+2\widehat{\alpha}_{1CVM}\widehat{\alpha}_{2CVM}\right)^{3}}$$

and

$$\tilde{R}_{\text{CVM-RSS}}=\frac{\tilde{\alpha}_{1\text{CVM}}^{2}\left(8\tilde{\alpha}_{1\text{CVM}}\tilde{\alpha}_{2\text{CVM}}^{2}+4\tilde{\alpha}_{2\text{CVM}}^{2}+3\tilde{\alpha}_{2\text{CVM}}+6\tilde{\alpha}_{1\text{CVM}}\tilde{\alpha}_{2\text{CVM}}+\tilde{\alpha}_{1\text{CVM}}\right)}{\left(\tilde{\alpha}_{2\text{CVM}}+\tilde{\alpha}_{1\text{CVM}}+2\tilde{\alpha}_{1\text{CVM}}\tilde{\alpha}_{2\text{CVM}}\right)^{3}}$$

using the CVM estimators.

3 Bootstrap confidence intervals

With the advancement of computer technology, non-parametric BCIs are becoming increasingly common. The appeal of the bootstrap method lies in its lack of dependence on theoretical assumptions. Especially in SSR analyses, non-parametric bootstrap methods offer significant advantages. These methods allow for calculating confidence intervals (CIs) by taking repeated samples from data sets and increasing the reliability of the results. In SSR analyses, bootstrap methods provide CIs without requiring prior knowledge of the distribution shape or parameters. Thus, non-parametric bootstrap stands out as a powerful tool due to its adaptability to various data sets and situations.

In this part of the study, six different BCIs were used to compare SRS and RSS in terms of four predictors (ML, LS, WLS, and CVM). Among the main types of BCIs, standard (std-boot), percentile (p-boot), bias-corrected percentile (Bcp-boot), student (t-boot), basic, and normal (N-boot) are used.

Algorithm A1. Consider a samples from $PoA\left(\alpha_1\right)$ distribution, denoted as $X_1, X_2, ..., X_n$ and another sample $PoA\left(\alpha_2\right)$ distribution, denoted as $Y_1, Y_2, ..., Y_n$. Obtain the estimates for the parameters α_1 and α_2 based on this sample, denoted as $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$.

A2. Calculate the estimate of SSR by $\widehat{R} = R\left(\widehat{\alpha}_1, \widehat{\alpha}_2\right)$.

A3. Generate the sample $X_1^* < X_2^* < ... < X_{n_1}^*$ from the $PoA\left(\alpha_1\right)$ distribution and generate the sample $Y_1^* < Y_2^* < ... < Y_{n_2}^*$ from the $PoA\left(\alpha_2\right)$ distribution.

A4. Obtain the estimate of SSR based on the bootstrap sample $X_1^* < X_2^* < ... < X_n^*$ and $Y_1^* < Y_2^* < ... < Y_n^*$ denote it by $\widehat{R}^*$.

A5. Repeat Steps 2-4 B times, and obtain $\widehat{R}_1^*$, $\widehat{R}_2^*, ...,$ $\widehat{R}_B^*$. These can be treated as a copy of $\widehat{R}^*$.

The several BCIs based on SSR are given in below. In this section, six **non parametric** bootstrap methods are provided to construct the BCIs for SSR.

3.1 Standart boostrap

This method calculates CIs using the standard deviation of statistics obtained from bootstrap samples. $\overline{\widehat{R}^*}$ and Se^* be the sample mean and sample standard deviation of $\left\{\widehat{R}^{*(j)}; j = 1, 2, ..., B\right\}$,i.e.,

$$\overline{\widehat{R}^*} = \frac{1}{B}\sum_{j=1}^{B}\widehat{R}^{*(j)}$$

and

$$Se^* = \sqrt{\frac{1}{(B-1)}\sum_{j=1}^{B}\left(\widehat{R}^{*(j)} - \widehat{R}^*\right)^2},$$

where given sequentially. A $100(1-\alpha)\%$ std-boot CI of R is defined by

$$\left\{\overline{\widehat{R}^*} - z_{(\alpha/2)}Se^*, \overline{\widehat{R}^*} + z_{(\alpha/2)}Se^*\right\}.$$

where, $z_{(\alpha/2)}$ is acquired by using upper $(\alpha/2) - th$ point of the standard normal distribution.

3.2 Percentile bootstrap

Let $\widehat{R}^{*(\tau)}$ be the τ percentile of $\left\{\widehat{R}^{*(j)}; j=1,2,...,B\right\}$,i.e., $\widehat{R}^{*(\tau)}$ is such that

$$\frac{1}{B}\sum_{j=1}^{B} I\left(\widehat{R}^{*(j)} \leq \widehat{R}^{*(\tau)}\right) = \tau; \qquad 0<\tau<1,$$

where, $I(.)$ is the indicator function. A $100(1-\alpha)\%$ p-boot CI of R is defined by

$$\left\{\widehat{R}^{*(B_{\alpha/2})}, \widehat{R}^{*(B_{(1-\alpha/2)})}\right\}.$$

3.3 Bias-corrected percentile bootstrap

The Bcp-boot can be defined as follows. The first step is to find the observed $\widehat{R}$ in the bootstrap order statistics $\widehat{R}^{*(1)} \leq \widehat{R}^{*(2)} \leq ... \leq \widehat{R}^{*(B)}$. First, using the ordered distributions of $\left\{\widehat{R}^{*(j)}; j=1,2,...,B\right\}$, compute the probability

$$P_0 = \frac{1}{B}\sum_{j=1}^{B} I\left(\widehat{R}^{*(j)} \leq \widehat{R}\right).$$

where, $I(.)$ is the indicator function. Then we calculate $Z_0 = \Phi^{-1}(P_0)$, where $\Phi^{-1}(.)$ is the standard normal cdf and this worth is used to compute the probabilities P_l and P_u, described as

$$P_l = \Phi^{-1}\left(2Z_0 - z_{(\alpha/2)}\right) \quad \text{and} \quad P_u = \Phi\left(2Z_0 + z_{(\alpha/2)}\right).$$

A $100(1-\alpha)\%$ BC_p-boot CI of R is obtain by

$$\left\{\widehat{R}^{*(B.P_l)}, \widehat{R}^{*(B.P_u)}\right\}.$$

3.4 Studentized bootstrap

The t-boot CI is given by the following algorithm:

A1. Compute the statistics

$$T_i^* = \frac{\left(\widehat{R}_i^* - \widehat{R}\right)}{\sqrt{S\left(\widehat{R}_i^*\right)}}, i=1,2,...,B,$$

A2. The t-boot CI is given by

$$CI_{t-boot}^{1-\alpha}(R) = \left(\widehat{R} + Q_{T_i^*}(\alpha/2)\sqrt{S\left(\widehat{R}^*\right)}, \widehat{R} + Q_{T_i^*}(1-\alpha/2)\sqrt{S\left(\widehat{R}^*\right)}\right),$$

where $Q_{T^*}(p)$ is p_{th} sample quantile based on data $T_i^* = (T_1^*, T_2^*, ..., T_B^*)$.

3.5 Basic bootstrap

The basic bootstrap is based on the idea that the quantity $\widehat{R}^* - \widehat{R}$ has roughly the same distribution as $\widehat{R} - R$. Then the basic boot CI is given by:

$$CI_{basic}^{1-\alpha}(R) = \left(2\widehat{R} - Q_{\widehat{R}^*}(1-\alpha/2), 2\widehat{R} + Q_{\widehat{R}^*}(1-\alpha/2)\right)$$

where $Q_{\widehat{R}^*}(p)$ is p^{th} sample quantile based on data $\widehat{R}^*$.

3.6 Normal bootstrap

The N-Boot CI is calculated as

$$CI_{normal}^{1-\alpha}(R) = \left(2\widehat{R} - \overline{\widehat{R}^*} - z_{1-(\alpha/2)}\sqrt{S\left(\widehat{R}^*\right)}, 2\widehat{R} + \overline{\widehat{R}^*} - z_{1-(\alpha/2)}\sqrt{S\left(\widehat{R}^*\right)}\right)$$

where z_p is the p_{th} the quantile of the standard normal distribution,

$$\overline{\widehat{R}^*} = \frac{1}{NBoot}\sum_{i=1}^{NBoot}\widehat{R}^*,$$

and

$$S\left(\widehat{R}^*\right) = \frac{1}{NBoot-1}\sum_{i=1}^{NBoot}\left(\widehat{R}^* - \overline{\widehat{R}^*}\right)^2.$$

4 Simulation Study

In this section, we conduct a comprehensive Monte Carlo simulation study to observe the behavior of sampling methods on different estimators. Within this study, we examine point and interval estimates of SSR in terms of RSS and SRS methods. By comparing the performance of RSS-based estimators with traditional SRS-based estimators, we evaluate the effectiveness and reliability of both methods. This allows us to analyze in detail the impact of different sampling methods on prediction performance. In our simulation setup, we can set the set sizes and the number of cycles $(m_x, m_y) = (2,2),(2,3),(3,3),(3,4),(4,4),(4,5),(5,5)$ and $r_x = r_y = 2,3,4$ and 5, respectively. Therefore, in the context of RSS, sample sizes for X and Y are obtained as $n_1 = m_x r_x$ and $n_2 = m_y r_y$. Note that we use (n_1, n_2) as the sample size for SRS samples, i.e. $(n_1, n_2) = (4,4),(4,6),(6,6),(6,8),(8,8),(8,10),(10,10),(6,6),(6,9),(9,9),(9,12),(12,12),(12,15),(15,15),(8,8),(8,12),(12,12),(12,16),(16,16),(16,20),(20,20),(10,10),(10,15),(15,15),(15,20),(20,20),(20,25),(25,25)$. In the context of parameter adjustment, we take $(\alpha_1, \alpha_2) = (0.5, 1.5)$. To compare the performance of point estimates from two sampling methods across different estimators, mean square errors (MSEs) and mean relative errors (MREs) have been computed. Additionally, for interval estimates, the coverage probabilities (CPs) and average lengths (ALs) of BCIs have been examined in detail using six different bootstrap methods. For each design, $B = 2000$ bootstrap samples with each of size n are taken from the original sample and this process is repeated $K = 5000$ times. The 95% BCIs are created by each of the six methods. MREs and MSE are calculated respectively, by

$$\widehat{MSE}_R = \frac{1}{K}\sum_{i=1}^{K}\left(\widehat{R} - R\right)^2,$$

$$\widehat{MRE}_R = \frac{1}{K}\sum_{i=1}^{K}\frac{\left|\widehat{R} - R\right|}{R}.$$

The ALs and estimated CPs are given by

$$ALs = \frac{\sum_{i=1}^{K}(U_i - L_i)}{K},$$

and

$$CPs = \frac{\sum_{i=1}^{K} I(L_i \leq R \leq U_i)}{K}$$

where, I is a indicator function, L_i and U_i are lower and upper bounds of the $100(1-\alpha)\%$ CIs based on 5000 replicates.

Table 1. Estimation of different parameter methods using SRS and RSS sampling methods (Real $R = 0.1603$)

(m_x, m_y)	r		ML		LS		WLS		CVM	
			SRS	RSS	SRS	RSS	SRS	RSS	SRS	RSS
$(2,2)$	2	Est	0.1831	0.1697	0.1890	0.1798	0.2073	0.1933	0.1965	0.1840
		MSE	0.0136	0.0108	0.0136	0.0108	0.0210	0.0155	0.0137	0.0106
		MRE	0.5835	0.5067	0.5713	0.5105	0.6998	0.5886	0.5730	0.5142
$(2,3)$	2	Est	0.1939	0.1721	0.2057	0.1897	0.2344	0.2106	0.2103	0.1935
		MSE	0.0127	0.0077	0.0142	0.0092	0.0249	0.0152	0.0142	0.0095
		MRE	0.5536	0.4238	0.5840	0.4597	0.7479	0.5703	0.5828	0.4682
$(3,3)$	2	Est	0.1830	0.1643	0.1895	0.1720	0.2013	0.1796	0.1947	0.1758
		MSE	0.0099	0.0060	0.0108	0.0066	0.0156	0.0083	0.0108	0.0066
		MRE	0.4897	0.3813	0.5043	0.3922	0.5917	0.4418	0.5059	0.3978
$(3,4)$	2	Est	0.1827	0.1668	0.1930	0.1791	0.1933	0.1780	0.1963	0.1814
		MSE	0.0079	0.0047	0.0095	0.0057	0.0129	0.0068	0.0096	0.0059
		MRE	0.4295	0.3381	0.4734	0.3645	0.5357	0.3882	0.4778	0.3729
$(4,4)$	2	Est	0.1787	0.1640	0.1856	0.1705	0.1880	0.1728	0.1888	0.1730
		MSE	0.0073	0.0039	0.0086	0.0044	0.0111	0.0049	0.0084	0.0045
		MRE	0.4204	0.3094	0.4472	0.3232	0.5015	0.3391	0.4467	0.3291
$(4,5)$	2	Est	0.1809	0.1627	0.1885	0.1717	0.1902	0.1756	0.1905	0.1732
		MSE	0.0064	0.0030	0.0079	0.0036	0.0127	0.0050	0.0078	0.0038
		MRE	0.3904	0.2701	0.4267	0.2921	0.5409	0.3380	0.4300	0.3003

Table 2. Estimation of different parameter methods using SRS and RSS sampling methods (Real $R = 0.1603$)

(m_x, m_y)	r		ML		LS		WLS		CVM	
			SRS	RSS	SRS	RSS	SRS	RSS	SRS	RSS
(2, 2)	3	Est	0.1808	0.1642	0.1870	0.1724	0.1960	0.1775	0.1922	0.1768
		MSE	0.0093	0.0073	0.0103	0.0078	0.0135	0.0091	0.0103	0.0079
		MRE	0.4785	0.4217	0.4924	0.4297	0.5437	0.4549	0.4940	0.4336
(2, 3)	3	Est	0.1831	0.1683	0.1936	0.1806	0.2014	0.1835	0.1963	0.1822
		MSE	0.0073	0.0053	0.0090	0.0064	0.0127	0.0077	0.0090	0.0065
		MRE	0.4130	0.3565	0.4557	0.3839	0.5223	0.4111	0.4602	0.3913
(3, 3)	3	Est	0.1755	0.1642	0.1810	0.1706	0.1840	0.1734	0.1838	0.1727
		MSE	0.0060	0.0040	0.0073	0.0046	0.0086	0.0050	0.0072	0.0047
		MRE	0.3823	0.3141	0.4140	0.3318	0.4459	0.3474	0.4158	0.3389
(3, 4)	3	Est	0.1846	0.1642	0.1922	0.1731	0.2001	0.1809	0.1934	0.1743
		MSE	0.0061	0.0030	0.0075	0.0037	0.0106	0.0051	0.0074	0.0038
		MRE	0.3752	0.2691	0.4110	0.2950	0.4890	0.3395	0.4147	0.3020
(4, 4)	3	Est	0.1758	0.1623	0.1807	0.1673	0.1841	0.1734	0.1815	0.1690
		MSE	0.0057	0.0026	0.0068	0.0026	0.0089	0.0042	0.0065	0.0030
		MRE	0.3661	0.2516	0.3939	0.2632	0.4568	0.3146	0.3941	0.2711
(4, 5)	3	Est	0.1767	0.1624	0.1830	0.1684	0.1814	0.1702	0.1836	0.1696
		MSE	0.0047	0.0021	0.0060	0.0024	0.0074	0.0032	0.0059	0.0026
		MRE	0.3339	0.2271	0.3677	0.2423	0.4112	0.2743	0.3719	0.2500

Table 3. Estimation of different parameter methods using SRS and RSS sampling methods (Real $R = 0.1603$)

(m_x, m_y)	r		ML		LS		WLS		CVM	
			SRS	RSS	SRS	RSS	SRS	RSS	SRS	RSS
(2, 2)	4	Est	0.1772	0.1668	0.1833	0.1740	0.1891	0.1774	0.1866	0.1770
		MSE	0.0069	0.0057	0.0082	0.0064	0.0100	0.0073	0.0081	0.0066
		MRE	0.4113	0.3749	0.4404	0.3922	0.4726	0.4109	0.4415	0.3981
(2, 3)	4	Est	0.1872	0.1679	0.1947	0.1776	0.1959	0.1785	0.1955	0.1786
		MSE	0.0066	0.0037	0.0080	0.0046	0.0101	0.0054	0.0080	0.0047
		MRE	0.3942	0.2973	0.4308	0.3233	0.4719	0.3395	0.4337	0.3302
(3, 3)	4	Est	0.1741	0.1617	0.1799	0.1662	0.1786	0.1667	0.1807	0.1679
		MSE	0.0056	0.0030	0.0068	0.0033	0.0073	0.0036	0.0065	0.0035
		MRE	0.3647	0.2730	0.3932	0.2855	0.4076	0.2937	0.3927	0.2932
(3, 4)	4	Est	0.1770	0.1627	0.1845	0.1687	0.1832	0.1692	0.1844	0.1697
		MSE	0.0049	0.0023	0.0063	0.0026	0.0069	0.0029	0.0060	0.0028
		MRE	0.3381	0.2356	0.3746	0.2495	0.3950	0.2617	0.3746	0.2566
(4, 4)	4	Est	0.1746	0.1615	0.1783	0.1649	0.1784	0.1663	0.1797	0.1659
		MSE	0.0042	0.0020	0.0049	0.0022	0.0058	0.0025	0.0050	0.0023
		MRE	0.3146	0.2191	0.3369	0.2307	0.3648	0.2450	0.3450	0.2382
(4, 5)	4	Est	0.1745	0.1614	0.1798	0.1660	0.1849	0.1710	0.1810	0.1667
		MSE	0.0035	0.0015	0.0043	0.0017	0.0059	0.0023	0.0044	0.0018
		MRE	0.2893	0.1900	0.3187	0.2023	0.3695	0.2349	0.3258	0.2100

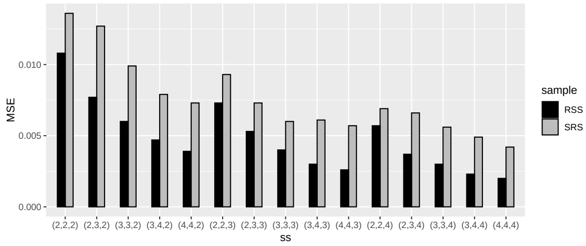

Fig. 1. MSE comparison of $\widehat{R}_{SRS}$ and $\widehat{R}_{RSS}$ sampling methods with ML estimation

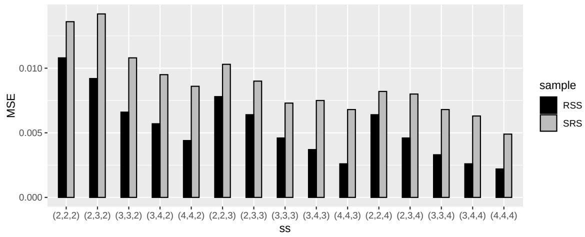

Fig. 2. MSE comparison of $\widehat{R}_{SRS}$ and $\widehat{R}_{RSS}$ sampling methods with LS estimation

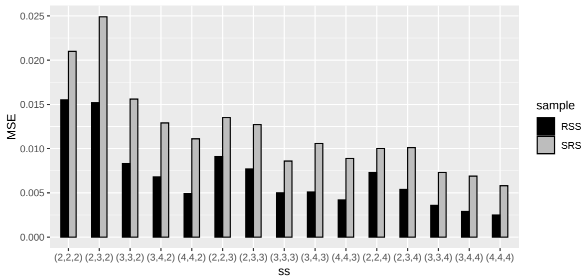

Fig. 3. MSE comparison of $\widehat{R}_{SRS}$ and $\widehat{R}_{RSS}$ sampling methods with WLS estimation

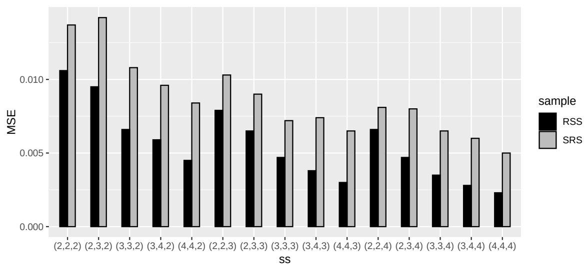

Fig. 4. MSE comparison of $\widehat{R}_{SRS}$ and $\widehat{R}_{RSS}$ sampling methods with CVM estimation

The MSE values of SSR obtained in the simulation study are shown in Fig. 1-Fig.4. When these graphs are examined, it is seen that the MSE values of the RSS method are always smaller than the SRS method for the estimators used. This result shows that the RSS method produces more consistent and reliable predictions. Lower MSE values indicate that the predictions are closer to reality and the model performs better. Therefore, based on the results of the simulation study, it can be said that the RSS method exhibits superior performance compared to the SRS method.

Table 4. CPs using BCIs for ML estimation.

(m_x, m_y)	r	sample	Std-boot	p-boot	Bcp-boot	t-boot	basic	N-boot
$(2,2)$	2	SRS	0.9637	0.9519	0.9510	0.9492	0.8218	0.8014
		RSS	0.9544	0.9506	0.9490	0.9502	0.8810	0.8684
$(2,3)$	2	SRS	0.9661	0.9530	0.9305	0.9491	0.8165	0.8364
		RSS	0.9585	0.9505	0.9503	0.9503	0.8901	0.9405
$(3,3)$	2	SRS	0.9613	0.9513	0.9341	0.9504	0.8586	0.8977
		RSS	0.9586	0.9505	0.9490	0.9503	0.9231	0.9742
$(3,4)$	2	SRS	0.9625	0.9510	0.9321	0.9504	0.8849	0.9470
		RSS	0.9565	0.9505	0.9503	0.9504	0.9108	0.9872
$(4,4)$	2	SRS	0.9678	0.9506	0.9399	0.9503	0.8985	0.9669
		RSS	0.9531	0.9505	0.9498	0.9504	0.9377	0.9878
$(4,5)$	2	SRS	0.9568	0.9507	0.9222	0.9504	0.8872	0.9734
		RSS	0.9517	0.9505	0.9500	0.9504	0.9382	0.9899
$(5,5)$	2	SRS	0.9537	0.9506	0.9434	0.9505	0.9166	0.9822
		RSS	0.9543	0.9505	0.9486	0.9505	0.9434	0.9858
$(2,2)$	3	SRS	0.9637	0.9508	0.9355	0.9501	0.8696	0.9113
		RSS	0.9569	0.9505	0.9485	0.9502	0.9203	0.9449
$(2,3)$	3	SRS	0.9624	0.9510	0.9199	0.9507	0.8730	0.9622
		RSS	0.9582	0.9505	0.9503	0.9503	0.9048	0.9816
$(3,3)$	3	SRS	0.9576	0.9506	0.9361	0.9504	0.9188	0.9805
		RSS	0.9540	0.9505	0.9501	0.9504	0.9361	0.9909
$(3,4)$	3	SRS	0.9527	0.9507	0.9071	0.9505	0.8546	0.9789
		RSS	0.9530	0.9505	0.9500	0.9505	0.9277	0.9939
$(4,4)$	3	SRS	0.9535	0.9505	0.9420	0.9503	0.8862	0.9803
		RSS	0.9518	0.9505	0.9498	0.9505	0.9384	0.9880
$(4,5)$	3	SRS	0.9542	0.9506	0.9303	0.9505	0.8703	0.9922
		RSS	0.9516	0.9505	0.9499	0.9505	0.9392	0.9915
$(5,5)$	3	SRS	0.9492	0.9505	0.9424	0.9504	0.9041	0.9952
		RSS	0.9514	0.9505	0.9500	0.9504	0.9432	0.9859
$(2,2)$	4	SRS	0.9634	0.9508	0.9407	0.9501	0.9075	0.9645
		RSS	0.9599	0.9505	0.9502	0.9503	0.9246	0.9782
$(2,3)$	4	SRS	0.9541	0.9506	0.9034	0.9504	0.8489	0.9777
		RSS	0.9516	0.9505	0.9502	0.9505	0.9162	0.9938
$(3,3)$	4	SRS	0.9563	0.9506	0.9428	0.9504	0.8983	0.9812
		RSS	0.9550	0.9505	0.9494	0.9505	0.9403	0.9914
$(3,4)$	4	SRS	0.9527	0.9505	0.9371	0.9504	0.8779	0.9917
		RSS	0.9499	0.9505	0.9499	0.9505	0.9413	0.9906
$(4,4)$	4	SRS	0.9482	0.9505	0.9366	0.9504	0.8893	0.9966
		RSS	0.9518	0.9505	0.9491	0.9505	0.9415	0.9888
$(4,5)$	4	SRS	0.9525	0.9505	0.9255	0.9505	0.8914	0.9981
		RSS	0.9506	0.9505	0.9498	0.9504	0.9445	0.9891
$(5,5)$	4	SRS	0.9511	0.9505	0.9466	0.9505	0.9063	0.9964
		RSS	0.9537	0.9505	0.9502	0.9504	0.9467	0.9862
$(2,2)$	5	SRS	0.9516	0.9506	0.9423	0.9505	0.9098	0.9803
		RSS	0.9519	0.9505	0.9490	0.9504	0.9277	0.9877
$(2,3)$	5	SRS	0.9492	0.9506	0.9064	0.9505	0.8494	0.9894
		RSS	0.9534	0.9505	0.9501	0.9505	0.9244	0.9948
$(3,3)$	5	SRS	0.9514	0.9505	0.9451	0.9504	0.8976	0.9955
		RSS	0.9512	0.9505	0.9490	0.9505	0.9438	0.9871
$(3,4)$	5	SRS	0.9527	0.9505	0.9366	0.9505	0.8873	0.9977
		RSS	0.9530	0.9505	0.9503	0.9504	0.9386	0.9909
$(4,4)$	5	SRS	0.9526	0.9505	0.9488	0.9505	0.9122	0.9959
		RSS	0.9532	0.9505	0.9497	0.9504	0.9458	0.9841
$(4,5)$	5	SRS	0.9535	0.9505	0.9476	0.9505	0.9248	0.9961
		RSS	0.9509	0.9505	0.9502	0.9505	0.9477	0.9884
$(5,5)$	5	SRS	0.9573	0.9505	0.9492	0.9505	0.9367	0.9918
		RSS	0.9517	0.9505	0.9503	0.9504	0.9439	0.9820

Table 5. ALs using BCIs for ML estimation.

(m_x, m_y)	r	sample	Std-boot	p-boot	Bcp-boot	t-boot	basic	N-boot
$(2,2)$	2	SRS	0.4484	0.4147	0.3899	0.4147	0.4147	0.3661
		RSS	0.4050	0.3851	0.3997	0.3851	0.3851	0.3394
$(2,3)$	2	SRS	0.4220	0.4010	0.3580	0.4010	0.4010	0.3879
		RSS	0.3409	0.3387	0.3389	0.3387	0.3387	0.3442
$(3,3)$	2	SRS	0.3804	0.3659	0.3273	0.3659	0.3659	0.3661
		RSS	0.3026	0.2985	0.3054	0.2985	0.2985	0.3286
$(3,4)$	2	SRS	0.3376	0.3382	0.2970	0.3382	0.3382	0.3654
		RSS	0.2687	0.2644	0.2650	0.2644	0.2644	0.3337
$(4,4)$	2	SRS	0.3194	0.3101	0.2831	0.3101	0.3101	0.3533
		RSS	0.2405	0.2376	0.2394	0.2376	0.2376	0.3256
$(4,5)$	2	SRS	0.3038	0.3026	0.2660	0.3026	0.3026	0.3618
		RSS	0.2135	0.2149	0.2158	0.2149	0.2149	0.3255
$(5,5)$	2	SRS	0.2852	0.2885	0.2710	0.2885	0.2885	0.3474
		RSS	0.2004	0.1962	0.1995	0.1962	0.1962	0.3207
$(2,2)$	3	SRS	0.3702	0.3536	0.3204	0.3536	0.3536	0.3615
		RSS	0.3336	0.3309	0.3399	0.3309	0.3309	0.3284
$(2,3)$	3	SRS	0.3238	0.3203	0.2829	0.3203	0.3203	0.3662
		RSS	0.2843	0.2809	0.2816	0.2809	0.2809	0.3366
$(3,3)$	3	SRS	0.2971	0.2944	0.2698	0.2944	0.2944	0.3510
		RSS	0.2484	0.2483	0.2498	0.2483	0.2483	0.3283
$(3,4)$	3	SRS	0.2899	0.2895	0.2536	0.2895	0.2895	0.3693
		RSS	0.2151	0.2136	0.2152	0.2136	0.2136	0.3284
$(4,4)$	3	SRS	0.2886	0.2883	0.2692	0.2883	0.2883	0.3517
		RSS	0.1989	0.1966	0.1976	0.1966	0.1966	0.3245
$(4,5)$	3	SRS	0.2622	0.2593	0.2437	0.2593	0.2593	0.3534
		RSS	0.1790	0.1796	0.1806	0.1796	0.1796	0.3249
$(5,5)$	3	SRS	0.2489	0.2517	0.2414	0.2517	0.2517	0.3447
		RSS	0.1662	0.1660	0.1662	0.1660	0.1660	0.3225
$(2,2)$	4	SRS	0.3192	0.3182	0.2877	0.3182	0.3182	0.3544
		RSS	0.2958	0.2948	0.2957	0.2948	0.2948	0.3335
$(2,3)$	4	SRS	0.3007	0.2961	0.2665	0.2961	0.2961	0.3744
		RSS	0.2382	0.2383	0.2367	0.2383	0.2383	0.3358
$(3,3)$	4	SRS	0.2882	0.2855	0.2681	0.2855	0.2855	0.3483
		RSS	0.2148	0.2111	0.2139	0.2111	0.2111	0.3234
$(3,4)$	4	SRS	0.2659	0.2643	0.2497	0.2643	0.2643	0.3540
		RSS	0.1871	0.1873	0.1887	0.1873	0.1873	0.3253
$(4,4)$	4	SRS	0.2484	0.2506	0.2379	0.2506	0.2506	0.3493
		RSS	0.1731	0.1725	0.1734	0.1725	0.1725	0.3230
$(4,5)$	4	SRS	0.2254	0.2242	0.2112	0.2242	0.2242	0.3489
		RSS	0.1503	0.1510	0.1518	0.1510	0.1510	0.3228
$(5,5)$	4	SRS	0.2121	0.2133	0.2067	0.2133	0.2133	0.3401
		RSS	0.1433	0.1425	0.1430	0.1425	0.1425	0.3222
$(2,2)$	5	SRS	0.2873	0.2913	0.2761	0.2913	0.2913	0.3499
		RSS	0.2581	0.2589	0.2617	0.2589	0.2589	0.3270
$(2,3)$	5	SRS	0.2658	0.2646	0.2385	0.2646	0.2646	0.3683
		RSS	0.2132	0.2108	0.2117	0.2108	0.2108	0.3283
$(3,3)$	5	SRS	0.2501	0.2488	0.2423	0.2488	0.2488	0.3434
		RSS	0.1895	0.1901	0.1925	0.1901	0.1901	0.3222
$(3,4)$	5	SRS	0.2201	0.2211	0.2059	0.2211	0.2211	0.3479
		RSS	0.1663	0.1643	0.1646	0.1643	0.1643	0.3244
$(4,4)$	5	SRS	0.2114	0.2097	0.2073	0.2097	0.2097	0.3373
		RSS	0.1525	0.1503	0.1514	0.1503	0.1503	0.3218
$(4,5)$	5	SRS	0.1865	0.1849	0.1815	0.1849	0.1849	0.3331
		RSS	0.1316	0.1337	0.1329	0.1337	0.1337	0.3243
$(5,5)$	5	SRS	0.1833	0.1787	0.1763	0.1787	0.1787	0.3290
		RSS	0.1259	0.1246	0.1247	0.1246	0.1246	0.3216

Table 6. CPs using BCIs for LS estimation.

(m_x, m_y)	r	sample	Std-boot	p-boot	Bcp-boot	t-boot	basic	N-boot
(2, 2)	2	SRS	0.9579	0.9521	0.9512	0.9495	0.8091	0.8042
		RSS	0.9624	0.9515	0.9514	0.9506	0.8503	0.8639
(2, 3)	2	SRS	0.9652	0.9513	0.9039	0.9497	0.7657	0.8187
		RSS	0.9603	0.9508	0.9253	0.9506	0.8230	0.9270
(3, 3)	2	SRS	0.9604	0.9508	0.9361	0.9502	0.8406	0.8847
		RSS	0.9609	0.9505	0.9501	0.9503	0.9005	0.9657
(3, 4)	2	SRS	0.9631	0.9506	0.9076	0.9504	0.8427	0.9278
		RSS	0.9532	0.9505	0.9359	0.9503	0.8658	0.9837
(4, 4)	2	SRS	0.9654	0.9506	0.9307	0.9504	0.8616	0.9395
		RSS	0.9501	0.9505	0.9469	0.9504	0.9177	0.9913
(4, 5)	2	SRS	0.9594	0.9505	0.9200	0.9504	0.8608	0.9567
		RSS	0.9543	0.9505	0.9436	0.9505	0.9124	0.9952
(5, 5)	2	SRS	0.9593	0.9505	0.9330	0.9504	0.8995	0.9673
		RSS	0.9535	0.9505	0.9494	0.9505	0.9313	0.9955
(2, 2)	3	SRS	0.9631	0.9507	0.9360	0.9501	0.8508	0.9006
		RSS	0.9582	0.9506	0.9502	0.9503	0.8663	0.9376
(2, 3)	3	SRS	0.9604	0.9506	0.9054	0.9504	0.8499	0.9419
		RSS	0.9567	0.9505	0.9420	0.9503	0.8655	0.9723
(3, 3)	3	SRS	0.9621	0.9506	0.9360	0.9504	0.8935	0.9633
		RSS	0.9538	0.9505	0.9484	0.9504	0.9079	0.9925
(3, 4)	3	SRS	0.9532	0.9505	0.9010	0.9505	0.8300	0.9628
		RSS	0.9573	0.9505	0.9419	0.9005	0.8884	0.9958
(4, 4)	3	SRS	0.9590	0.9505	0.9385	0.9503	0.8684	0.9676
		RSS	0.9499	0.9505	0.9466	0.9505	0.9262	0.9929
(4, 5)	3	SRS	0.9536	0.9505	0.9264	0.9504	0.8554	0.9493
		RSS	0.9525	0.9505	0.9461	0.9505	0.9124	0.9974
(5, 5)	3	SRS	0.9535	0.9505	0.9389	0.9503	0.8904	0.9879
		RSS	0.9501	0.9505	0.9489	0.9505	0.9342	0.9910
(2, 2)	4	SRS	0.9651	0.9506	0.9345	0.9503	0.8730	0.9458
		RSS	0.9607	0.9505	0.9478	0.9504	0.9055	0.9702
(2, 3)	4	SRS	0.9525	0.9505	0.8961	0.9503	0.8181	0.9596
		RSS	0.9539	0.9505	0.9338	0.9505	0.8767	0.9914
(3, 3)	4	SRS	0.9610	0.9505	0.9393	0.9503	0.8740	0.9669
		RSS	0.9512	0.9505	0.9502	0.9504	0.9278	0.9954
(3, 4)	4	SRS	0.9586	0.9505	0.9249	0.9504	0.8558	0.9753
		RSS	0.9501	0.9505	0.9458	0.9505	0.9205	0.9956
(4, 4)	4	SRS	0.9511	0.9505	0.9313	0.9504	0.8801	0.9909
		RSS	0.9527	0.9505	0.9498	0.9505	0.9323	0.9926
(4, 5)	4	SRS	0.9537	0.9505	0.9180	0.9505	0.8654	0.9957
		RSS	0.9528	0.9505	0.9460	0.9505	0.9273	0.9939
(5, 5)	4	SRS	0.9529	0.9505	0.9429	0.9505	0.8930	0.9977
		RSS	0.9522	0.9505	0.9488	0.9505	0.9388	0.9919
(2, 2)	5	SRS	0.9606	0.9505	0.9312	0.9504	0.8975	0.9676
		RSS	0.9550	0.9505	0.9492	0.9503	0.9134	0.9861
(2, 3)	5	SRS	0.9519	0.9505	0.8904	0.9504	0.8157	0.9713
		RSS	0.9546	0.9505	0.9454	0.9505	0.8890	0.9958
(3, 3)	5	SRS	0.9572	0.9505	0.9422	0.9504	0.8799	0.9872
		RSS	0.9520	0.9505	0.9501	0.9505	0.9286	0.9936
(3, 4)	5	SRS	0.9553	0.9505	0.9256	0.9505	0.8565	0.9950
		RSS	0.9551	0.9505	0.9450	0.9505	0.9150	0.9966
(4, 4)	5	SRS	0.9531	0.9505	0.9446	0.9505	0.8988	0.9947
		RSS	0.9523	0.9505	0.9490	0.9504	0.9396	0.9923
(4, 5)	5	SRS	0.9541	0.9505	0.9449	0.9505	0.9043	0.9970
		RSS	0.9507	0.9505	0.9464	0.9505	0.9234	0.9948
(5, 5)	5	SRS	0.9576	0.9505	0.9460	0.9505	0.9202	0.9955
		RSS	0.9475	0.9505	0.9493	0.9504	0.9427	0.9874

Table 7. ALs using BCIs for LS estimation.

(m_x, m_y)	r	sample	Std-boot	p-boot	Bcp-boot	t-boot	basic	N-boot
$(2, 2)$	2	SRS	0.4439	0.4051	0.3969	0.4051	0.4051	0.3780
		RSS	0.3998	0.3767	0.3803	0.3767	0.3767	0.3597
$(2, 3)$	2	SRS	0.4318	0.4109	0.3391	0.4109	0.4109	0.4114
		RSS	0.3587	0.3517	0.3056	0.3517	0.3517	0.3794
$(3, 3)$	2	SRS	0.3912	0.3774	0.3410	0.3774	0.3774	0.3790
		RSS	0.3144	0.3091	0.3037	0.3091	0.3091	0.3439
$(3, 4)$	2	SRS	0.3608	0.3540	0.3001	0.3540	0.3540	0.3860
		RSS	0.2853	0.2811	0.2683	0.2811	0.2811	0.3582
$(4, 4)$	2	SRS	0.3493	0.3376	0.3056	0.3376	0.3376	0.3713
		RSS	0.2565	0.2527	0.2521	0.2527	0.2527	0.3410
$(4, 5)$	2	SRS	0.3297	0.3308	0.2885	0.3308	0.3308	0.3769
		RSS	0.2307	0.2304	0.2201	0.2304	0.2304	0.3433
$(5, 5)$	2	SRS	0.3154	0.3223	0.2838	0.3223	0.3223	0.3605
		RSS	0.2136	0.2134	0.2106	0.2134	0.2134	0.3320
$(2, 2)$	3	SRS	0.3835	0.3747	0.3303	0.3747	0.3747	0.3740
		RSS	0.3433	0.3387	0.3350	0.3387	0.3387	0.3447
$(2, 3)$	3	SRS	0.3477	0.3492	0.2936	0.3492	0.3492	0.3873
		RSS	0.3022	0.3024	0.2800	0.3024	0.3024	0.3612
$(3, 3)$	3	SRS	0.3246	0.3216	0.2922	0.3216	0.3216	0.3620
		RSS	0.2620	0.2555	0.2517	0.2555	0.2555	0.3411
$(3, 4)$	3	SRS	0.3160	0.3167	0.2695	0.3167	0.3167	0.3845
		RSS	0.2343	0.2296	0.2170	0.2296	0.2296	0.3463
$(4, 4)$	3	SRS	0.3134	0.3077	0.2861	0.3077	0.3077	0.3614
		RSS	0.2079	0.2078	0.2042	0.2078	0.2078	0.3347
$(4, 5)$	3	SRS	0.2896	0.2873	0.2619	0.2873	0.2873	0.3661
		RSS	0.1905	0.1900	0.1852	0.1900	0.1900	0.3368
$(5, 5)$	3	SRS	0.2685	0.2699	0.2542	0.2699	0.2699	0.3530
		RSS	0.1738	0.1740	0.1727	0.1740	0.1740	0.3301
$(2, 2)$	4	SRS	0.3426	0.3346	0.3047	0.3346	0.3346	0.3365
		RSS	0.3097	0.3102	0.2973	0.3102	0.3102	0.3480
$(2, 3)$	4	SRS	0.3244	0.3203	0.2812	0.3203	0.3203	0.3894
		RSS	0.2579	0.2589	0.2372	0.2589	0.2589	0.3551
$(3, 3)$	4	SRS	0.3146	0.3061	0.2833	0.3061	0.3061	0.3598
		RSS	0.2253	0.2254	0.2245	0.2254	0.2254	0.3324
$(3, 4)$	4	SRS	0.2959	0.2947	0.2584	0.2947	0.2947	0.3689
		RSS	0.1986	0.2012	0.1947	0.2012	0.2012	0.3374
$(4, 4)$	4	SRS	0.2655	0.2664	0.2474	0.2664	0.2664	0.3566
		RSS	0.1818	0.1819	0.1797	0.1819	0.1819	0.3297
$(4, 5)$	4	SRS	0.2457	0.2418	0.2255	0.2418	0.2418	0.3596
		RSS	0.1596	0.1592	0.1548	0.1592	0.1592	0.3319
$(5, 5)$	4	SRS	0.2271	0.2250	0.2187	0.2250	0.2250	0.3448
		RSS	0.1522	0.1512	0.1498	0.1512	0.1512	0.3278
$(2, 2)$	5	SRS	0.3169	0.3219	0.2846	0.3219	0.3219	0.3631
		RSS	0.2744	0.2716	0.2653	0.2719	0.2719	0.3403
$(2, 3)$	5	SRS	0.2992	0.3041	0.2529	0.3041	0.3041	0.3850
		RSS	0.2340	0.2300	0.2224	0.2300	2300	0.3450
$(3, 3)$	5	SRS	0.2743	0.2705	0.2559	0.2705	0.2705	0.3522
		RSS	0.1984	0.1975	0.1971	0.1975	0.1975	0.3301
$(3, 4)$	5	SRS	0.2433	0.2421	0.2154	0.2421	0.2421	0.3588
		RSS	0.1769	0.1745	0.1687	0.1745	0.1745	0.3349
$(4, 4)$	5	SRS	0.2285	0.2253	0.2182	0.2253	0.2253	0.3438
		RSS	0.1602	0.1593	0.1581	0.1593	0.1593	0.3277
$(4, 5)$	5	SRS	0.2019	0.1989	0.1931	0.1989	0.1989	0.3400
		RSS	0.1402	0.1415	0.1375	0.1415	0.1415	0.3315
$(5, 5)$	5	SRS	0.1953	0.1903	0.1863	0.1903	0.1903	0.3341
		RSS	0.1348	0.1357	0.1356	0.1357	0.1357	0.3257

Table 8. CPs using BCIs for WLS estimation.

(m_x, m_y)	r	sample	Std-boot	p-boot	Bcp-boot	t-boot	basic	N-boot
(2, 2)	2	SRS	0.9425	0.9518	0.9359	0.9497	0.7987	0.7153
		RSS	0.9494	0.9517	0.9459	0.9488	0.8449	0.7945
(2, 3)	2	SRS	0.9656	0.9515	0.8562	0.9505	0.7275	0.6497
		RSS	0.9417	0.9507	0.8870	0.9503	0.7391	0.8181
(3, 3)	2	SRS	0.9513	0.9508	0.9186	0.9498	0.8497	0.7887
		RSS	0.9562	0.9506	0.9446	0.9503	0.8614	0.9447
(3, 4)	2	SRS	0.9673	0.9507	0.9251	0.9499	0.8806	0.8554
		RSS	0.9447	0.9505	0.9442	0.9503	0.8852	0.9692
(4, 4)	2	SRS	0.9708	0.9506	0.9242	0.9503	0.9042	0.8908
		RSS	0.9452	0.9505	0.9428	0.9503	0.9014	0.9684
(4, 5)	2	SRS	0.9727	0.9506	0.8948	0.9502	0.9034	0.8587
		RSS	0.9350	0.9505	0.9217	0.9503	0.9401	0.9524
(5, 5)	2	SRS	0.9730	0.9505	0.9277	0.9502	0.9352	0.9197
		RSS	0.9301	0.9505	0.9450	0.9503	0.9447	0.9531
(2, 2)	3	SRS	0.9570	0.9509	0.9248	0.9499	0.8519	0.8365
		RSS	0.9608	0.9506	0.9486	0.9504	0.8805	0.9168
(2, 3)	3	SRS	0.9507	0.9506	0.8972	0.9500	0.8662	0.8760
		RSS	0.9506	0.9505	0.9402	0.9503	0.8585	0.9568
(3, 3)	3	SRS	0.9590	0.9506	0.9283	0.9504	0.9213	0.9433
		RSS	0.9498	0.9505	0.9437	0.9503	0.8971	0.9845
(3, 4)	3	SRS	0.9157	0.9505	0.8811	0.9502	0.8930	0.9210
		RSS	0.9520	0.9505	0.9230	0.9505	0.8421	0.9811
(4, 4)	3	SRS	0.9302	0.9505	0.9337	0.9502	0.9168	0.9299
		RSS	0.9535	0.9505	0.9391	0.9504	0.8728	0.9817
(4, 5)	3	SRS	0.9160	0.9505	0.9264	0.9503	0.9457	0.9625
		RSS	0.9554	0.9505	0.9406	0.9505	0.8908	0.9882
(5, 5)	3	SRS	0.9158	0.9505	0.9344	0.9502	0.9608	0.9787
		RSS	0.9561	0.9505	0.9473	0.9505	0.9243	0.9895
(2, 2)	4	SRS	0.9606	0.9506	0.9296	0.9502	0.8534	0.9142
		RSS	0.9602	0.9505	0.9460	0.9503	0.8934	0.9578
(2, 3)	4	SRS	0.9430	0.9506	0.8959	0.9501	0.8907	0.9267
		RSS	0.9494	0.9505	0.9321	0.9504	0.8876	0.9839
(3, 3)	4	SRS	0.9314	0.9505	0.9370	0.9502	0.9327	0.9626
		RSS	0.9458	0.9505	0.9491	0.9504	0.9175	0.9777
(3, 4)	4	SRS	0.9290	0.9505	0.9249	0.9502	0.9450	0.9708
		RSS	0.9544	0.9505	0.9436	0.9505	0.9071	0.9902
(4, 4)	4	SRS	0.9286	0.9505	0.9291	0.9502	0.9627	0.9843
		RSS	0.9554	0.9505	0.9486	0.9505	0.9162	0.9921
(4, 5)	4	SRS	0.9387	0.9505	0.9064	0.9502	0.9519	0.9885
		RSS	0.9574	0.9505	0.9377	0.9505	0.8886	0.9961
(5, 5)	4	SRS	0.9383	0.9505	0.9414	0.9503	0.9394	0.9676
		RSS	0.9508	0.9505	0.9467	0.9505	0.9067	0.9980
(2, 2)	5	SRS	0.9523	0.9505	0.9228	0.9502	0.9071	0.9621
		RSS	0.9539	0.9505	0.9469	0.9503	0.9092	0.9838
(2, 3)	5	SRS	0.9403	0.9505	0.8889	0.9504	0.8588	0.9647
		RSS	0.9496	0.9505	0.9441	0.9505	0.8847	0.9928
(3, 3)	5	SRS	0.9454	0.9505	0.9420	0.9503	0.9077	0.9833
		RSS	0.9519	0.9505	0.9500	0.9505	0.9231	0.9921
(3, 4)	5	SRS	0.9428	0.9505	0.9292	0.9504	0.8819	0.9746
		RSS	0.9567	0.9505	0.9478	0.9505	0.9177	0.9952
(4, 4)	5	SRS	0.9411	0.9505	0.9429	0.9504	0.9488	0.9702
		RSS	0.9552	0.9505	0.9490	0.9504	0.9345	0.9926
(4, 5)	5	SRS	0.9527	0.9505	0.9400	0.9505	0.8862	0.9882
		RSS	0.9521	0.9505	0.9451	0.9504	0.9192	0.9956
(5, 5)	5	SRS	0.9579	0.9505	0.9448	0.9505	0.9051	0.9881
		RSS	0.9519	0.9505	0.9481	0.9504	0.9401	0.9899

Table 9. ALs using BCIs for WLS estimation.

(m_x, m_y)	r	sample	Std-boot	p-boot	Bcp-boot	t-boot	basic	N-boot
$(2, 2)$	2	SRS	0.5370	0.4973	0.4661	0.4973	0.4973	0.4147
		RSS	0.4711	0.4752	0.4524	0.4752	0.4752	0.3865
$(2, 3)$	2	SRS	0.5466	0.5490	0.3806	0.5490	0.5490	0.4688
		RSS	0.4411	0.4401	0.3246	0.4401	0.4401	0.4210
$(3, 3)$	2	SRS	0.4618	0.4831	0.3720	0.4831	0.4831	0.4027
		RSS	0.3489	0.3375	0.3132	0.3375	0.3375	0.3592
$(3, 4)$	2	SRS	0.4260	0.4537	0.3634	0.4537	0.4537	0.3867
		RSS	0.3162	0.3114	0.3254	0.3114	0.3114	0.3560
$(4, 4)$	2	SRS	0.3990	0.3971	0.3351	0.3971	0.3971	0.3760
		RSS	0.2701	0.2702	0.2918	0.2702	0.2702	0.3456
$(4, 5)$	2	SRS	0.4251	0.4059	0.3163	0.4059	0.4059	0.3804
		RSS	0.2716	0.3081	0.2737	0.3081	0.3081	0.3513
$(5, 5)$	2	SRS	0.3802	0.3676	0.3184	0.3676	0.3676	0.3512
		RSS	0.2554	0.2953	0.2768	0.2953	0.2953	0.3347
$(2, 2)$	3	SRS	0.4333	0.4558	0.3498	0.4558	0.4558	0.3922
		RSS	0.3688	0.3620	0.3515	0.3620	0.3620	0.3550
$(2, 3)$	3	SRS	0.4115	0.4464	0.3263	0.4464	0.4464	0.4028
		RSS	0.3322	0.3241	0.3148	0.3241	0.3241	0.3670
$(3, 3)$	3	SRS	0.3520	0.3583	0.3144	0.3583	0.3583	0.3681
		RSS	0.2723	0.2657	0.2777	0.2657	0.2657	0.3467
$(3, 4)$	3	SRS	0.3727	0.4021	0.3093	0.4021	0.4021	0.4001
		RSS	0.2676	0.2581	0.2817	0.2581	0.2581	0.3619
$(4, 4)$	3	SRS	0.3585	0.3753	0.3345	0.3753	0.3753	0.3681
		RSS	0.2480	0.2374	0.2749	0.2374	0.2374	0.3468
$(4, 5)$	3	SRS	0.3266	0.3596	0.3090	0.3596	0.3596	0.3627
		RSS	0.2184	0.2105	0.2298	0.2105	0.2105	0.3403
$(5, 5)$	3	SRS	0.3054	0.3351	0.3030	0.3351	0.3351	0.3509
		RSS	0.1935	0.1864	0.1859	0.1864	0.1864	0.3314
$(2, 2)$	4	SRS	0.3748	0.3730	0.3321	0.3730	0.3730	0.3781
		RSS	0.3294	0.3285	0.3041	0.3285	0.3285	0.3547
$(2, 3)$	4	SRS	0.3685	0.4021	0.3078	0.4021	0.4021	0.3917
		RSS	0.2794	0.2718	0.2777	0.2718	0.2718	0.3569
$(3, 3)$	4	SRS	0.3264	0.3440	0.3120	0.3440	0.3440	0.3571
		RSS	0.2354	0.2360	0.2572	0.2360	0.2360	0.3334
$(3, 4)$	4	SRS	0.3122	0.3480	0.3012	0.3480	0.3480	0.3664
		RSS	0.2096	0.2062	0.2046	0.2062	0.2062	0.3385
$(4, 4)$	4	SRS	0.2898	0.3304	0.2913	0.3304	0.3304	0.3568
		RSS	0.1949	0.1910	0.1878	0.1910	0.1910	0.3326
$(4, 5)$	4	SRS	0.2857	0.3302	0.2843	0.3302	0.3302	0.3697
		RSS	0.1843	0.1773	0.1742	0.1773	0.1773	0.3420
$(5, 5)$	4	SRS	0.2692	0.3004	0.2982	0.3004	0.3004	0.3505
		RSS	0.1801	0.1781	0.1755	0.1781	0.1781	0.3366
$(2, 2)$	5	SRS	0.3237	0.3349	0.2866	0.3349	0.3349	0.3671
		RSS	0.2791	0.2767	0.2645	0.2767	0.2767	0.3438
$(2, 3)$	5	SRS	0.3142	0.3269	0.2852	0.3269	0.3269	0.3855
		RSS	0.2432	0.2403	0.2337	0.2403	0.2403	0.3470
$(3, 3)$	5	SRS	0.2833	0.2879	0.2927	0.2879	0.2879	0.3515
		RSS	0.2059	0.2035	0.2032	0.2035	0.2035	0.3307
$(3, 4)$	5	SRS	0.2517	0.2545	0.2733	0.2545	0.2545	0.3542
		RSS	0.1791	0.1751	0.1702	0.1751	0.1751	0.3323
$(4, 4)$	5	SRS	0.2431	0.2853	0.2771	0.2853	0.2853	0.3422
		RSS	0.1633	0.1596	0.1593	0.1596	0.1596	0.3281
$(4, 5)$	5	SRS	0.2159	0.2075	0.2330	0.2075	0.2075	0.3422
		RSS	0.1470	0.1476	0.1422	0.1476	0.1476	0.3330
$(5, 5)$	5	SRS	0.2117	0.2001	0.2087	0.2001	0.2001	0.3361
		RSS	0.1443	0.1431	0.1419	0.1431	0.1431	0.3279

Table 10. CPs using BCIs for CVM estimation.

(m_x, m_y)	r	sample	Std-boot	p-boot	Bcp-boot	t-boot	basic	N-boot
(2, 2)	2	SRS	0.9557	0.9516	0.9458	0.9503	0.7698	0.8075
		RSS	0.9622	0.9511	0.9491	0.9499	0.8058	0.8620
(2, 3)	2	SRS	0.9643	0.9512	0.8936	0.9502	0.7164	0.8274
		RSS	0.9608	0.9508	0.9128	0.9505	0.7954	0.9217
(3, 3)	2	SRS	0.9575	0.9508	0.9281	0.9501	0.7848	0.8869
		RSS	0.9592	0.9505	0.9488	0.9504	0.8707	0.9669
(3, 4)	2	SRS	0.9616	0.9506	0.9044	0.9504	0.7900	0.9339
		RSS	0.9595	0.9505	0.9314	0.9504	0.8489	0.9825
(4, 4)	2	SRS	0.9636	0.9506	0.9209	0.9504	0.8282	0.9480
		RSS	0.9561	0.9505	0.9434	0.9504	0.9050	0.9929
(4, 5)	2	SRS	0.9635	0.9505	0.9162	0.9503	0.8166	0.9598
		RSS	0.9540	0.9505	0.9405	0.9505	0.9067	0.9971
(5, 5)	2	SRS	0.9597	0.9505	0.9270	0.9503	0.8675	0.9753
		RSS	0.9536	0.9505	0.9472	0.9505	0.9282	0.9975
(2, 2)	3	SRS	0.9601	0.9507	0.9282	0.9502	0.8005	0.9039
		RSS	0.9617	0.9505	0.9482	0.9503	0.8647	0.9356
(2, 3)	3	SRS	0.9632	0.9506	0.8998	0.9504	0.8006	0.9440
		RSS	0.9591	0.9505	0.9386	0.9503	0.8474	0.9699
(3, 3)	3	SRS	0.9642	0.9506	0.9302	0.9504	0.8541	0.9656
		RSS	0.9569	0.9505	0.9464	0.9503	0.8985	0.9936
(3, 4)	3	SRS	0.9577	0.9505	0.8987	0.9503	0.8114	0.9653
		RSS	0.9570	0.9505	0.9412	0.9505	0.8847	0.9975
(4, 4)	3	SRS	0.9602	0.9505	0.9354	0.9503	0.8605	0.9715
		RSS	0.9516	0.9505	0.9443	0.9505	0.9246	0.9958
(4, 5)	3	SRS	0.9540	0.9505	0.9239	0.9504	0.8514	0.9824
		RSS	0.9537	0.9505	0.9447	0.9505	0.9109	0.9980
(5, 5)	3	SRS	0.9549	0.9505	0.9339	0.9504	0.8827	0.9889
		RSS	0.9494	0.9505	0.9479	0.9505	0.9300	0.9924
(2, 2)	4	SRS	0.9636	0.9506	0.9282	0.9503	0.8414	0.9506
		RSS	0.9631	0.9505	0.9445	0.9503	0.8787	0.9700
(2, 3)	4	SRS	0.9597	0.9505	0.8935	0.9503	0.8025	0.9616
		RSS	0.9548	0.9505	0.9307	0.9504	0.8712	0.9928
(3, 3)	4	SRS	0.9617	0.9505	0.9354	0.9503	0.8689	0.9728
		RSS	0.9536	0.9505	0.9492	0.9504	0.9225	0.9981
(3, 4)	4	SRS	0.9590	0.9505	0.9231	0.9504	0.8506	0.9824
		RSS	0.9495	0.9505	0.9430	0.9505	0.9158	0.9965
(4, 4)	4	SRS	0.9543	0.9505	0.9303	0.9504	0.8755	0.9903
		RSS	0.9525	0.9505	0.9491	0.9505	0.9279	0.9942
(4, 5)	4	SRS	0.9920	0.9505	0.9150	0.9504	0.8610	0.9974
		RSS	0.9532	0.9505	0.9444	0.9504	0.9258	0.9944
(5, 5)	4	SRS	0.9532	0.9505	0.9408	0.9505	0.8925	0.9992
		RSS	0.9525	0.9505	0.9479	0.9504	0.9359	0.9926
(2, 2)	5	SRS	0.9626	0.9505	0.9231	0.9504	0.8634	0.9708
		RSS	0.9589	0.9505	0.9471	0.9503	0.9057	0.9867
(2, 3)	5	SRS	0.9545	0.9505	0.8858	0.9504	0.8067	0.9770
		RSS	0.9506	0.9505	0.9434	0.9505	0.8877	0.9987
(3, 3)	5	SRS	0.9573	0.9505	0.9401	0.9504	0.8750	0.9891
		RSS	0.9524	0.9505	0.9490	0.9505	0.9259	0.9952
(3, 4)	5	SRS	0.9542	0.9505	0.9222	0.9505	0.8530	0.9970
		RSS	0.9542	0.9505	0.9448	0.9505	0.9115	0.9974
(4, 4)	5	SRS	0.9523	0.9505	0.9425	0.9505	0.8945	0.9976
		RSS	0.9533	0.9505	0.9477	0.9504	0.9374	0.9930
(4, 5)	5	SRS	0.9530	0.9505	0.9427	0.9505	0.9003	0.9982
		RSS	0.9510	0.9505	0.9457	0.9504	0.9211	0.9952
(5, 5)	5	SRS	0.9565	0.9505	0.9449	0.9505	0.9172	0.9965
		RSS	0.9487	0.9505	0.9487	0.9504	0.9433	0.9884

Table 11. ALs using BCIs for CVM estimation.

(m_x, m_y)	r	sample	Std-boot	p-boot	Bcp-boot	t-boot	basic	N-boot
$(2,2)$	2	SRS	0.4374	0.3926	0.3785	0.3926	0.3926	0.3930
		RSS	0.3981	0.3660	0.3519	0.3660	0.3660	0.3733
$(2,3)$	2	SRS	0.4250	0.3949	0.3252	0.3949	0.3949	0.4205
		RSS	0.3597	0.3474	0.2969	0.3474	0.3474	0.3869
$(3,3)$	2	SRS	0.3853	0.3698	0.3225	0.3698	0.3698	0.3893
		RSS	0.3138	0.3047	0.2955	0.3047	0.3047	0.3516
$(3,4)$	2	SRS	0.3572	0.3454	0.2823	0.3454	0.3454	0.3926
		RSS	0.2886	0.2840	0.2618	0.2840	0.2840	0.3628
$(4,4)$	2	SRS	0.3414	0.3283	0.2818	0.3283	0.3283	0.3774
		RSS	0.2584	0.2521	0.2449	0.2521	0.2521	0.3461
$(4,5)$	2	SRS	0.3262	0.3135	0.2774	0.3135	0.3135	0.3809
		RSS	0.2355	0.2366	0.2229	0.2366	0.2366	0.3464
$(5,5)$	2	SRS	0.3031	0.2965	0.2631	0.2965	0.2965	0.3634
		RSS	0.2188	0.2182	0.2130	0.2182	0.2182	0.3360
$(2,2)$	3	SRS	0.3774	0.3665	0.3087	0.3665	0.3665	0.3843
		RSS	0.3421	0.3331	0.3172	0.3331	0.3331	0.3535
$(2,3)$	3	SRS	0.3445	0.3339	0.2810	0.3339	0.3339	0.3927
		RSS	0.3043	0.3026	0.2764	0.3026	0.3026	0.3645
$(3,3)$	3	SRS	0.3186	0.3073	0.2764	0.3073	0.3073	0.3677
		RSS	0.2643	0.2592	0.2517	0.2592	0.2592	0.3454
$(3,4)$	3	SRS	0.3121	0.3112	0.2574	0.3112	0.3112	0.3868
		RSS	0.2368	0.2345	0.2205	0.2345	0.2345	0.3485
$(4,4)$	3	SRS	0.3054	0.3009	0.2711	0.3009	0.3009	0.3630
		RSS	0.2130	0.2126	0.2063	0.2126	0.2126	0.3379
$(4,5)$	3	SRS	0.2860	0.2856	0.2592	0.2856	0.2856	0.3673
		RSS	0.1965	0.1964	0.1897	0.1964	0.1964	0.3391
$(5,5)$	3	SRS	0.2700	0.2710	0.2507	0.2710	0.2710	0.3560
		RSS	0.1793	0.1785	0.1783	0.1785	0.1785	0.3322
$(2,2)$	4	SRS	0.3365	0.3259	0.2870	0.3259	0.3259	0.3734
		RSS	0.3112	0.3029	0.2848	0.3029	0.3029	0.3541
$(2,3)$	4	SRS	0.3214	0.3127	0.2687	0.3127	0.3127	0.3910
		RSS	0.2585	0.2572	0.2391	0.2572	0.2572	0.3571
$(3,3)$	4	SRS	0.3061	0.2988	0.2725	0.2988	0.2988	0.3613
		RSS	0.2312	0.2299	0.2265	0.2299	0.2299	0.3358
$(3,4)$	4	SRS	0.2877	0.2863	0.2537	0.2863	0.2863	0.3688
		RSS	0.2031	0.2059	0.1987	0.2059	0.2059	0.3393
$(4,4)$	4	SRS	0.2679	0.2694	0.2461	0.2694	0.2694	0.3594
		RSS	0.1872	0.1860	0.1836	0.1860	0.1860	0.3318
$(4,5)$	4	SRS	0.2478	0.2457	0.2256	0.2457	0.2457	0.3619
		RSS	0.1650	0.1640	0.1608	0.1640	0.1640	0.3334
$(5,5)$	4	SRS	0.2317	0.2303	0.2229	0.2303	0.2303	0.3473
		RSS	0.1573	0.1567	0.1538	0.1567	0.1567	0.3294
$(2,2)$	5	SRS	0.3084	0.3030	0.2652	0.3030	0.3030	0.3670
		RSS	0.2738	0.2739	0.2604	0.2739	0.2739	0.3439
$(2,3)$	5	SRS	0.2934	0.2948	0.2450	0.2948	0.2948	0.3854
		RSS	0.2350	0.2356	0.2264	0.2356	0.2356	0.3464
$(3,3)$	5	SRS	0.2736	0.2725	0.2555	0.2725	0.2725	0.3547
		RSS	0.2034	0.2030	0.2007	0.2030	0.2030	0.3328
$(3,4)$	5	SRS	0.2435	0.2446	0.2173	0.2446	0.2446	0.3605
		RSS	0.1814	0.1791	0.1740	0.1791	0.1791	0.3365
$(4,4)$	5	SRS	0.2314	0.2302	0.2214	0.2302	0.2302	0.3460
		RSS	0.1650	0.1634	0.1615	0.1634	0.1634	0.3294
$(4,5)$	5	SRS	0.2068	0.2048	0.1974	0.2048	0.2048	0.3422
		RSS	0.1455	0.1466	0.1416	0.1466	0.1466	0.3326
$(5,5)$	5	SRS	0.1997	0.1958	0.1906	0.1958	0.1958	0.3364
		RSS	0.1393	0.1406	0.1393	0.1406	0.1406	0.3267

All computations were performed using an R program based on 5000 Monte Carlo simulations. The simulation results are presented in Tables 1 -11. Tables 1-3 present point estimates, while Table 4 - 11 include interval estimate results. It was observed that, in both SRS and RSS methods, as n_1 and n_2 increase, the mean SSR estimates approach the real R-value and the MSEs decrease. Furthermore, among all the estimator methods examined, the RSS method exhibits smaller MSE values compared to the SRS method. Overall, there is strong evidence that the RSS-based method performs better than other estimators as a point estimate of SSR. According to the simulation results, the BCIs for all cases indicated that the 95% CIs obtained from the RSS method were generally closer to the true value than those obtained from the SRS method. Moreover, in terms of ALs, the RSS method exhibited lower values across all bootstrap methods compared to the SRS method.

5 Real Data Applications

In this section, we analyze three real data sets to present the implementation of the proposed methods. Here, we analyze practical real datasets with PoA fit to demonstrate the developed procedure. The main difference between this application and others is that we treat dataset I and dataset II as populations of interest and select data from these populations using SRS and RSS with the same sample sizes. In other words, we use perfect ranking. The RSS sampling units are selected from dataset I and dataset II using the following procedure: first, 9 observations are individually selected from each dataset I and dataset II. Then randomly divide them into 3 groups of 3 individuals each and rank the observations in each group from smaller to larger. Select the smallest observations from the first row of clusters. Select the second smallest observations from the second row of clusters. Select the largest observations from the third set. So you get the set of sizes for X and Y as $m_x = m_y = 3$. Repeat this process $r_x = r_y = 7$ times for X and Y respectively. Finally, obtain the RSS samples for X and Y with sizes $n = m_x r_x = 21$ and $m = m_y r_y = 21$ respectively. In addition, the SRS sample units are obtained by randomly selecting $n = 21$ and $m = 21$ observations from dataset I and dataset II, respectively. Using the mentioned motivation, SSR based on SRS and RSS will be calculated on three real data sets and their performances will be compared.

Example I: The dataset comprises the waiting times before customer service for two banks: Bank A and Bank B. During the processing of the continuous data, each data point was rounded such that values with decimal parts greater than 0.5 were rounded up, and values with decimal parts less than or equal to 0.5 were rounded down. This rounding method was employed to ensure a more accurate and consistent representation within a specified range of the dataset It is first reported by Ghitany et al. [25]. In this section, we are interested in estimating the stress-strength parameter $R = P(X < Y)$, where X and Y represents the exact portions of wait times prior to the customer service period in Bank A (dataset I, sample size = 100) and Bank B (dataset II, sample size = 60) presented in Tables 12 and 13, respectively. Before starting the analysis, we adapt the PoA distribution to the X and Y observations separately. We present the Kolmogorov-Smirnov (KS) values and the corresponding

Table 12. Waiting time (in minutes) before customer service at Bank A (X).

1	1	1	2	2	2	2	2	3	3	3	3	3	3	4
4	4	4	4	4	4	4	4	4	5	5	5	5	5	5
5	5	6	6	6	6	6	6	6	6	7	7	7	7	7
7	7	8	8	8	8	9	9	9	9	9	9	9	10	10
10	10	11	11	11	11	11	11	11	12	12	12	12	13	13
13	13	14	14	14	14	15	15	17	17	18	18	18	19	19
20	21	21	21	22	23	27	32	33	39					

Table 13. Waiting time (in minutes) before customer service in Bank B (Y).

0	0	0	1	1	1	1	2	2	2	2	2	2	2	3
3	3	3	3	3	3	3	3	3	4	4	4	4	5	5
5	6	6	6	6	7	7	7	8	8	8	8	8	9	9
9	10	11	11	11	12	12	13	13	13	14	15	16	17	28

p-values (in parentheses) for Bank A and Bank B are 0.1265 (0.0817) and 0.1150 (0.3259), respectively. Accordingly, both datasets fit the PoA distribution from KS statistics and Fig. 5.

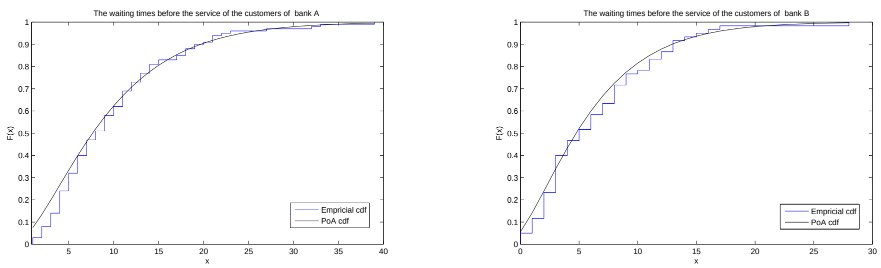

Fig. 5. Empirical and PoA distributions based on the Example I data set.

The RSS and SRS data sets obtained according to the motivation given in the introduction part of the real data are given in Table 14 and 15:

Table 14. Newly chosen data for waiting time before customer service in Bank SRS; $n = m = 21$.

X							Y						
14	9	5	15	6	6	3	6	5	2	7	8	11	2
11	12	3	3	12	15	15	4	5	2	9	10	3	2
10	3	4	4	4	4	1	3	0	11	12	8	3	5

Table 15. Newly chosen data for waiting time (in minutes) before customer service in Bank RSS; $n = m = 21$.

X				Y			
Set				Set			
Cycle	1	2	3	Cycle	1	2	3
1	4	6	14	1	6	4	13
2	2	14	6	2	2	5	7
3	2	6	11	3	0	4	10
4	6	7	21	4	2	3	7
5	7	7	13	5	11	6	8
6	3	6	18	6	2	10	8
7	11	5	8	7	1	1	5

Table 16. Estimates of $R = 0.3209$ Waiting time before customer service in Bank.

	Sample	$\widehat{R}$	std-boot	p-boot	Bcp-boot	t-boot	basic	N-boot
ML	SRS	0.3706	0.2075	0.1972	0.1322	0.1972	0.1972	0.8018
	RSS	0.3256	0.1950	0.1952	0.00003	0.1952	0.1952	0.6780
LS	SRS	0.3899	0.2409	0.2368	0.1585	0.2368	0.2368	0.7745
	RSS	0.3395	0.2023	0.2024	0.0016	0.2024	0.2024	0.6619
WLS	SRS	0.3862	0.2317	0.2287	0.1279	0.2287	0.2287	0.7695
	RSS	0.3288	0.1978	0.1984	0.1877	0.1984	0.1984	0.6602
CVM	SRS	0.3911	0.2527	0.2503	0.1759	0.2503	0.2503	0.7810
	RSS	0.3456	0.2130	0.2152	0.0045	0.2152	0.2152	0.6669

The results for estimates SSR and the width of the CIs for the std-boot, p-boot, BCp-boot, t-boot, basic, and N-boot BCI methods are presented in Table 16. Then, the following estimates for SSR under RSS and SRS are obtained: Accordingly, while the SSR obtained from the sample created by the SRS method from the same population is far from its true value, the SSR obtained from the sample created by the RSS method is quite close to the true value. Additionally, Table 16 presents the CI widths of the SSR estimates for the std-boot, p-boot, BCp-boot, t-boot, basic, and N-boot BCI methods.

Example II: Consider the real-life data set representing the monthly concentration of sulfur dioxide in Long Beach, California, from 1956 to 1974. It is reported by Roberts [26]. It has been analyzed by Wang and Ye [27] to compare between based on this scenario, let $X_1, ..., X_{20}$ and $Y_1, ..., Y_{20}$ are the concentrations of sulfur dioxide in March and August, respectively. We calculate the KS values and the corresponding p-value (in parentheses) using dataset I and dataset II are 0.2260 (0.2586) and 0.2345 (0.2211), respectively. Accordingly, both datasets fit the PoA distribution from Fig. 6 and KS statistics.

The RSS and SRS data sets were obtained according to the motivation given in the introduction part of the data is reported in Table 18 and 19:

Table 17. The monthly concentration of sulfur dioxide.

Data Set I	21	16	20	15	9	10	10	4	25	18
	18	26	25	17	40	55	19	16	9	19
Data Set II	44	20	20	20	23	20	15	27	3	9
	25	32	18	55	10	20	18	8	9	20

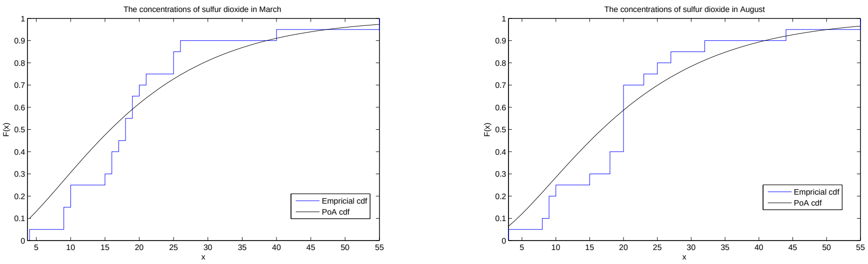

Fig. 6. Empirical and PoA distributions based on the Example II data set.

Table 18. Newly chosen data for concentration of sulfur dioxide SRS; $n = m = 21$.

X							Y						
19	9	18	40	16	18	55	9	18	27	32	20	9	32
40	4	20	19	18	19	10	20	18	44	10	20	10	32
21	10	19	25	4	21	19	32	55	55	3	20	3	20

Table 19. Newly chosen data for concentration of sulfur dioxide RSS; $n = m = 21$.

X				Y			
	Set				Set		
Cycle	1	2	3	Cycle	1	2	3
1	10	20	20	1	9	10	32
2	17	16	40	2	15	23	55
3	17	25	20	3	15	18	32
4	19	18	19	4	3	18	55
5	4	15	25	5	18	25	32
6	10	16	25	6	10	20	25
7	10	19	21	7	20	20	32

The results for estimates SSR and the width of the CIs for the std-boot, p-boot, BCp-boot, t-boot, basic, and N-boot BCI methods are presented in Table 20. Then, the following estimates for SSR under RSS and SRS are obtained: Accordingly, while the

Table 20. Estimates of $R = 0.5690$ concentration of sulfur dioxide.

	Sample	$\widehat{R}$	std-boot	p-boot	Bcp-boot	t-boot	basic	N-boot
ML	SRS	0.5484	0.2437	0.2462	0.2467	0.2462	0.2462	1.1308
	RSS	0.5508	0.1489	0.1490	0.0855	0.1490	0.1490	1.0307
LS	SRS	0.5241	0.2528	0.2631	0.1767	0.2631	0.2631	1.0694
	RSS	0.5324	0.1674	0.1638	0.0611	0.1638	0.1638	0.9783
WLS	SRS	0.5275	0.2679	0.2756	0.2638	0.2756	0.2756	1.0899
	RSS	0.5430	0.1770	0.1768	0.0826	0.1769	0.1768	0.9910
CVM	SRS	0.5292	0.2465	0.2547	0.1792	0.2547	0.2547	1.0710
	RSS	0.5339	0.1601	0.1602	0.0604	0.1602	0.1602	0.9801

SSR obtained from the sample created by the SRS method from the same population is far from its true value, the SSR obtained from the sample created by the RSS method is quite close to the true value. Additionally, Table 20 presents the confidence interval widths of the SSR estimates for the std-boot, p-boot, BCp-boot, t-boot, basic, and N-boot BCI methods.

Example III: For a real application, we consider the successive failure times (in hours) of the air conditioning system of the jet airplanes, initially reported by Proschan [28]. Here, we represent two jet airplane data sets considered for empirical analysis have the following observations in Table 21. Before starting the analysis, We calculate the KS values and the corresponding p-value (in parentheses) using for Plane 7913 and Plane 7909 are 0.1694 (0.4207) and 0.1113 (0.8653), respectively. Accordingly, both datasets fit the PoA distribution from Fig. 7 and KS statistics.

Table 21. The air conditioning system of jet airplanes.

Plane 7913 (X)	97	51	11	4	141	18	142	24	191	
	68	77	80	1	16	106	206	163	18	
	82	54	31	216	46	111	39	18	63	
Plane 7909 (Y)	90	10	60	186	61	49	14	24	208	130
	56	20	79	84	44	59	29	118	101	208
	25	156	310	76	26	44	23	62	70	

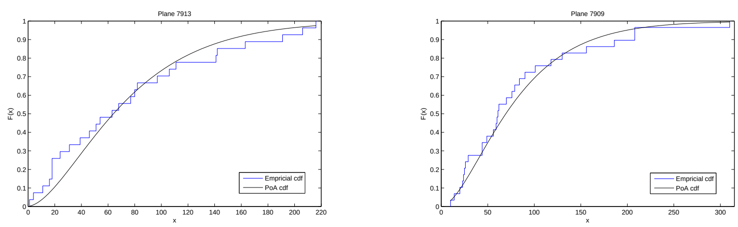

Fig. 7. Empirical and PoA distributions based on the Example III data set.

The RSS and SRS data sets were obtained according to the motivation given in the introduction part of the data are reported in Table 22 and 23:

Table 22. Newly chosen data for Plane 7913 and Plane 7909 SRS; $n = m = 21$.

X							Y						
54	18	39	68	54	46	77	186	61	59	20	61	10	84
1	24	80	82	54	46	24	90	20	56	56	59	56	44
1	54	4	111	18	141	46	49	62	49	62	79	186	26

Table 23. Newly chosen data for Plane 7913 and Plane 7909 RSS; $n = m = 21$.

	X				Y		
	Set				Set		
Cycle	1	2	3	Cycle	1	2	3
1	18	39	46	1	24	76	59
2	1	106	191	2	56	70	79
3	51	68	163	3	20	56	118
4	80	31	77	4	29	25	186
5	16	46	163	5	20	90	208
6	18	97	80	6	44	101	130
7	16	77	142	7	44	101	62

Table 24. Estimates of $R = 0.5241$ air conditioning system of jet airplanes.

	Sample	$\widehat{R}$	std-boot	p-boot	Bcp-boot	t-boot	basic	N-boot
ML	SRS	0.5798	0.2830	0.2842	0.2835	0.2842	0.2842	1.0467
	RSS	0.5166	0.2483	0.2461	0.2475	0.2461	0.2461	1.0412
LS	SRS	0.5888	0.2972	0.2999	0.2838	0.2999	0.2999	1.1209
	RSS	0.5369	0.2928	0.2906	0.2857	0.2906	0.2906	1.0805
WLS	SRS	0.5697	0.3337	0.3472	0.3215	0.3472	0.3472	1.1046
	RSS	0.5280	0.3116	0.3125	0.3123	0.3125	0.3125	1.0557
CVM	SRS	0.5689	0.3391	0.3511	0.3381	0.3511	0.3511	1.0926
	RSS	0.5231	0.3136	0.3170	0.3187	0.3170	0.3170	1.0424

The results for estimates SSR and the width of the CIs for the std-boot, p-boot, BCp-boot, t-boot, basic, and N-boot BCI methods are presented in Table 24. Then, the following estimates for SSR under RSS and SRS are obtained: Accordingly, while the SSR obtained from the sample created by the SRS method from the same population is far from its true value, the SSR obtained from the sample created by the RSS method is quite close to the true value. Additionally, Table 24 presents the confidence interval widths of the SSR estimates for the std-boot, p-boot, BCp-boot, t-boot, basic, and N-boot BCI methods. According to these results, the SSR obtained from the sample created using the SRS method from the same population is far from its true value, while the SSR obtained from the sample created using the RSS method is quite close to the true value. Additionally, for three data sets, the as of the estimates using the BCI RSS method and SSR were found to be narrower compared to the SRS method. This indicates that the RSS method provides more reliable results.

6 Conclusion

In this article, we examine point estimates of R=P(X<Y) using ML, LS, WLS, and CVM, as well as interval estimates using BCI methods (std-boot, N-boot, basic, t-boot, BCp-boot, and p-boot) based on different sampling methods, SRS and RSS, when X and Y are independent PoA random variables with different parameters. The performance of these sampling methods is compared using Monte Carlo simulations with MRE and MSE criteria. Simulation results under the assumption of perfect ranking show that RSS-based ML, LS, WLS, and CVM estimators outperform the corresponding SRS-based ML, LS, WLS, and CVM estimators. Additionally, as observed from these three real data sets, the RSS method provides better SSR estimation compared to the SRS method. The applicability of the results obtained theoretically and supported by simulations has been demonstrated with three real data applications. As a result, when examining point and interval estimates for the SRS and RSS methods under different m and r conditions, the RSS method has provided more consistent results compared to the SRS method. As with previous studies, it can be said that modifications to RSS reduce the likelihood of errors in rankings and thus increase the efficiency of estimators. The fact that SSR has rarely been studied in discrete models and the use of RSS as a sample makes this study unique. This study will be an inspiration for future works.

7 Compliance with Ethical Standards

Conflicts of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Funding Declaration The authors did not receive support from any organization for the submitted work.

Data availability statement The data sets have been taken from the literature and references are given at the end.

Ethical approval This article does not contain any studies with animals performed by any of the authors.

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